# A Priori Error Analysis

- Given $\mathcal{I}_{\mathbf{p}} : V \to V^{\mathbf{p}}(\mathcal{T}_h)$, write

$$u - u_h = u - \mathcal{I}_{\mathbf{p}} u + \mathcal{I}_{\mathbf{p}} u - u_h \equiv \eta + \xi,$$

where $\eta = u - \mathcal{I}_{\mathbf{p}} u$ and $\xi = \mathcal{I}_{\mathbf{p}} u - u_h (\in V^{\mathbf{p}}(\mathcal{T}_h))$.

- Hence,

$$|||u - u_h|||_{\mathrm{DG}} \leq |||\eta|||_{\mathrm{DG}} + |||\xi|||_{\mathrm{DG}}$$

- Employing Galerkin orthogonality, we deduce that

$$B_{ar}(u - u_h, v_h) \equiv B_{ar}(\eta + \xi, v_h) = 0 \qquad \forall v_h \in V^{\mathbf{p}}(\mathcal{T}_h).$$

- Hence

$$(|||\xi|||^2_{\mathrm{DG}} =) \; B_{ar}(\xi, \xi) = -B_{ar}(\eta, \xi).$$

- Idea: Bound the right-hand side in terms of $|||\xi|||_{\mathrm{DG}}$ and employ approximation results.

Consider

$$
\begin{aligned}
B_{ar}(\eta,\xi) &= \sum_{\kappa\in\mathcal{T}_h}\left\{\int_\kappa \mathcal{L}\eta\,(\xi+\delta\mathcal{L}\xi)\mathrm{d}\mathbf{x} - \int_{\partial_-\kappa\setminus\partial\Omega} \mathbf{b}\cdot\mathbf{n}_\kappa\,(\eta^+-\eta^-)\xi^+\,\mathrm{d}s\right.\\
&\qquad\left. - \int_{\partial_-\kappa\cap\partial\Omega} \mathbf{b}\cdot\mathbf{n}_\kappa\,\eta^+\xi^+\,\mathrm{d}s\right\}.
\end{aligned}
$$

Integrating by parts gives

$$
\int_\kappa \mathcal{L}\eta\,\xi\mathrm{d}\mathbf{x} = \int_{\partial\kappa}\mathbf{b}\cdot\mathbf{n}_\kappa\eta^+\xi^+\mathrm{d}s + 2\int_\kappa (c - {}^1\!/_2\nabla\cdot\mathbf{b})\xi\eta\mathrm{d}\mathbf{x} - \int_\kappa \eta\mathcal{L}\xi\mathrm{d}\mathbf{x}.
$$

Rewriting the boundary terms, cf. above, gives

$$
\begin{aligned}
B_{ar}(\eta,\xi) &= \sum_{\kappa\in\mathcal{T}_h}\left\{\int_\kappa \delta\mathcal{L}\eta\mathcal{L}\xi\mathrm{d}\mathbf{x} + 2\int_\kappa c_0^2\xi\eta\mathrm{d}\mathbf{x} - {\color{red}\int_\kappa \eta\mathcal{L}\xi\mathrm{d}\mathbf{x}}\right.\\
&\qquad\left. - \int_{\partial_-\kappa\setminus\partial\Omega} \mathbf{b}\cdot\mathbf{n}_\kappa\,(\xi^+-\xi^-)\eta^-\,\mathrm{d}s - \int_{\partial_+\kappa\cap\partial\Omega} \mathbf{b}\cdot\mathbf{n}_\kappa\,\eta^+\xi^+\,\mathrm{d}s\right\}.
\end{aligned}
$$

- How to deal with the term $\int_\kappa \eta \mathcal{L} \xi \, \mathrm{d}\mathbf{x}$:

1. Assume $\delta > 0$:

$$
\begin{aligned}
B_{ar}(\eta, \xi) \quad = \quad & \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa \left( \sqrt{\delta} \mathcal{L} \eta - \frac{1}{\sqrt{\delta}} \eta \right) \sqrt{\delta} \mathcal{L} \xi ) \mathrm{d}\mathbf{x} + 2 \int_\kappa c_0^2 \xi \eta \mathrm{d}\mathbf{x} \right. \\
& \left. - \int_{\partial_- \kappa \backslash \partial \Omega} \mathbf{b} \cdot \mathbf{n}_\kappa \, (\xi^+ - \xi^-) \eta^- \, \mathrm{d}s - \int_{\partial_+ \kappa \cap \partial \Omega} \mathbf{b} \cdot \mathbf{n}_\kappa \, \eta^+ \xi^+ \, \mathrm{d}s \right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|||\xi|||_{\mathrm{DG}}^2 \quad \leq \quad & \left\| \sqrt{\delta} \mathcal{L} \eta - \frac{1}{\sqrt{\delta}} \eta \right\|_{L^2(\Omega)}^2 + 4 \| c_0 \eta \|_{L^2(\Omega)}^2 \\
& + 2 \sum_{\kappa \in \mathcal{T}_h} \left\{ \| \eta^- \|_{\partial_- \kappa \backslash \partial \Omega}^2 + \| \eta^+ \|_{\partial_+ \kappa \cap \partial \Omega}^2 \right\}.
\end{aligned}
$$

Thereby,

$$
\begin{aligned}
|||u-u_h|||_{\mathrm{DG}}^2 \quad \leq \quad & 2|||\eta|||_{\mathrm{DG}}^2 + 2\left\|\sqrt{\delta}\mathcal{L}\eta - \frac{1}{\sqrt{\delta}}\eta\right\|_{L^2(\Omega)}^2 + 8\|c_0\eta\|_{L^2(\Omega)}^2 \\
& + 4\sum_{\kappa\in\mathcal{T}_h}\left\{\|\eta^-\|_{\partial_-\kappa\setminus\partial\Omega}^2 + \|\eta^+\|_{\partial_+\kappa\cap\partial\Omega}^2\right\}.
\end{aligned}
$$

Selecting $\mathcal{I}_{\boldsymbol{p}}|_\kappa = \Pi_p$, $\kappa\in\mathcal{T}_h$, (Babuška-Suri projector), then on standard element shapes, assuming $v|_\kappa \in H^l(\kappa)$, $l\geq 0$, we have that $v|_T\in H^l(T)$,

$$
\begin{aligned}
\|v-\Pi_p v\|_{H^q(\kappa)} \quad &\leq \quad C_{\mathrm{I},1}\frac{h_\kappa^{s-q}}{p^{l-q}}\|v\|_{H^l(\kappa)}, \quad l\geq 0, \\
\|v-\Pi_p v\|_{L^2(F)} \quad &\leq \quad C_{\mathrm{I},3}\frac{h_\kappa^{s-1/2}}{p^{l-1/2}}\|v\|_{H^l(\kappa)}, \quad l\geq 1.
\end{aligned}
$$

Here, $F\subset\partial\kappa$, $s=\min\{p+1,l\}$ and $C_{\mathrm{I},1}$ and $C_{\mathrm{I},3}$ are positive constants which depend on the shape-regularity of $\kappa$, but are independent of $v$, $h_\kappa$, and $p$; see Lemmas 6 & 7.

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The University of Nottingham
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For uniform orders and uniform mesh sizes, we deduce that

$$|||u - u_h|||_{\mathrm{DG}} \leq C \left( \sqrt{\delta} \frac{h^{s-1}}{p^{l-1}} + \frac{h^s}{p^l} \left( 1 + \frac{1}{\sqrt{\delta}} + \sqrt{\delta} \right) + \frac{h^{s-1/2}}{p^{l-1/2}} \right) \|u\|_{H^l(\Omega)}.$$

Hence, selecting $\delta = C_\delta h/p$, gives

$$|||u - u_h|||_{\mathrm{DG}} \leq C \frac{h^{s-1/2}}{p^{l-1/2}} \|u\|_{H^l(\Omega)}.$$

H., Schwab & Suli 2000

$\Rightarrow$ Analogous result holds on polytopic meshes (Proof: exercise).

- How to deal with the term $\int_\kappa \eta \mathcal{L} \xi \, \mathrm{d}\mathbf{x}$:

2. Assume $\delta \equiv 0$. In this case, we have that

$$|||v|||^2_{\mathrm{DG}} := \sum_{\kappa \in \mathcal{T}_h} \left( \|c_0 v\|^2_{L^2(\kappa)} + \frac{1}{2} \|v^+\|^2_{\partial_-\kappa \cap \partial_-\Omega} + \frac{1}{2} \|v^+ - v^-\|^2_{\partial_-\kappa \setminus \partial\Omega} + \frac{1}{2} \|v^+\|^2_{\partial_+\kappa \cap \partial\Omega} \right),$$

Proceeding as before, we have that

$$\begin{aligned} B_{ar}(\eta, \xi) &= \sum_{\kappa \in \mathcal{T}_h} \left\{ 2 \int_\kappa c_0^2 \xi \eta \, \mathrm{d}\mathbf{x} - \int_\kappa \eta \mathcal{L} \xi \, \mathrm{d}\mathbf{x} \right. \\ &\quad \left. - \int_{\partial_-\kappa \setminus \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa \, (\xi^+ - \xi^-) \eta^- \, \mathrm{d}s - \int_{\partial_+\kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa \, \eta^+ \xi^+ \, \mathrm{d}s \right\}. \end{aligned}$$