Proof: exploiting Assumption 4, together with Lemma 7, the arithmetic-geometric mean inequality (with $\epsilon = p/h_\kappa$), and Lemma 8, we deduce that

$$
\begin{aligned}
\|v - \tilde{\Pi}_p v\|^2_{L^2(\partial\kappa)} &\leq \sum_{F\subset\partial\kappa} C_{\rm t}|F| \left(\frac{1}{|\kappa_\flat^F|}\|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa_\flat^F)} \right. \\
&\qquad \left. + \frac{h_{\kappa_\flat^F}}{|\kappa_\flat^F|}\|v - \tilde{\Pi}_p v\|_{L^2(\kappa_\flat^F)}\|\nabla(v - \tilde{\Pi}_p v)\|_{L^2(\kappa_\flat^F)} \right) \\
&\leq C_{\rm t}\, C_s\, d \sum_{F\subset\partial\kappa} \left(\frac{1}{h_\kappa}\|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa_\flat^F)} \right. \\
&\qquad \left. + \|v - \tilde{\Pi}_p v\|_{L^2(\kappa_\flat^F)}\|\nabla(v - \tilde{\Pi}_p v)\|_{L^2(\kappa_\flat^F)} \right) \\
&\leq C_{\rm t}\, C_s\, d \sum_{F\subset\partial\kappa} \left(\frac{p+1}{h_\kappa}\|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa_\flat^F)} + \frac{h_k}{4p}\|\nabla(v - \tilde{\Pi}_p v)\|^2_{L^2(\kappa_\flat^F)} \right) \\
&\leq C_{\rm t}\, C_s\, d \left(\frac{(p+1)}{h_\kappa}\|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa)} + \frac{h_k}{4p}\|\nabla(v - \tilde{\Pi}_p v)\|^2_{L^2(\kappa)} \right) \\
&\leq \frac{9}{4} C_{\rm t}\, C_s\, d\, C^2_{{\rm I},3} \frac{h_\kappa^{2s_\kappa - 1}}{p^{2l_\kappa - 1}}\|\mathcal{E}v\|^2_{H^{l_\kappa}(\mathcal{K})}.
\end{aligned}
$$

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Lemma 13

Given that Assumption 4 holds, we define the $\sigma : \mathcal{F}_h \to \mathbb{R}$ by

$$\sigma(\mathbf{x}) := \begin{cases} C_\sigma \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left\{ C_{\mathrm{inv},1} \frac{p_\kappa^2}{h_\kappa} \right\}, & \mathbf{x} \in F \in \mathcal{F}_h^{\mathcal{I}},\ F = \partial\kappa^+ \cap \partial\kappa^-, \\ C_\sigma C_{\mathrm{inv},1} \frac{p_\kappa^2}{h_\kappa}, & \mathbf{x} \in F \in \mathcal{F}_h^{\mathcal{B}},\ F \subset \partial\kappa, \end{cases}$$

where C_σ is a sufficiently large positive constant, which is independent of the number of faces per element. Then, the bilinear form $\tilde{B}_{\mathrm{d}}(\cdot,\cdot)$ is coercive and continuous over $\mathcal{V} \times \mathcal{V}$, i.e.,

$$\tilde{B}_{\mathrm{d}}(v,v) \geq C_{\mathrm{coer}} |||v|||_{\mathrm{DG}}^2 \quad \text{for all} \quad v \in \mathcal{V},$$

and

$$\tilde{B}_{\mathrm{d}}(w,v) \leq C_{\mathrm{cont}} |||w|||_{\mathrm{DG}}\ |||v|||_{\mathrm{DG}} \quad \text{for all} \quad w, v \in \mathcal{V},$$

respectively.

Theorem 2

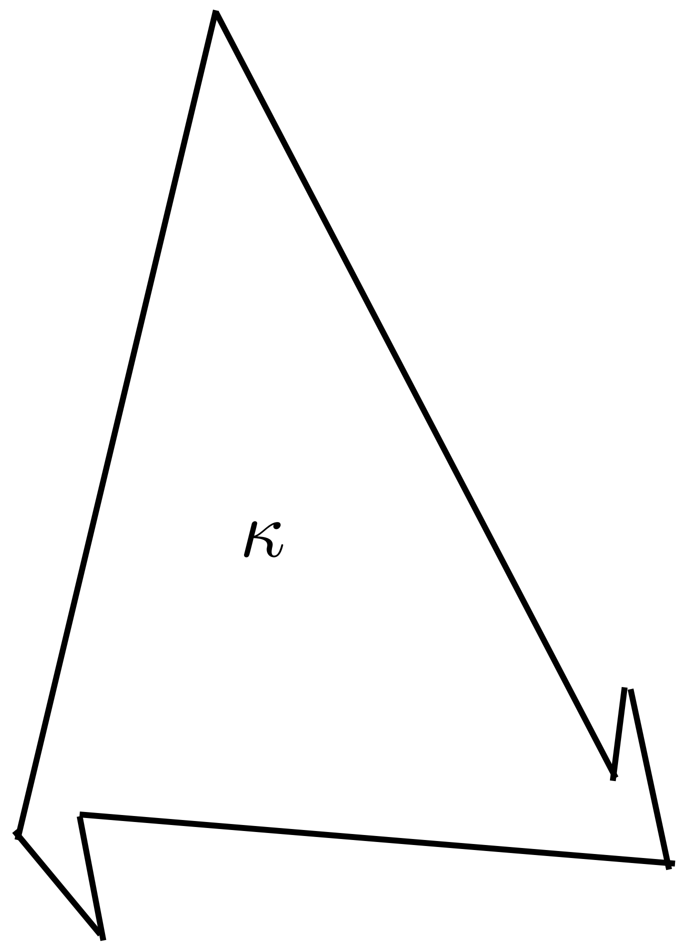
Given that $\mathcal{T}_h = \{\kappa\}$ satisfies Assumptions 3 and 4. Assuming $u|_\kappa \in H^{l_\kappa}(\kappa)$, $l_\kappa > 3/2$, for each $\kappa \in \mathcal{T}_h$, such that $\mathfrak{E}u|_{\mathcal{K}} \in H^{l_\kappa}(\mathcal{K})$, where $\mathcal{K} \in \mathcal{T}_h^\sharp$ with $\kappa \subset \mathcal{K}$, then

$$|||u - u_h|||_{\mathrm{DG}}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \left(1 + \mathcal{G}_\kappa(h_\kappa, p_\kappa)\right) \|\mathfrak{E}u\|_{H^{l_\kappa}(\mathcal{K})}^2,$$

where, $s_\kappa = \min\{p_\kappa + 1, l_\kappa\}$,

$$\begin{aligned} \mathcal{G}_\kappa(h_\kappa, p_\kappa) \quad := \quad & p_\kappa h_\kappa^{-1} \max_{F \subset \partial\kappa} \sigma|_F^{-1} + p_\kappa^2 h_\kappa^{-1} \max_{F \subset \partial\kappa} \sigma|_F^{-1} \\ & + p_\kappa^{-1} h_\kappa \max_{F \subset \partial\kappa} \sigma|_F, \end{aligned}$$

and C is a positive constant, independent of the discretization parameters and the number of faces per element.



- Note that κ *does not* satisfy Assumption 4 as the faces degenerate, but is *p*-coverable

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Implementation Aspects

Polynomial basis and Quadrature Construction

Element Basis Construction

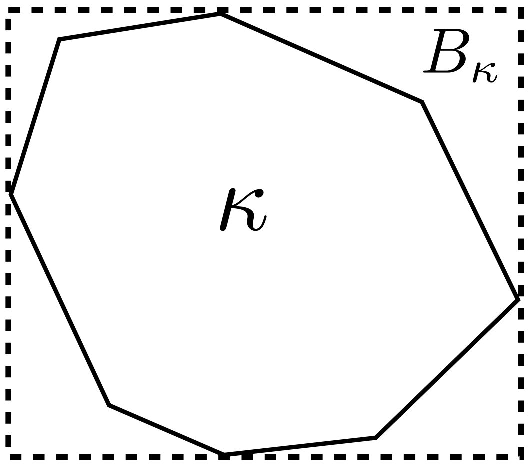
- **Composite meshes (overlapping refined meshes):** Coarse level basis functions generated based on application of the restriction operator applied to the fine level space: for $\phi_{\mathtt{CFE},i} \in V(\mathcal{T}_{\mathtt{CFE}}, \mathbf{p})$

$$\phi_{\mathtt{CFE},i} := \sum_{j=1,\ldots,\dim(V(\mathcal{T}_{h_\ell},p))} \alpha_{i,j}\phi_{h_\ell,j}\ , \quad \phi_{h_\ell,j} \in V(\mathcal{T}_{h_\ell}, p).$$

Hackbusch & Sauter 1997, Antonietti, Giani, & H 2013

- **General polytopes:** Basis may be constructed in global coordinates, for example, based on restricting the support of a set of polynomials defined over the bounding box of each element.

$$\mathcal{P}_{p_\kappa}(B_\kappa) = \mathrm{span}\{L_{i_1}(x_1)\cdots L_{i_d}(x_d)\}_{i=1}^{\dim(\mathcal{P}_{p_\kappa}(B_\kappa))}.$$



Cangiani, Georgoulis, & H 2014

Gram Schmidt Orthogonalisation: Bassi, Botti & Colombo 2013

- ☑ Order of approximation is independent of the shape of the element;
- ☐ Computational overhead: each element is unique, and hence all operations must be undertaken `on-the-fly'.

Numerical Integration

- Element Sub-Tessellation: Quadrature employed on the sub-elements.

  Johansson and Larson 2013, Cangiani, Georgoulis, & H 2014

- Moment Quadratures: Computationally expensive since a quadrature much be generated for every element.

  Mousavi, Xiao, & Sukumar 2010, Xiao & Gimbutas 2010, Mousavi & Sukumar 2011

- Integration of Homogeneous Functions: Exploit Stokes' theorem, together with Euler's homogeneous function theorem.

  Lasserre 1998, Chin, Lasserre, & Sukumar 2015

Integration of Homogeneous Functions

- Given a polytopic domain $\mathcal{P}$, and a sufficiently regular function *f*, defined over $\mathcal{P}$, we wish to evaluate

$$\int_{\mathcal{P}} f \mathrm{d}\mathbf{x}.$$

- Assuming that *f* is a positively homogeneous function of degree *q*, i.e.,

$$f(\lambda \mathbf{x}) = \lambda^q f(\mathbf{x}),$$

for $\lambda > 0$, then assuming *f* is continuously differentiable, Euler's homogeneous function theorem states that

$$q f(\mathbf{x}) = \mathbf{x} \cdot \nabla f(\mathbf{x}).$$

[Proof: Compute $d/d\lambda|_{\lambda=1}$].

Lasserre 1998, Chin, Lasserre, & Sukumar 2015

Integration of Homogeneous Functions

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

- Given a vector-valued function $\mathbf{g}$, Stokes' theorem states that

$$\int_{\mathcal{P}} (\nabla \cdot \mathbf{g}) f \mathrm{d}\mathbf{x} = \int_{\partial\mathcal{P}} (\mathbf{g} \cdot \mathbf{n}) f \mathrm{d}s - \int_{\mathcal{P}} \mathbf{g} \cdot \nabla f \mathrm{d}\mathbf{x},$$

where $\mathbf{n}$ denotes the unit outward normal vector to the boundary $\partial\mathcal{P}$ of $\mathcal{P}$.

- Thereby, selecting $\mathbf{g} = \mathbf{x}$ and employing Euler's homogeneous function theorem, we deduce that

$$\int_{\mathcal{P}} f \mathrm{d}\mathbf{x} = \frac{1}{d+q} \int_{\partial\mathcal{P}} (\mathbf{x} \cdot \mathbf{n}) f \mathrm{d}s.$$

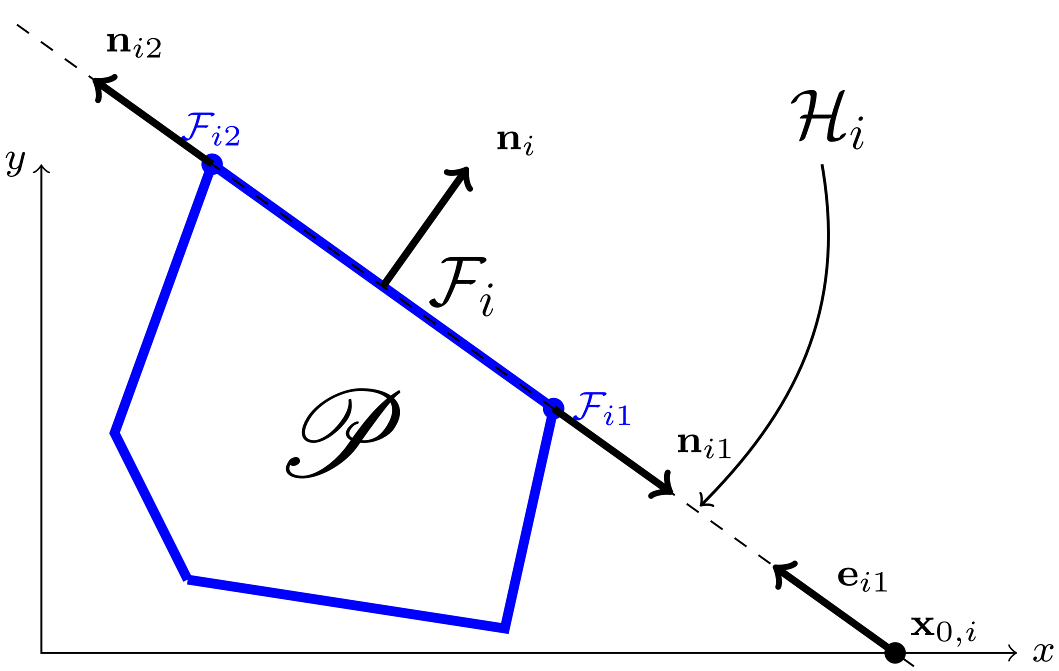
$\Rightarrow$ The integral over $\mathcal{P}$ is reduced to an integration over the boundary $\partial\mathcal{P}$.

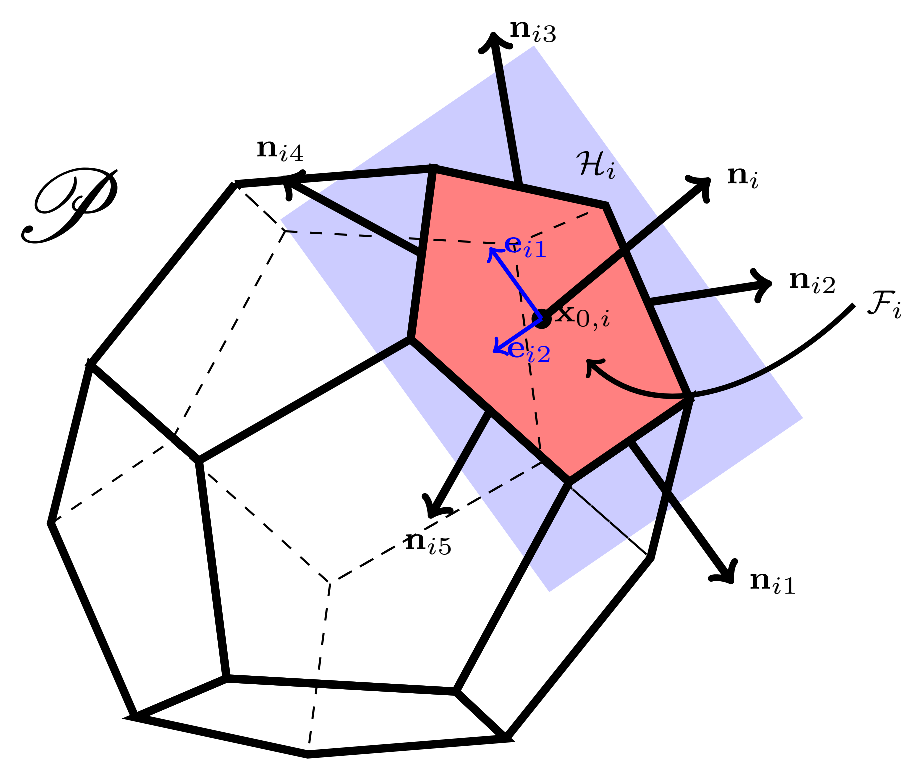
Integration of Homogeneous Functions

- Writing $\partial\mathcal{P} = \bigcup_{i=1}^{m} \mathcal{F}_i$, where $\mathcal{F}_i$, $i = 1, \ldots, m$, denote the planar $(d-1)$-dimensional facets which form the boundary of $\mathcal{P}$, we have

$$\int_{\mathcal{P}} f \mathrm{d}\mathbf{x} = \frac{1}{d+q} \sum_{i=1}^{m} \int_{\mathcal{F}_i} (\mathbf{x} \cdot \mathbf{n}_i) f \mathrm{d}s,$$

where $\mathbf{n}_i$ denotes the restriction of the unit outward normal vector $\mathbf{n}$ to the facet $\mathcal{F}_i$, $i = 1, \ldots, m$.

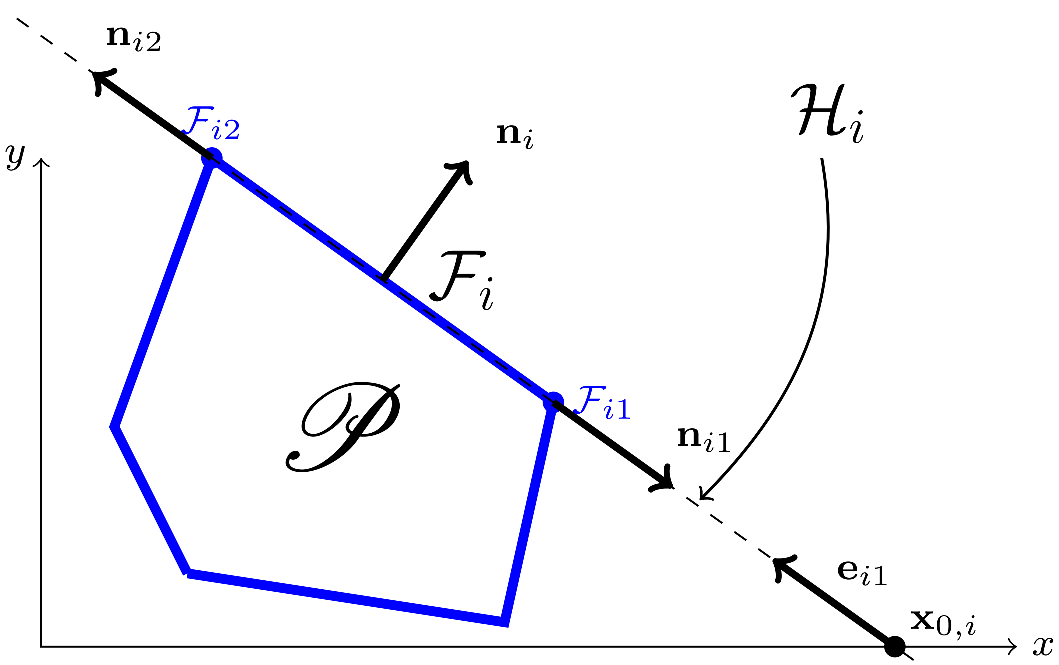


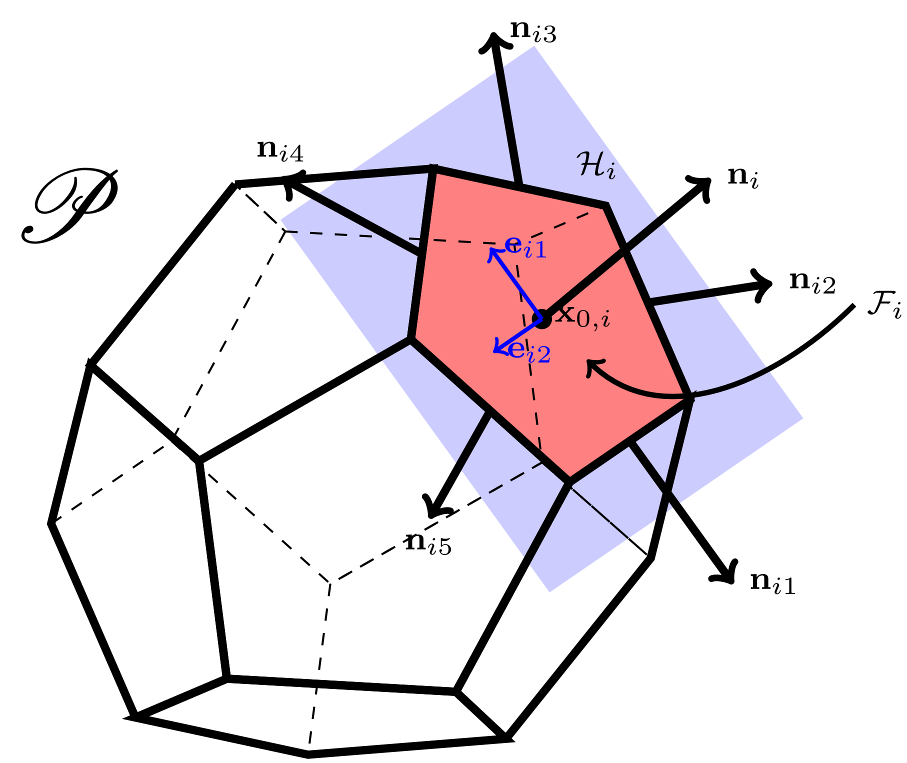


- This process can be repeated in order to yield a formula which involves integration on lower dimensional facets. For example, given $\mathcal{F}_i$, for some (fixed) i, $1 \leq i \leq m$, we write

$$\partial \mathcal{F}_i = \{\mathcal{F}_{ij} = \mathcal{F}_i \cap \mathcal{F}_j : \mathcal{F}_i \cap \mathcal{F}_j \neq \emptyset,\ i \neq j\}$$

to denote the set of $(d-2)$-dimensional facets of $\mathcal{P}$.





- We define $\mathbf{n}_{ij}$ to be the unit normal vector to $\mathcal{F}_{ij}$ which lies in the plane $\mathcal{F}_i$.
- Given an arbitrary point $\mathbf{x}_i \in \mathcal{F}_i$ and a $(d-1)$-dimensional orthonormal basis $\{\mathbf{e}_j^i\}_{j=1}^{d-1}$ on the facet $\mathcal{F}_i$, i.e., any $\mathbf{x} \in \mathcal{F}_i$ may be written in the form

$$\mathbf{x} = \mathbf{x}_i + \sum_{k=1}^{d-1} \alpha_k \mathbf{e}_k,$$

for some scalars α_k, $k = 1, \ldots, d-1$.

- Upon application of Stokes Theorem to a given facet $\mathcal{F}_i$, $1 \le i \le m$, with $\mathbf{g} = \mathbf{x} - \mathbf{x}_i$, we deduce that

$$\int_{\mathcal{F}_i} f \mathrm{d}s = \frac{1}{d+q-1} \left(\sum_{\mathcal{F}_{ij} \subset \partial \mathcal{F}_i} \int_{\mathcal{F}_{ij}} ((\mathbf{x} - \mathbf{x}_j) \cdot \mathbf{n}_{\mathcal{F}_{ij}}) f \mathrm{d}s + \int_{\mathcal{F}_i} (\mathbf{x}_i \cdot \nabla f) \mathrm{d}s \right).$$