Second-order pseudo-stationary random fields and point processes on graphs and their edges

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(in collaboration with Ethan Anderes and Jakob G. Rasmussen)

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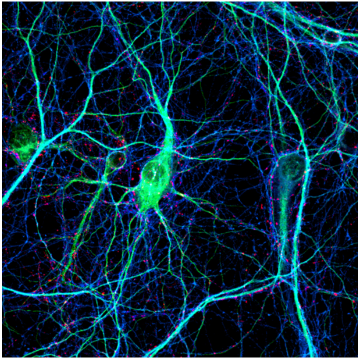
Graph with edges = dendrite networks of neurons (green lines):



The dendrites carry information from other neurons to the cell body.

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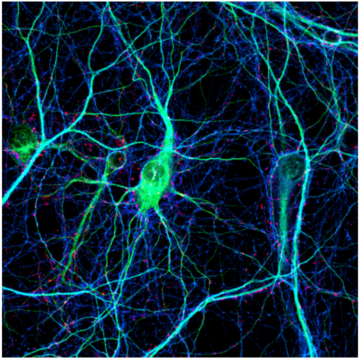

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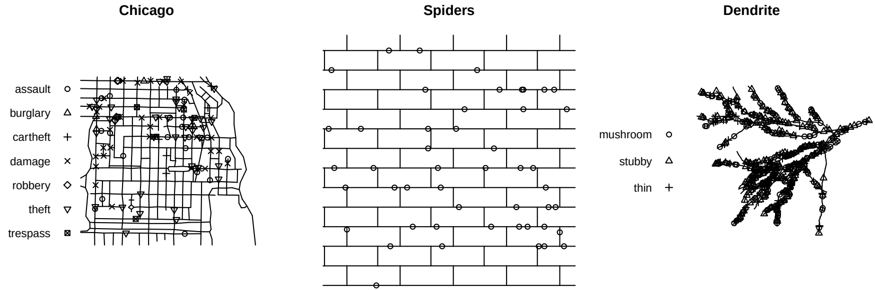
Want to model the random field = diameter along this graph with edges.

Point patterns on graphs with edges:

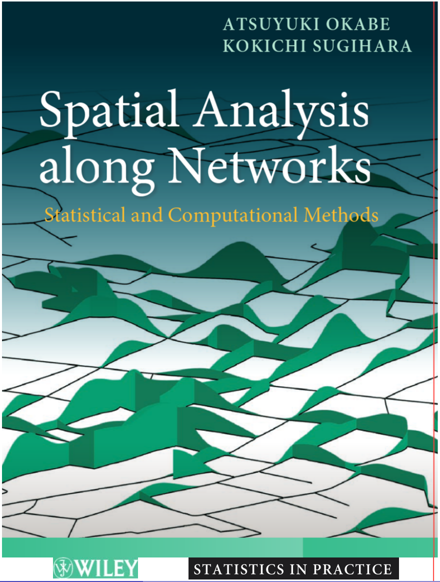


Is there
- clustering in street crimes?
- interaction between spider webs on mortar lines of a brick wall?
- interaction within and between different types of spines (small protusions)?

Snow's (1855) cholera map: Point pattern on a graph with edges = street network around the Broad Street pump:

Conclusion: cause of the victims' illness was contamination of the water from the Broad Street pump.

Textbook on ...


ATSUYUKI OKABE
KOKICHI SUGIHARA
Spatial Analysis along Networks
Statistical and Computational Methods
WILEY
STATISTICS IN PRACTICE

Some other research:

Cressie, Frey, Harch & Smith (2006). Spatial prediction on a river network. *Journal of Agricultural, Biological, and Environmental Statistics*.

Ver Hoef, Peterson & Theobald (2006). Spatial statistical models that use flow and stream distance. *Environmental and Ecological Statistics*.

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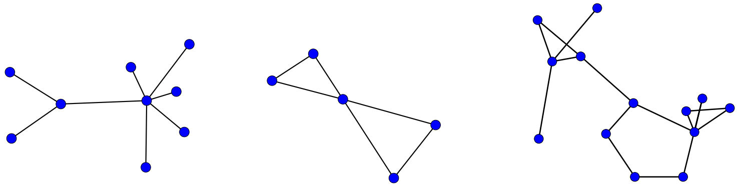
Ang, Baddeley & Nair (2012). Geometrically corrected second order analysis of events on a linear network, with applications to ecology and criminology. *Scandinavian Journal of Statistics*.

Baddeley, Jammalamadaka & Nair (2014). Multitype point process analysis of spines on the dendrite network of a neuron. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*.

Existing literature

... considers only the case of a
linear network:
edges = straight line segments, only meeting at vertices.

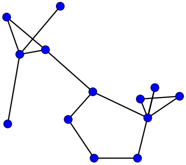


(Left and middle panels: linear networks. Right panel: *not* a linear network.)

A more general definition is needed

... in order to model

- crossing points (e.g. bridges/tunnels),
- multiple crossings (e.g. multiple roads),
- vertices disconnected from their adjacent edges (e.g. a ferry connecting two roads),
- curved or more general edges,
- different length measures on edges (e.g. different speeds on roads).



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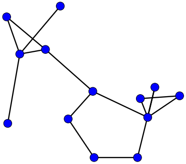

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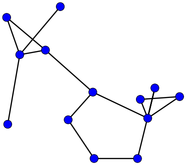

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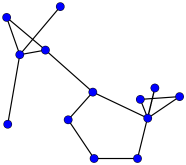

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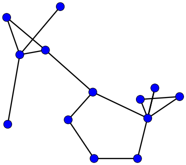

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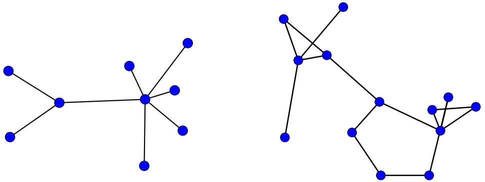
(c) $\varphi_i : e_i \mapsto (a_i, b_i)$ is a bijection called an **edge-coordinate**. ("Constant speed": φ_i^{-1} = natural parametrization of e_i.)

$L =$ **index set for random fields/space for point processes on** $\mathcal{G}$:



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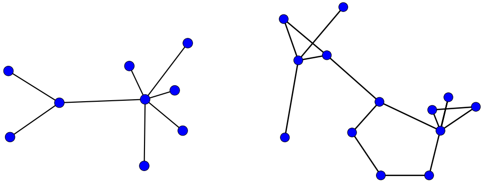
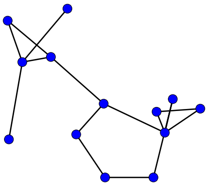


If no overlap (left panel): $L = \mathcal{V} \cup \bigcup_{i \in I} e_i$.

If overlap (right panel): $L = (\{0\} \times \mathcal{V}) \cup \bigcup_{i \in I} (\{i\} \times e_i)$.



Geodesic distance:
$d_{\mathcal{G}}(u, v) =$ **infimum of length of paths in** $\mathcal{G}$ **between** $u, v \in L$
(where "length" is induced by edge-coordinates and usual length on the intervals (a_i, b_i) — e.g. $d_{\mathcal{G}}(u, v)$ could be shortest time when driving from u to v with different speed limits).

Open problems:

- How do we construct **covariance functions** of the form

$$c(u, v) = c_0(d_{\mathcal{G}}(u, v))$$

for $u, v \in L$? Say then that c is **pseudo-stationary**.

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- How do we construct **point processes on** L with pair correlation function of the form

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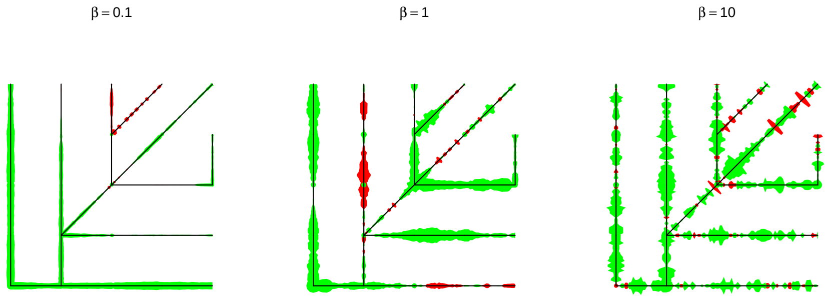
PART 1: PSEUDO-STATIONARY COVARIANCE FUNCTIONS AND RANDOM FIELDS



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- A graph with Euclidean edges $\mathcal{G}$ is said to **support the PDEFs** if for any $\beta > 0$,

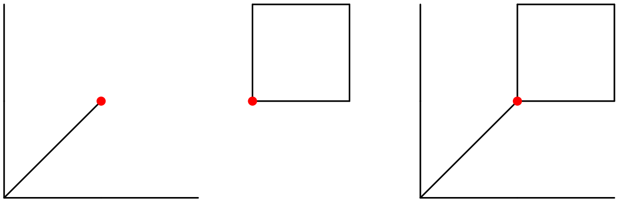
$$c(u, v) = \exp(-\beta d_{\mathcal{G}}(u, v))$$

is positive semi-definite for $u, v \in L$.

Definition 3:

Suppose $\mathcal{G}_1 = (\{\mathcal{V}_1, \{e_i : i \in I_1\}, \{\varphi_i : i \in I_1\})$ and $\mathcal{G}_2 = (\{\mathcal{V}_2, \{e_i : i \in I_2\}, \{\varphi_i : i \in I_2\})$ have only one vertex v_0 in common, but no common edges, and disjoint index sets I_1 and I_2.

The **1-sum** of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the graph with Euclidean edges given by $\mathcal{G} = (\mathcal{V}_1 \cup \mathcal{V}_2, \{e_i : i \in I_1 \cup I_2\}, \{\varphi_i : i \in I_1 \cup I_2\})$.



Graphs with Euclidean edges supporting the PDEFs:

Theorem 1. If $\mathcal{G}_1, \mathcal{G}_2, \ldots$ support the PDEFs, then the 1-sum of $\mathcal{G}_1, \mathcal{G}_2, \ldots$ supports the PDEFs. In fact $\sigma^2 \exp(-\beta d_{\mathcal{G}}(u, v))$ is (strictly) positive definite for all $\beta, \sigma^2 > 0$.

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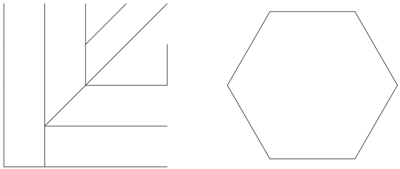
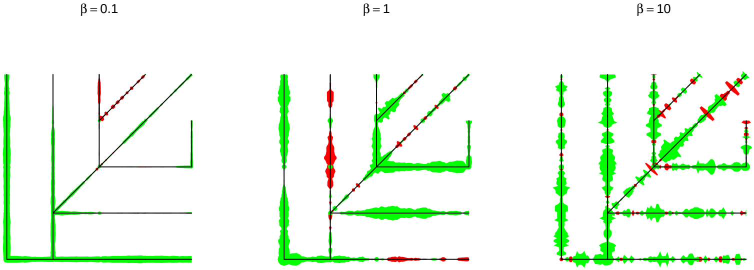
Theorem 2. Cycles and trees support the PDEFs, and so do countable 1-sums of these.

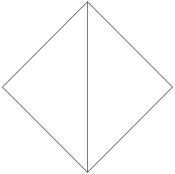


Sim. of GRF on $\mathcal{G}$ with $c(u, v) = \sigma^2 \exp(-\beta d_{\mathcal{G}}(u, v))$



Forbidden subgraph:

Theorem 3. Suppose G is a graph with Euclidean edges that has three paths which have common endpoints but are otherwise pairwise disjoint.



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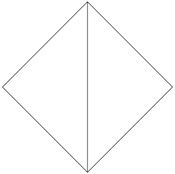

Then there exists a $\beta > 0$ s.t.

$$c(u, v) = \exp(-\beta d_{\mathcal{G}}(u, v)), \quad u, v \in L,$$

is **not** positive semi-definite.

Completely monotonic covariance functions:

$c_0 : [0, \infty) \mapsto [0, \infty)$ is **completely monotonic** if it is continuous and $(-1)^k c_0^{(k)}(t) \geq 0$ for all $t \in (0, \infty)$ and $k = 1, 2, \ldots$.

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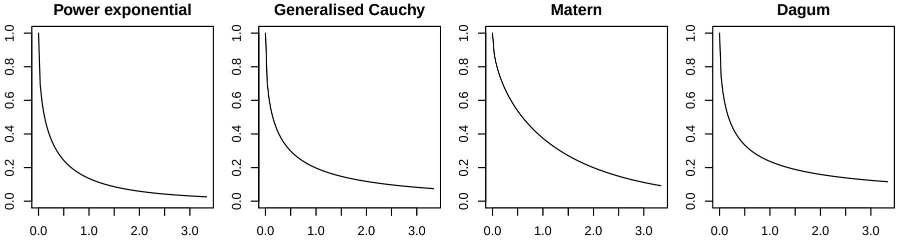
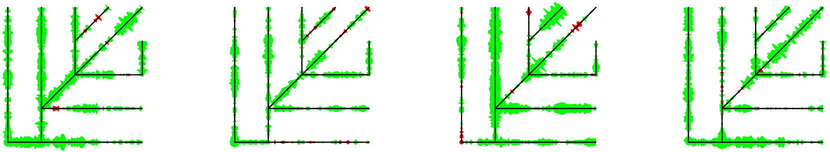
Theorem 4. If $\mathcal{G}$ supports the PDEFs and c_0 is completely monotonic and non-constant, then $c(u,v) = c_0(d_{\mathcal{G}}(u,v))$ is (strictly) pos. def.

Simulations using completely monotonic covariance fcts:





Examples of completely monotonic covariance functions:

Theorem 5. Suppose $\mathcal{G}$ supports the PDEFs. Then for $\sigma^2, \beta > 0$, we have parametric families of pos. def. cov. fcts. $c(u,v) = c_0(d_{\mathcal{G}}(u,v))$:

- **Power exponential covariance function:**

$$c_0(s) = \sigma^2 \exp\left(-\beta s^{\alpha}\right), \quad \alpha \in (0,1].$$

- **Generalized Cauchy covariance function:**

$$c_0(s) = \sigma^2 \left(\beta s^{\alpha} + 1\right)^{-\xi/\alpha}, \quad \alpha \in (0,1],\ \xi > 0.$$

- **The Matérn covariance function:**

$$c_0(s) = \sigma^2 \frac{(\beta s)^{\alpha} K_{\alpha}(\beta s)}{\Gamma(\alpha) 2^{\alpha-1}}, \quad \alpha \in (0, 1/2].$$

- **The Dagum covariance function:**

$$c_0(s) = \sigma^2 \left[1 - \left(\frac{\beta s^{\alpha}}{1 + \beta s^{\alpha}}\right)^{\xi/\alpha}\right], \quad \alpha, \xi \in (0,1].$$

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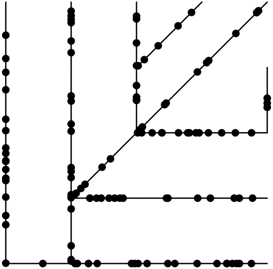
Theorem 6. For any of the functions $c(u, v)$ given in Theorem 5 but with $\alpha > 0$ outside the parameter range given in Theorem 5,

- there exists a graph with Euclidean edges $\mathcal{G}$ which supports the PDEFs (and is not necessarily a cycle),
- but $c(u, v)$ is **not** a covariance function.

PART 2: PSEUDO-STATIONARY POINT PROCESSES



Definitions for point processes on $\mathcal{G}$:

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• X has n^{th} **order joint intensity function** $\rho^{(n)}$ if
for small pairwise disjoint sets $B_1, \ldots, B_n \subset L$ and $u_1 \in B_1, \ldots, u_n \in B_n$,

$$\mathrm{P}(X \text{ has a point in each of } B_1, \ldots, B_n) \approx \rho^{(n)}(u_1, \ldots, u_n)\, \lambda_{\mathcal{G}}(B_1) \cdots \lambda_{\mathcal{G}}(B_n)\,.$$

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• If $\rho(u)$ is locally integrable, then for $\rho(u) > 0$ there exists a point process $X_u^!$ on $\mathcal{G}$ which follows the **reduced Palm distribution at** u, i.e.

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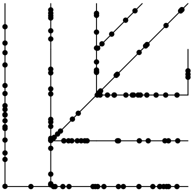
If $\rho(u) \equiv \rho > 0$ and $g(u,v) = g_0(d_{\mathcal{G}}(u,v))$, then for any $u \in L$,

$$\rho K(r) = \mathrm{E}\#\{v \in X_u^! : d_{\mathcal{G}}(u,v) \leq r\}.$$

Poisson processes:

• X is a **Poisson process** on $\mathcal{G}$ with (locally integrable) intensity function $\rho : L \mapsto [0, \infty)$, if for any $B \subseteq L$ with $\mu(B) := \int_B \rho(u)\,\mathrm{d}\lambda_{\mathcal{G}}(u) < \infty$,

- $\#(X \cap B) \sim \mathrm{Poisson}(\mu(B))$,
- cond. on $\#(X \cap B)$, the points in $X \cap B$ are iid with density $\propto \rho$.



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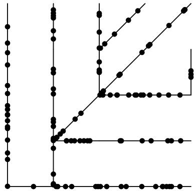

• Then $\rho^{(n)}(u_1, \ldots, u_n) = \rho(u_1) \cdots \rho(u_n)$, so $g(u, v) = 1$, i.e. X is pseudo-stationary and $K(r) = r$.

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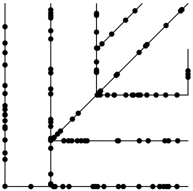

• Then $\rho^{(n)}(u_1, \ldots, u_n) = \rho(u_1) \cdots \rho(u_n)$, so $g(u, v) = 1$, i.e. X is pseudo-stationary and $K(r) = r$. Moreover, $X_u^! \sim X$ whenever $\rho(u) > 0$.

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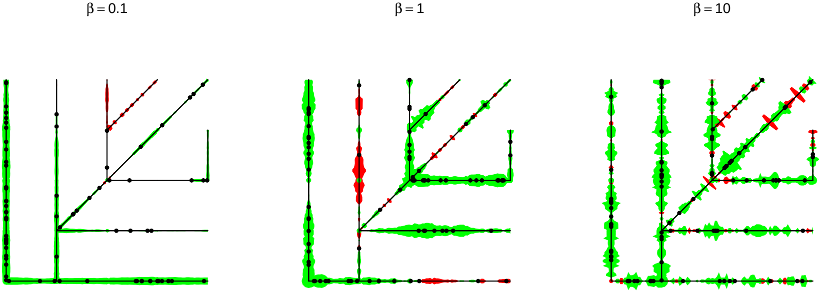
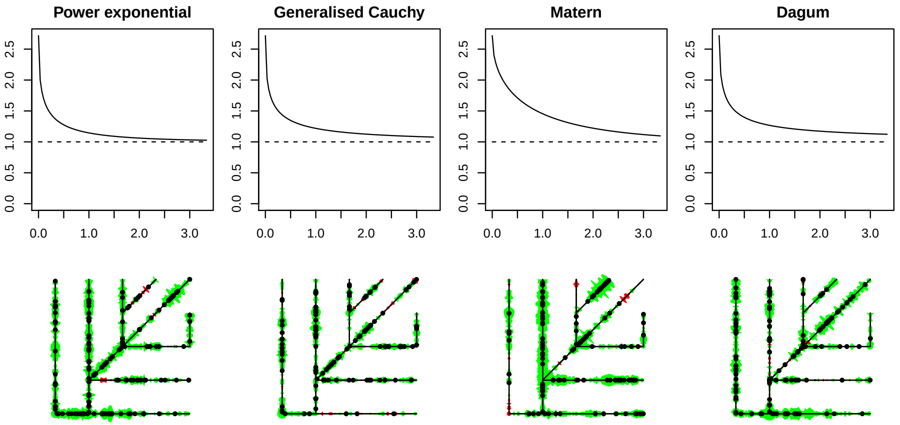
• $X_u^!$ is a LGCP with underlying GRF having mean function $m_u(v) = m(v) + c(u, v)$ and covariance function c.

Simulations of LGCPs using exponential covariance fcts:

Given a realisation of the GRF Z on $\mathcal{G}$, we simulate a Poisson process with intensity function $\exp(Z)$ to obtain a simulation of the LGCP X on $\mathcal{G}$.



Simulations of LGCPs using other covariance fcts:



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- However, they only work on trees, cycles and countable 1-sums of these.
- For other graphs even something simple like the exponential function is not (necessarily) a covariance function.
- The covariance functions we have established are all completely monotonic, so they cannot e.g. be negative.

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THANK YOU!