Structure of the proofs:

- Derivation of an error representation formula using duality;
- Use of Galerkin orthogonality;

$$|J(u) - J(u_h)| \leq \sum_{\kappa \in \mathcal{T}_h} |(R(u_h), z - z_h)_\kappa|$$

$\Rightarrow$ (Weighted) Type I Error Bound

Becker & Rannacher 1996, Rannacher et al. 1996 …

- Local interpolation error estimates for the dual solution;
- Stability estimates for the dual problem.

$$|J(u) - J(u_h)| \leq C_{\mathrm{int}} C_{\mathrm{stab}} \|(h/p)^s R(u_h)\|, \quad s > 0$$

$\Rightarrow$ (Unweighted) Type II Error Bound

Johnson et al. 1995 …

Example: DGFEM for Hyperbolic PDEs

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Transport Equation

Consider the PDE problem

$$\begin{aligned} \mathcal{L}u \equiv \nabla \cdot (\mathbf{b}u) &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \partial_-\Omega. \end{aligned}$$

DGFEM Discretization

Find $u_h \in V^{\mathbf{p}}(\mathcal{T}_h)$ such that

$$A_h(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V^{\mathbf{p}}(\mathcal{T}_h),$$

where

$$\begin{aligned} A_h(u_h, v_h) &= \sum_{\kappa \in \mathcal{T}_h} \left\{ - \int_\kappa (\mathbf{b} u_h) \cdot \nabla v_h \mathrm{d}\mathbf{x} \right. \\ &\left. + \int_{\partial_+ \kappa} (\mathbf{b} \cdot \mathbf{n}_\kappa) \, u_h^+ v_h^+ \mathrm{d}s + \int_{\partial_- \kappa \setminus \partial \Omega} (\mathbf{b} \cdot \mathbf{n}_\kappa) \, u_h^- v_h^+ \mathrm{d}s \right\}, \\ \ell(v_h) &= \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa f v_h \mathrm{d}\mathbf{x} - \int_{\partial_- \kappa \cap \partial \Omega} (\mathbf{b} \cdot \mathbf{n}_\kappa) \, g \, v_h^+ \mathrm{d}s \right\}. \end{aligned}$$

Dual Problem

Find z such that

$$A_h(v, z) = J(v) \quad \forall v,$$

where $J(\cdot)$ is a given *linear functional*.

- Outflow flux: $J(u) = \int_{\partial_+\Omega} (\mathbf{b}\cdot\mathbf{n}u)\psi \, \mathrm{d}s$, $\psi \in L^2(\partial_+\Omega)$.

$$\begin{aligned} \mathcal{L}^* z \equiv -\mathbf{b}\cdot\nabla z &= 0 \ \text{ in } \Omega\,, \\ z &= \psi \ \text{ on } \partial_+\Omega\,. \end{aligned}$$

- Meanflow functional: $J(u) = \int_{\Omega} u\psi \, \mathrm{d}\mathbf{x}$, $\psi \in L^2(\Omega)$.

$$\begin{aligned} \mathcal{L}^* z \equiv -\mathbf{b}\cdot\nabla z &= \psi \ \text{ in } \Omega\,, \\ z &= 0 \ \text{ on } \partial_+\Omega\,. \end{aligned}$$

Dual Problem $\neq$ Adjoint Problem

- Artificial Viscosity:

  Discretization: find $u_h \in V^{\mathbf{p}}(\mathcal{T}_h)$ such that

$$\mathcal{A}_h(u_h, v_h) \equiv A_h(u_h, v_h) + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \varepsilon \nabla u_h \cdot \nabla v_h \mathrm{d}\mathbf{x} = \ell(v_h) \quad \forall v_h \in V^{\mathbf{p}}(\mathcal{T}_h).$$

  Dual Problem: find z such that

$$\mathcal{A}(v, z) = J(v) \quad \forall v,$$

$$\begin{aligned} -\mathbf{b} \cdot \nabla z - \nabla \cdot (\varepsilon \nabla z) &= \psi \text{ in } \Omega, \\ z + \varepsilon \mathbf{n} \cdot \nabla z = 0 \text{ on } \partial_+\Omega, \quad \varepsilon \mathbf{n} \cdot \nabla z &= 0 \text{ on } \partial_-\Omega. \end{aligned}$$

Dual Problem $\neq$ Adjoint Problem

- SUPG Stabilization:
  Discretization: find $u_h \in V^{\mathbf{p}}(\mathcal{T}_h)$ such that

$$\mathcal{A}_h(u_h, v_h) \equiv A_h(u_h, v_h) + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \delta \mathcal{L} u_h \mathcal{L} v_h \mathrm{d}\mathbf{x} = \ell(v_h) + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f \delta \mathcal{L} v_h \mathrm{d}\mathbf{x} \quad \forall v_h.$$

  Dual Problem: find z such that

$$\mathcal{A}(v, z) = J(v) \quad \forall v,$$

$$\begin{aligned} \mathcal{L}^*(z + \delta \mathcal{L} z) &= \psi \ \text{ in } \Omega\,, \\ z + \delta \mathcal{L} z = 0 \ \text{ on } \partial_+\Omega, \quad \mathcal{L} z &= 0 \ \text{ on } \partial_-\Omega. \end{aligned}$$

For further details, see H. 2017.

Error Representation Formula

$$
\begin{aligned}
J(u) - J(u_h) &= J(u - u_h) && \text{Linearity} \\
&= A_h(u - u_h, z) && \text{Dual Problem} \\
&= A_h(u - u_h, z - z_h) && \text{Galerkin Orthogonality} \\
&= \ell(z - z_h) - A_h(u_h, z - z_h) && \text{Consistency} \\
&= \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa (f - \nabla \cdot (\mathbf{b} u_h))(z - z_h)\,\mathrm{d}\mathbf{x} \right. \\
&\qquad + \int_{\partial_- \kappa \setminus \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u_h^+ - u_h^-)(z - z_h)^+ \,\mathrm{d}s \\
&\qquad \left. + \int_{\partial_- \kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u_h^+ - g)(z - z_h)^+ \,\mathrm{d}s \right\} \\
&\equiv \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa .
\end{aligned}
$$

Type I A Posteriori Error Bound

Proposition 1

Assuming the dual problem is well-posed, the following result holds:

$$|J(u) - J(u_h)| \leq \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^{(\mathrm{I})},$$

where $\eta_\kappa^{(\mathrm{I})} = |\eta_\kappa|$ and

$$\begin{aligned}
\eta_\kappa &= \int_\kappa (f - \nabla \cdot (\mathbf{b} u_h))(z - z_h)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_{\partial_- \kappa \setminus \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u_h^+ - u_h^-)(z - z_h)^+ \,\mathrm{d}s, \\
&\quad + \int_{\partial_- \kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n}_\kappa (u_h^+ - g)(z - z_h)^+ \,\mathrm{d}s.
\end{aligned}$$

Lemma 16 (Assume h-Version FEM on standard meshes)

Given $\kappa \in \mathcal{T}_h$, suppose that $v|_\kappa \in H^{k_\kappa}(\kappa)$, $0 \leq k_\kappa \leq p+1$. Then, there exists $\Pi_p v$ in the finite element space $V^{\mathbf{p}}(\mathcal{T}_h)$, such that

$$\|v - \Pi_p v\|_{L^2(\kappa)} + h_\kappa^{1/2} \|v - \Pi_p v\|_{L^2(\partial\kappa)} \leq C_{\mathcal{I}} h_\kappa^{k_\kappa} \|v\|_{H^{k_\kappa}(\kappa)},$$

where $C_{\mathcal{I}}$ is a positive constant, independent of the mesh size h.

Babuska & Suri 1987