**Title:** The Nash-Tognoli theorem over $\mathbb{Q}$ & the $\mathbb{Q}$-algebraicity problem for isolated singularities

**Speaker:** *Enrico Savi*

**Abstract.** In 1973, Tognoli proved that every compact smooth manifold $M$ is diffeomorphic to a nonsingular real algebraic set $M' \subset \mathbb{R}^m$, confirming a conjecture by Nash from 1952. This result, known as the Nash-Tognoli Theorem, led to major advances in real algebraic geometry, particularly regarding the *algebraicity problem*, which seeks to characterize stratified spaces that admit a homeomorphic real algebraic model.

Let $K$ be a subfield of $\mathbb{R}$. We refer to a real $K$*-algebraic set* as a real algebraic set $X \subset \mathbb{R}^n$ defined by polynomial equations over $K$. In 2020, Parusinski and Rond proved that every real algebraic set $X \subset \mathbb{R}^n$ is homeomorphic to a real $\overline{\mathbb{Q}}^r$-algebraic set $X' \subset \mathbb{R}^n$, where $\overline{\mathbb{Q}}^r = \overline{\mathbb{Q}} \cap \mathbb{R}$ denotes the real closure of $\mathbb{Q}$. Hence, Parusinski proposed the following open problem:

$\mathbb{Q}$-ALGEBRAICITY PROBLEM: (Parusiński, 2021) *Is every algebraic set $X \subset \mathbb{R}^n$ homeomorphic to some $\mathbb{Q}$-algebraic set $X' \subset \mathbb{R}^m$, with $m \geqslant n$?*

Compared to the result of Parusinski and Rond, the fact that $\mathbb{Q}$ is not a real closed field is a crucial difficulty.

The aim of this talk is to present a version over $\mathbb{Q}$ of the Nash-Tognoli theorem, obtained in collaboration with Ghiloni, emphasizing its deep connections to recent advances in algebraic geometry over subfields, as developed by Fernando and Ghiloni. As an application of resolution of singularities, the Nash-Tognoli theorem over $\mathbb{Q}$ and blowing-down techniques I will present a complete positive answer to the $\mathbb{Q}$-ALGEBRAICITY PROBLEM in the case of real algebraic sets with isolated singularities. More precisely, the main theorem I will present reads as follows:

**Theorem** ([GS23])**.** *Every real algebraic set with isolated singularities $X \subset \mathbb{R}^n$ is semialgebraically homeomorphic to a $\mathbb{Q}$-algebraic set $X' \subset \mathbb{R}^m$, with $m \geqslant n$.*

Time permitting, I will also discuss other affirmative answers to the $\mathbb{Q}$-ALGEBRAICITY PROBLEM , as well as some ongoing work.

The talk is mostly based on [GS23] and [Sav24].

## REFERENCES

[GS23] Riccardo Ghiloni and Enrico Savi. *The topology of real algebraic sets with isolated singularities is determined by the field of rational numbers.* 2023 (The final improved version will soon appear on arXiv). arXiv: `2302.04142 [math.AG]`.

[Sav24] Enrico Savi. *A relative Nash-Tognoli theorem over $\mathbb{Q}$ and application to the $\mathbb{Q}$-algebraicity problem.* 2024. arXiv: `2302.04673 [math.AG]`.