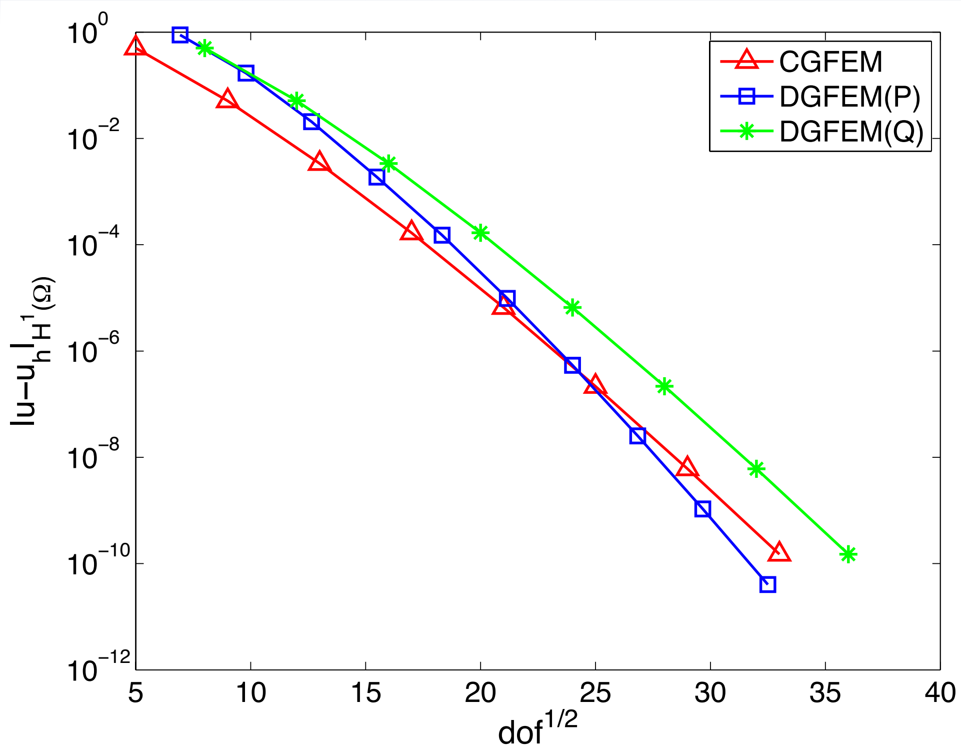
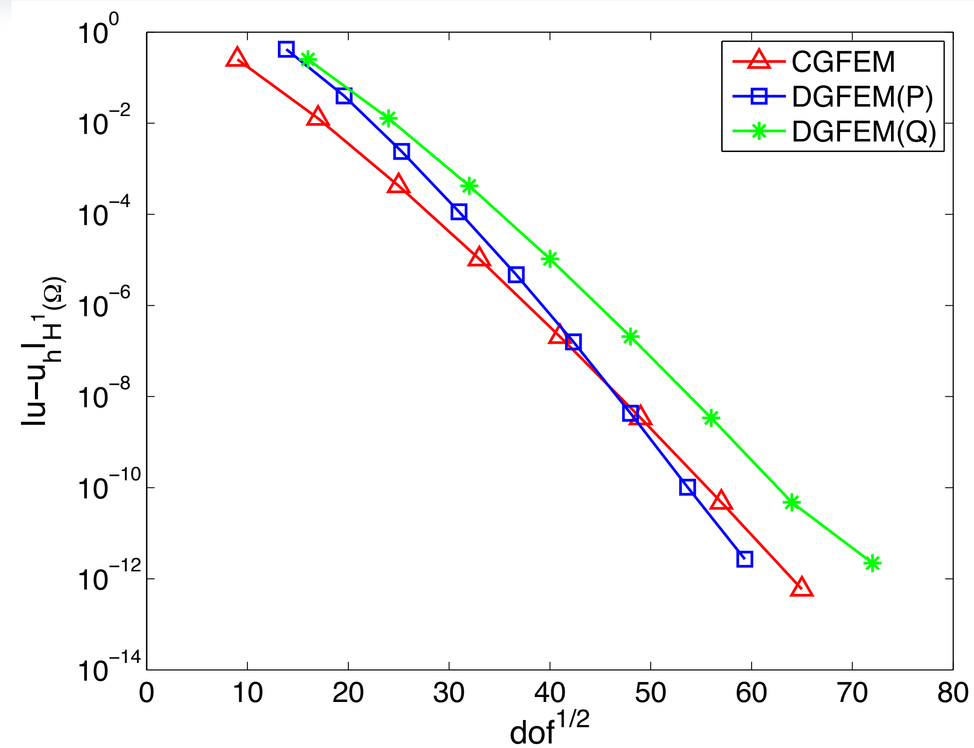
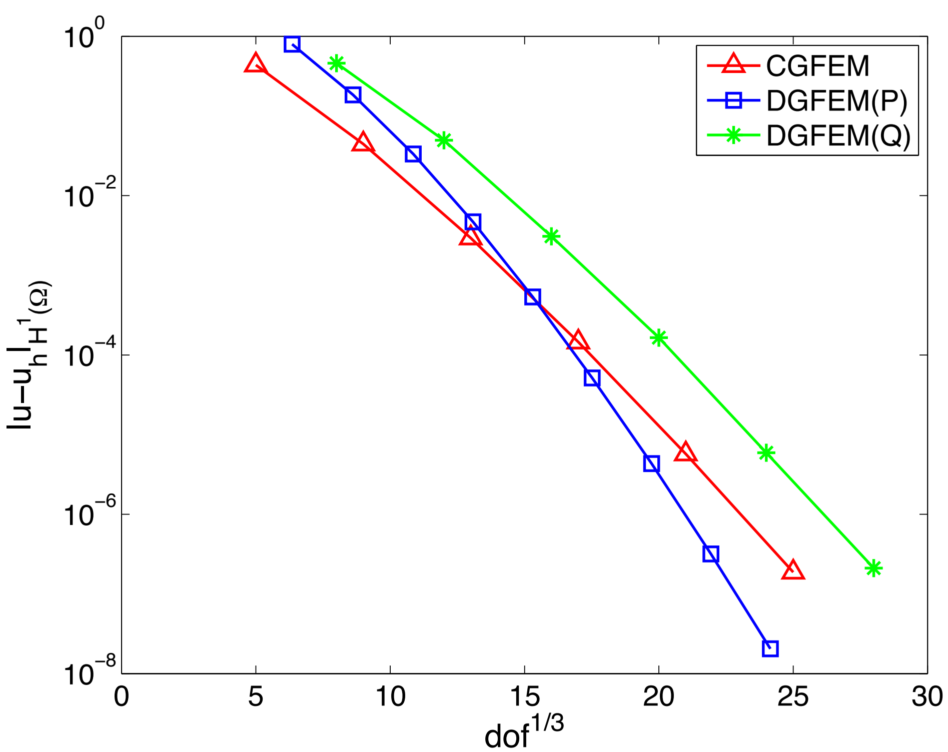
DGFEM vs CGFEM (Cangiani, Georgoulis, & H. 2014)



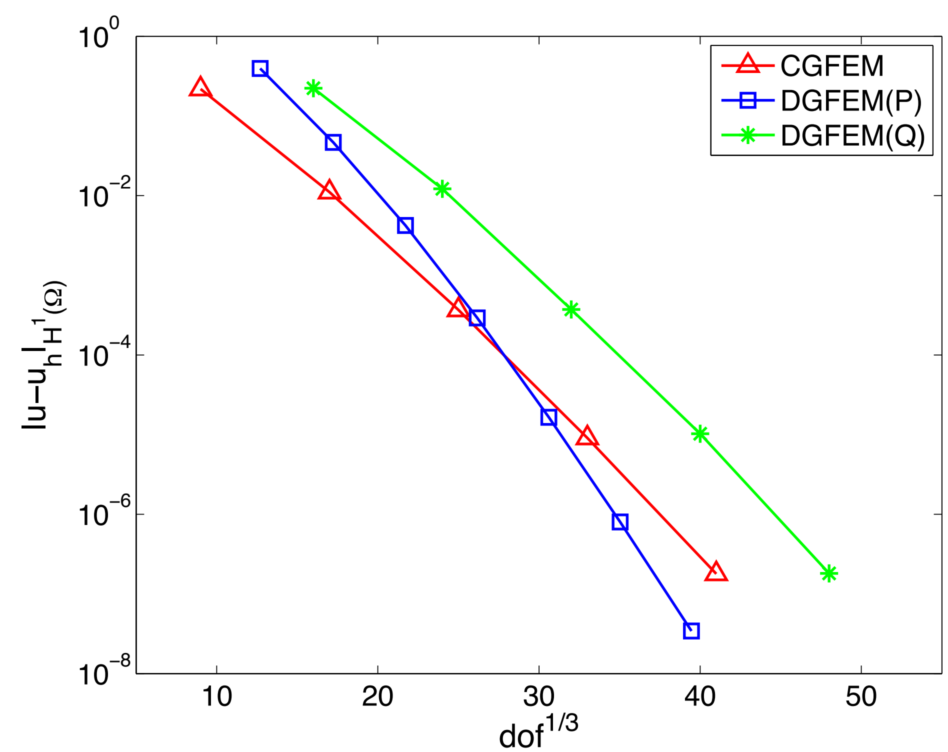
4 × 4 Mesh



8 × 8 Mesh

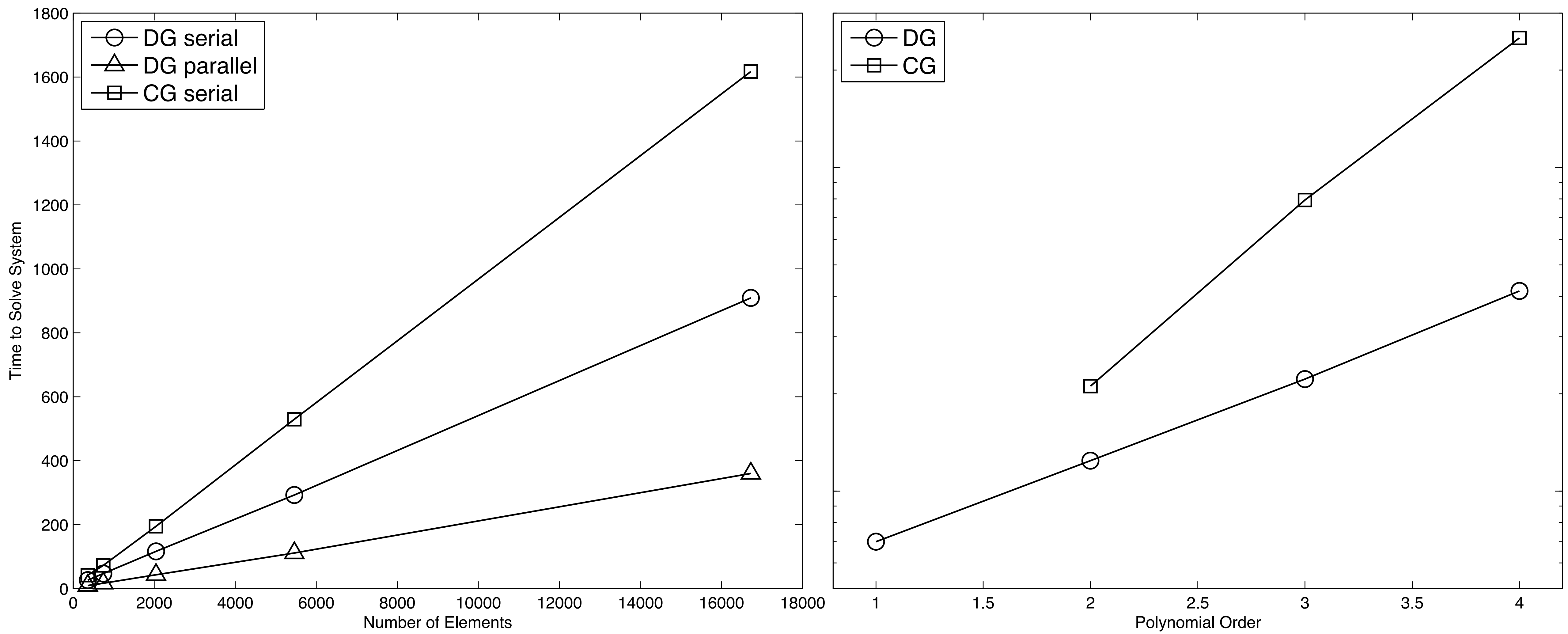


4 × 4 × 4 Mesh



8 × 8 × 8 Mesh

Chemically Reacting Flow Example



Joint work with Nathan Sime (Nottingham)

Discontinuous Galerkin FEMs

First-order Hyperbolic PDEs

Transport Equation

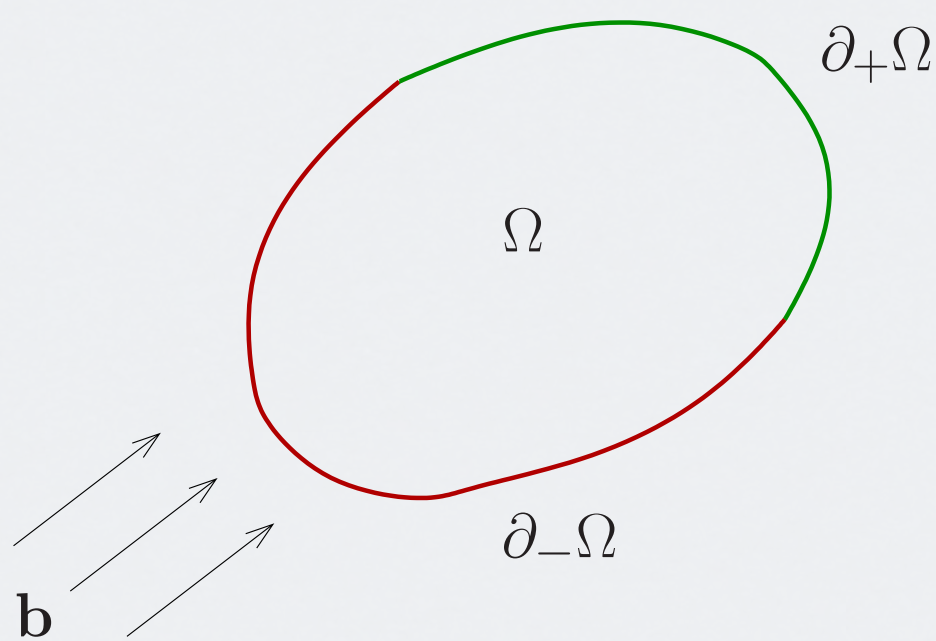
Consider the PDE problem

$$\begin{aligned} \mathbf{b} \cdot \nabla u &= f, \quad x \in \Omega, \\ u &= g, \quad x \in \partial_- \Omega. \end{aligned}$$

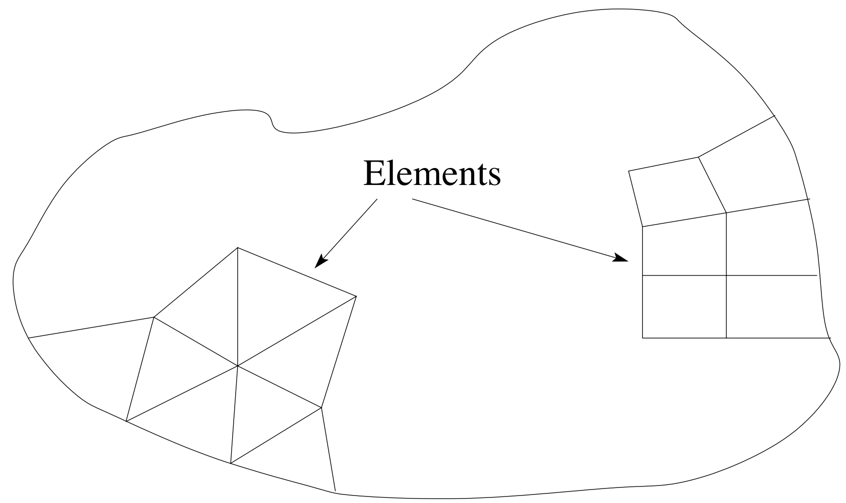
Inflow/Outflow boundaries

We define the inflow and outflow boundaries $\partial_- \Omega$ and $\partial_+ \Omega$, respectively, by

$$\begin{aligned} \partial_- \Omega &= \{x \in \partial \Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \\ \partial_+ \Omega &= \{x \in \partial \Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}. \end{aligned}$$

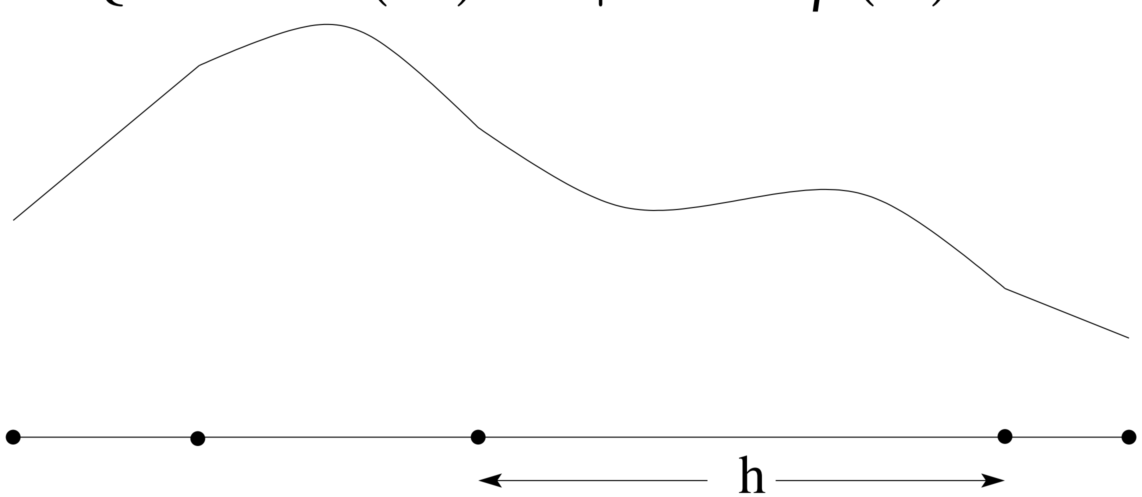


- We shall consider two variants:
  - Standard Galerkin with strongly imposed boundary conditions.
  - Standard Galerkin with weakly imposed boundary conditions.
- $\mathcal{T}_h$ is a non-degenerate mesh consisting of elements of granularity h.



- Finite element space:

$$V_h = \{v \in H^1(\Omega) : v|_\kappa \in \mathcal{S}_p(\kappa) \ \ \forall \kappa \in \mathcal{T}_h\}.$$



- Consider the following two additional finite element spaces:

$$\begin{aligned} V_{h,E} &= \{v \in V_h : v = g \text{ on } \partial_-\Omega\}, \\ V_{h,E_0} &= \{v \in V_h : v = 0 \text{ on } \partial_-\Omega\}. \end{aligned}$$

Standard Galerkin with strongly imposed boundary conditions

Find $u_h \in V_{h,E}$ such that

$$(\mathbf{b} \cdot \nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in V_{h,E_0}.$$

- Here, $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ inner-product, i.e., given $v, w \in L^2(\Omega)$,

$$(v, w) = \int_\Omega vw \mathrm{d}x.$$

- Variants include the streamline-diffusion/SUPG method ($v_h \mapsto v_h + \delta \mathbf{b} \cdot \nabla v_h$, where $\delta = C_\delta h$, or more generally, $\delta = C_\delta {}^h/_p$, $C_\delta > 0$).

Standard Galerkin with weakly imposed boundary conditions

Find $u_h \in V_h$ such that

$$(\mathbf{b} \cdot \nabla u_h, v_h) - < u_h, v_h >_- = (f, v_h) - < g, v_h >_- \quad \forall v_h \in V_h,$$

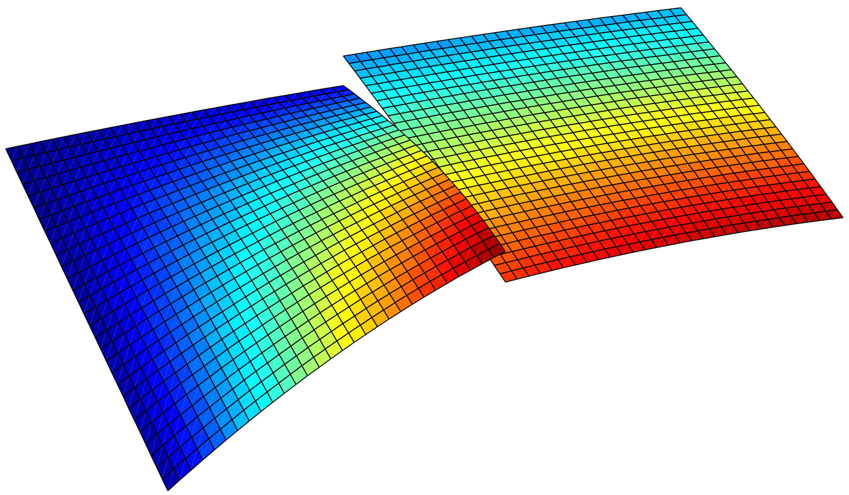
where

$$< v, w >_- = \int_{\partial_- \Omega} \mathbf{b} \cdot \mathbf{n} \, vw \, \mathrm{d}s,$$

and $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$.

- Finite Element Space:

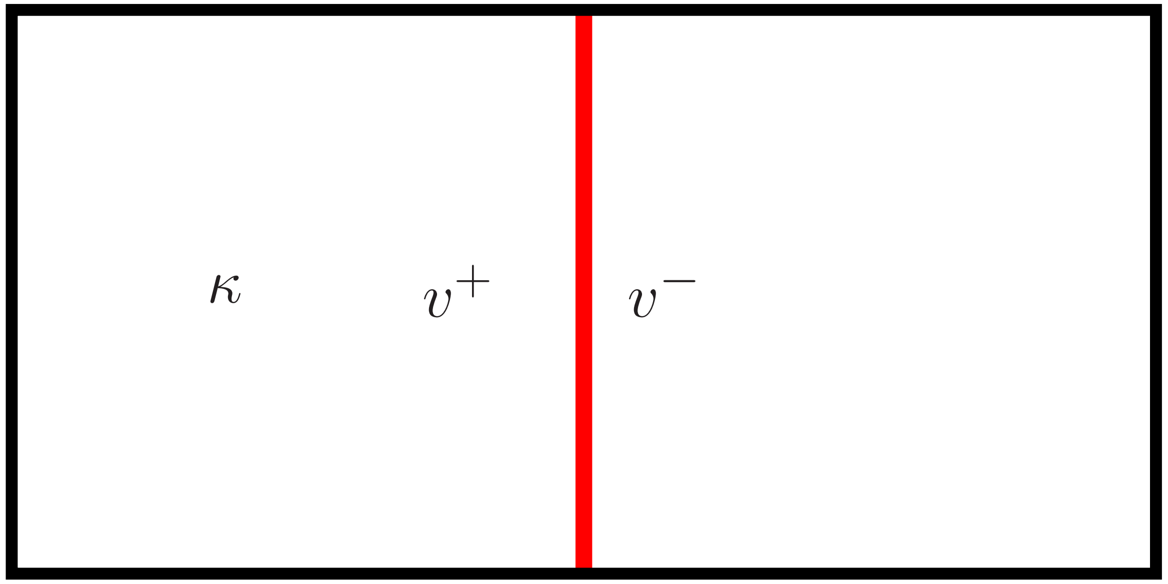
$$V_h = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_p(\kappa) \ \ \forall \kappa \in \mathcal{T}_h\},$$



where $\mathcal{T}_h$ is the mesh.

- Consider a local (elementwise) FE formulation with weakly imposed bcs.
- Notation: for $v \in H^1(\kappa)$, we write

$$\begin{aligned} v^+ &= \text{interior trace of } v \text{ on } \partial\kappa \text{ taken from within } \kappa \\ v^- &= \text{exterior trace of } v \text{ on } \partial\kappa \text{ taken from outside } \kappa \end{aligned}$$



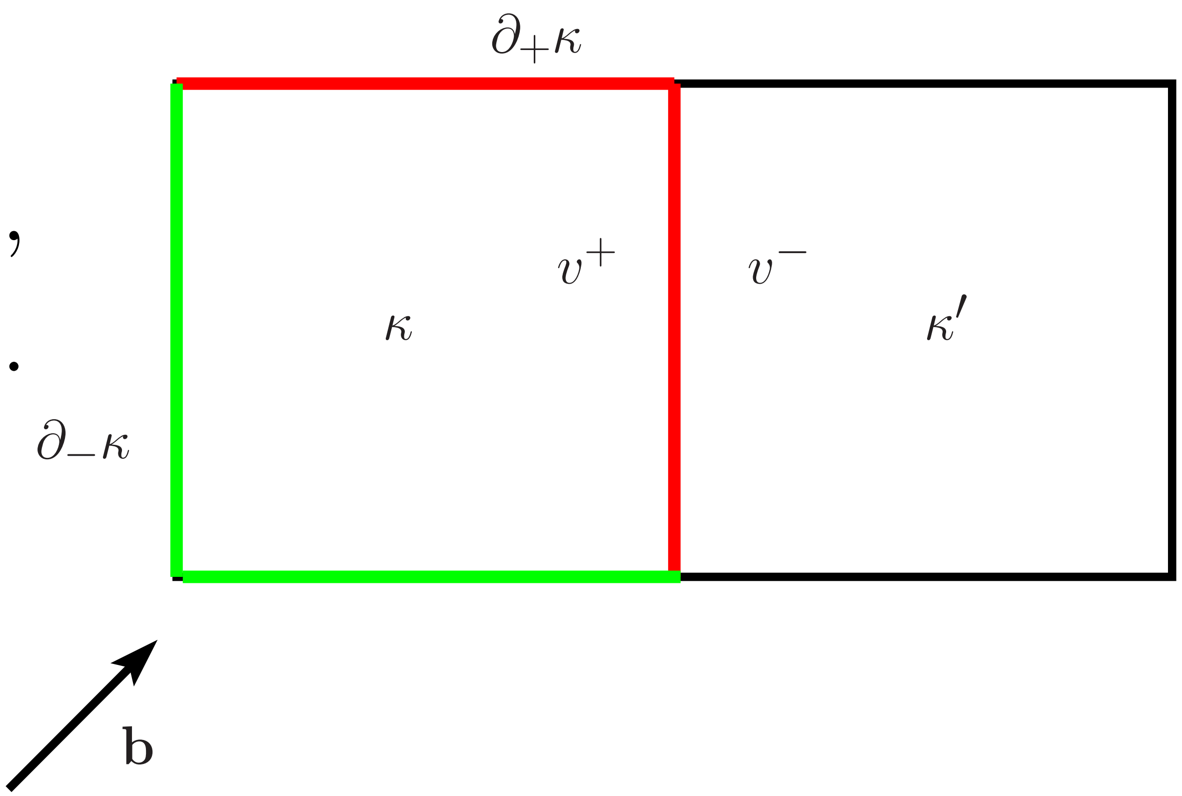
Discontinuous Galerkin Methods

- **Local FE formulation:** on each $\kappa \in \mathcal{T}_h$, find $u_h \in V_h$ such that

$$\int_\kappa \mathbf{b} \cdot \nabla u_h \, v_h \mathrm{d}x - \int_{\partial_- \kappa} \mathbf{b} \cdot \mathbf{n} \, u_h^+ v_h^+ \, \mathrm{d}s = \int_\kappa f v_h \mathrm{d}x - \int_{\partial_- \kappa} \mathbf{b} \cdot \mathbf{n} \, \hat{g} v_h^+ \, \mathrm{d}s$$

for all $v_h \in V_h$. Here,

$$\begin{aligned} \partial_- \kappa &= \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \\ \partial_+ \kappa &= \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}. \end{aligned}$$



Moreover, we define

$$\hat{g}(x) = \begin{cases} u_h^-(x) & x \in \partial_-\kappa \backslash \partial\Omega, \\ g(x) & x \in \partial_-\kappa \cap \partial\Omega. \end{cases}$$

- Sum over all elements $\kappa \in \mathcal{T}_h$:

DGFEM

Find $u_h \in V_h$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa \mathbf{b} \cdot \nabla u_h \, v_h \mathrm{d}x - \int_{\partial_- \kappa \backslash \partial\Omega} \mathbf{b} \cdot \mathbf{n} \, (u_h^+ - u_h^-) v_h^+ \, \mathrm{d}s \right.$$
$$\left. - \int_{\partial_- \kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n} \, u_h^+ v_h^+ \, \mathrm{d}s \right\} = \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa f v_h \mathrm{d}x - \int_{\partial_- \kappa \cap \partial\Omega} \mathbf{b} \cdot \mathbf{n} \, g v_h^+ \mathrm{d}s \right\}$$

for all $v_h \in V_h$.

Flux Formulation

The University of Nottingham
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Transport Equation in Conservative Form

Consider the PDE problem

$$
\begin{aligned}
\nabla \cdot (\mathbf{b}u) &= f, \quad x \in \Omega, \\
u &= g, \quad x \in \partial_{-}\Omega.
\end{aligned}
$$

[Assuming $\mathbf{b}$ is incompressible, i.e., $\nabla \cdot \mathbf{b} = 0$, the conservative and non-conservative forms are equivalent.]