EXERCISES ON SINGULARITIES OF PAIRS

Exercise 0.1. Let $C = (x^2 - y^3 = 0) \subseteq \mathbb{A}^2$. Find a log resolution of $(\mathbb{A}^2, C)$.

Exercise 0.2. Let (X, D) be a pair. Assume that (X, D) is not lc. Prove that $\mathrm{discrep}(X, D) = -\infty$.

Exercise 0.3. Let $(X, \sum_i D_i)$ be a log smooth pair.

(1) Prove that $(X, \sum_i d_i D_i)$ is klt, resp. lc if and only if $d_i < 1$ for every i, resp. $d_i \leq 1$ for every i.
(2) Prove that $(X, \sum_i d_i D_i)$ is terminal if and only if $d_i < 1$ for every i and $d_i + d_j < 1$ whenever $D_i \cap D_j \neq \emptyset$.

Exercise 0.4. Let X be a normal surface. Show that $(X, 0)$ is terminal if and only if $(X, 0)$ is smooth.

Exercise 0.5. Let $W \subseteq \mathbb{P}^{n-1}$ be a variety and let $Y \subseteq \mathbb{P}^n$ be the cone over W with vertex v. We assume that the blow up X of Y along v is a desingularization of C_W.

(1) Prove that if W is a normal curve of genus g, then Y is klt if and only if $g = 0$ and lc if and only if $g \leq 1$.
(2) What happens if $\dim W \geq 2$?

Exercise 0.6. Determine a log resolution and singularities of $(X, 0)$ where X is

(1) $(x^2 + y^3 + z^5 = 0) \subseteq \mathbb{A}^3$,
(2) $(xy + z^2 + t^k = 0) \subseteq \mathbb{A}^4$, for every $k \geq 1$;
(3) $(x^2 + y^3 + z^6 + t^6 = 0) \subseteq \mathbb{A}^4$.

Exercise 0.7. Determine the singularities of $(X, 0)$ where X is

(1) $\mathbb{A}^2/\langle g \rangle$ with $g(x, y) = (-x, -y)$;
(2) $\mathbb{A}^2/\langle h \rangle$ with $h(x, y) = (\omega x, \omega y)$ and $\omega^3 = 1$, $\omega \neq 1$.

Exercise 0.8. Let X be a variety. Let $D \sum_i d_i D_i$ and $D' \sum_i d'_i D_i$ be divisors on X such that (X, D) and (X, D') are pairs. Assume that $d'_i \geq d_i$ for every i. Then $\mathrm{discrep}(X, D') \leq \mathrm{discrep}(X, D)$.

Exercise 0.9. Let X be a smooth variety and B an effective divisor. Then (X, B) is lc if and only if for every $c \in (0, 1)$ the pair (X, cB) is klt. Find an example of lc pair (X, B) such that X is not smooth and (X, cB) is not klt for every $c \in (0, 1)$.

Exercise 0.10. Let X, Y be projective varieties and $p \colon X \to Y$ be a finite map. Let D_X and D_Y be divisors on X, Y such that (X, D_X) is a pair and $K_X + D_X = p^*(K_Y + D_Y)$. Then

(1) $1 + \mathrm{discrep}(Y, D_Y) \leq 1 + \mathrm{discrep}(X, D_X) \leq \deg(p)(1 + \mathrm{discrep}(Y, D_Y))$;
(2) (X, D_X) is lc, resp. klt if and only if (Y, D_Y) is lc, resp. klt.

Exercise 0.11. Let (X, D) be a pair. Then (X, D) is klt if and only if $\mathcal{I}(X, D) \cong \mathcal{O}_X$.

Exercise 0.12. Verify that the definition of $\mathcal{I}(X, D)$ does not depend on the log resolution.

Exercise 0.13. Let $X \subseteq \mathbb{C}^{n+1}$ be an n-dimensional hypersurface which is smooth except for an ordinary d-fold point at $P \in X$: thus $\mathrm{mult}_P X = d$, and the blowing-up $\mu\colon X_0 = Bl_P(X) \to X$ of P is a log-resolution of $(X, 0)$. Prove that

(1) X is Gorenstein (i.e. K_X is Cartier), and
(2) $\mathcal{I}((X, 0), 0) = m^{d-n}$.

Definition 0.14. Let X be a smooth divisor and $D = \sum a_i D_i$ be an effective $\mathbb{Q}$-divisor on X, and let $x \in X$ be a fixed point. The multiplicity $mult_x D$ is the rational number $\sum_i a_i mult_x D_i$.

Exercise 0.15. Let X be a smooth divisor and D an effective $\mathbb{Q}$-divisor on X. Suppose that $x \in X$ is a point at which $mult_x D < 1$, the multiplicity of a $\mathbb{Q}$-divisor being understood in the sense of Definition 0.14. Then the multiplier ideal $\mathcal{I}(X, D)$ associated to D is trivial at x, that is $\mathcal{I}(X, D)_x \cong \mathcal{O}_{X,x}$.

Exercise 0.16. Let $d, r \in \mathbb{N}$. Then there is $t \in \mathbb{R}$, $t = t(d, r)$ such that for every (X, L, A) with

- $\dim X = d$, X smooth projective;
- A very ample $A^d \leq t$;
- $L \geq 0$ and $\deg_A L \leq r$;

the pair (X, tL) is klt.

Exercise 0.17. Let X be a weak Fano variety (i.e. K_X Cartier, and $-K_X$ nef and big and $(X, 0)$ klt). Then

- $(X, 0)$ is canonical;
- $\chi(X) = 1$;
- there are no finite étale morphisms $\widetilde{X} \to X$.

Exercise 0.18. Let X be such that K_X Cartier, nef and big and $(X, 0)$ klt. Then K_X is semiample.

Useful (?) results

Theorem 0.1 (Hodge index theorem)**.** *Let X be a compact Kähler surface. Then the signature of the intersection form on $H^2(X, \mathbb{R}) \cap H^{1,1}(X)$ is $(1, h^{1,1} - 1)$.*

Lemma 0.2. *Let $\pi\colon X' \to X$ be a finite morphism between normal proper varieties, and let $\tilde{f}\colon \widetilde{Y} \to X'$ be a birational morphism. Then there exists a birational morphism $f\colon Y \to X$ such that, if Y' is the normalisation of the main component of the fibre product $X' \times_X Y$ and $f'\colon Y' \dashrightarrow \widetilde{Y}$ is the induced birational map, then $(f')^{-1}$ is an isomorphism at the generic point of each $\tilde{f}$-exceptional prime divisor on $\tilde{}\widetilde{Y}$.*

$$\begin{array}{ccccc} X' & \xleftarrow{\tilde{f}} & \widetilde{Y} & \xleftarrow{f'} & Y' \\ \downarrow{\scriptstyle \pi} & & & & \downarrow \\ X & & \xleftarrow{\quad f \quad} & & Y \end{array}$$

Moreover, we may assume that f is a composition of blowups along proper subvarieties.