Flux Formulation

- Local weak formulation: on each $\kappa \in \mathcal{T}_h$, find $u|_\kappa$ such that

$$-\int_\kappa (\mathbf{b}u) \cdot \nabla v \mathrm{d}x + \int_{\partial\kappa} (\mathbf{b}u^+) \cdot \mathbf{n}_\kappa v^+ \,\mathrm{d}s = 0.$$

- Inter-element continuity and bcs weakly enforced:

$$-\int_\kappa (\mathbf{b}u) \cdot \nabla v \mathrm{d}x + \int_{\partial\kappa} \mathcal{H}(u^+, u^-, \mathbf{n}_\kappa) v^+ \,\mathrm{d}s = \int_\kappa f v \mathrm{d}x.$$

- $\mathcal{H}(\cdot, \cdot, \mathbf{n})$ is a numerical flux function.
- Sum over all elements $\kappa \in \mathcal{T}_h$ and restrict to the FEM space V_h:

DGFEM

Find $u_h \in V_h$ such that

$$\sum_{\kappa \in \mathcal{T}_h} \left\{ -\int_\kappa (\mathbf{b}u_h) \cdot \nabla v_h \mathrm{d}x + \int_{\partial\kappa} \mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa) v_h^+ \,\mathrm{d}s \right\} = \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa f v_h \mathrm{d}x \right\}$$

for all $v_h \in V_h$.

Flux Formulation

- Properties of the numerical flux function $\mathcal{H}(\cdot,\cdot,\cdot)$.

  1. Consistency: for each κ in $\mathcal{T}_h$ we have that

  $$\mathcal{H}(v, v, \mathbf{n}_\kappa)|_{\partial\kappa} = (\mathbf{b}v)\cdot\mathbf{n}_\kappa \quad \forall\kappa\in\mathcal{T}_h.$$

  2. Conservation: given any two neighbouring elements κ and κ' from the finite element partition $\mathcal{T}_h$, at each point $\mathbf{x}\in\partial\kappa\cap\partial\kappa'\neq\emptyset$, noting that $\mathbf{n}_{\kappa'}=-\mathbf{n}_\kappa$, we have that

  $$\mathcal{H}(v, w, \mathbf{n}_\kappa) = -\mathcal{H}(w, v, -\mathbf{n}_\kappa).$$

- Choose the upwind numerical flux:

$$\mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa)|_{\partial\kappa} = \mathbf{b}\cdot\mathbf{n}_\kappa \lim_{s\to 0^+} u_h(x - s\mathbf{b}) \quad \text{for } \kappa\in\mathcal{T}_h.$$

  By convention, we set $u_h^-|_{\partial_-\kappa\cap\partial\Omega} = g$; thereby,

$$\mathcal{H}(u_h^+, u_h^-, \mathbf{n}_\kappa)|_{\partial_-\kappa\cap\partial\Omega} = \mathbf{b}\cdot\mathbf{n}_\kappa\, g \quad \text{for } \kappa\in\mathcal{T}_h.$$

- Note that both DGFEM formulations are equivalent (proof: exercise).
- Nonlinear Problems: Lax-Friedrichs flux, Roe's flux, Vijayasundaram flux, …

Discontinuous Galerkin FEMs

Second-order Elliptic PDEs

Poisson's Equation

Given $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and $f \in L^2(\Omega)$, find u such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

- **DG Discretization:**

1. Rewrite as a first-order system.
2. Derive an elemental weak formulation.
3. Introduce numerical flux functions $\Rightarrow$ Flux Formulation.
4. Eliminate the auxiliary variables $\Rightarrow$ Primal Formulation.

1. Rewrite as a first-order system:

$$s - \nabla u = 0, \qquad -\nabla \cdot s = f.$$

2. Elemental weak formulation: find (s, u) such that

$$\begin{aligned} \int_\kappa s \cdot \tau \,\mathrm{d}x + \int_\kappa u \nabla \cdot \tau \,\mathrm{d}x - \int_{\partial\kappa} u \tau \cdot n_\kappa \,\mathrm{d}s &= 0, \\ \int_\kappa s \cdot \nabla v \,\mathrm{d}x - \int_{\partial\kappa} s \cdot n_\kappa v \,\mathrm{d}s &= \int_\kappa f v \,\mathrm{d}x. \end{aligned}$$

- Notation: ∇_h denotes the broken gradient operator, defined elementwise.
- Numerical flux functions:
  - $\hat{u} = \hat{u}(u_h)$,
  - $\hat{s} = \hat{s}(u_h, \nabla_h u_h)$

  are approximations to u_h and $\nabla_h u_h$, respectively.

3. Flux Formulation: find $u_h \in V_h$ and $s_h \in \Sigma_h = [V_h]^d$ such that

$$\int_\Omega s_h \cdot \boldsymbol{\tau}_h \mathrm{d}x + \int_\Omega u_h \nabla_h \cdot \boldsymbol{\tau}_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{u} \boldsymbol{\tau}_h \cdot n_\kappa \mathrm{d}s = 0, \qquad (1)$$

$$\int_\Omega s_h \cdot \nabla v_h \mathrm{d}x - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{s} \cdot n_\kappa v_h \mathrm{d}s = \int_\Omega f v_h \mathrm{d}x \qquad (2)$$

for all $\boldsymbol{\tau}_h \in \Sigma_h$, $v_h \in V_h$.

- Setting $\boldsymbol{\tau}_h = \nabla_h v_h$ in (1) and integrating by parts gives

$$\int_\Omega \mathbf{s}_h \cdot \nabla_h v_h \mathrm{d}\mathbf{x} - \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (u_h^+ - \hat{u}) \nabla_h v_h^+ \cdot \mathbf{n}_\kappa \mathrm{d}s = 0.$$

4. **Primal Formulation.** Inserting into (2) gives: find $u_h \in V_h$ such that

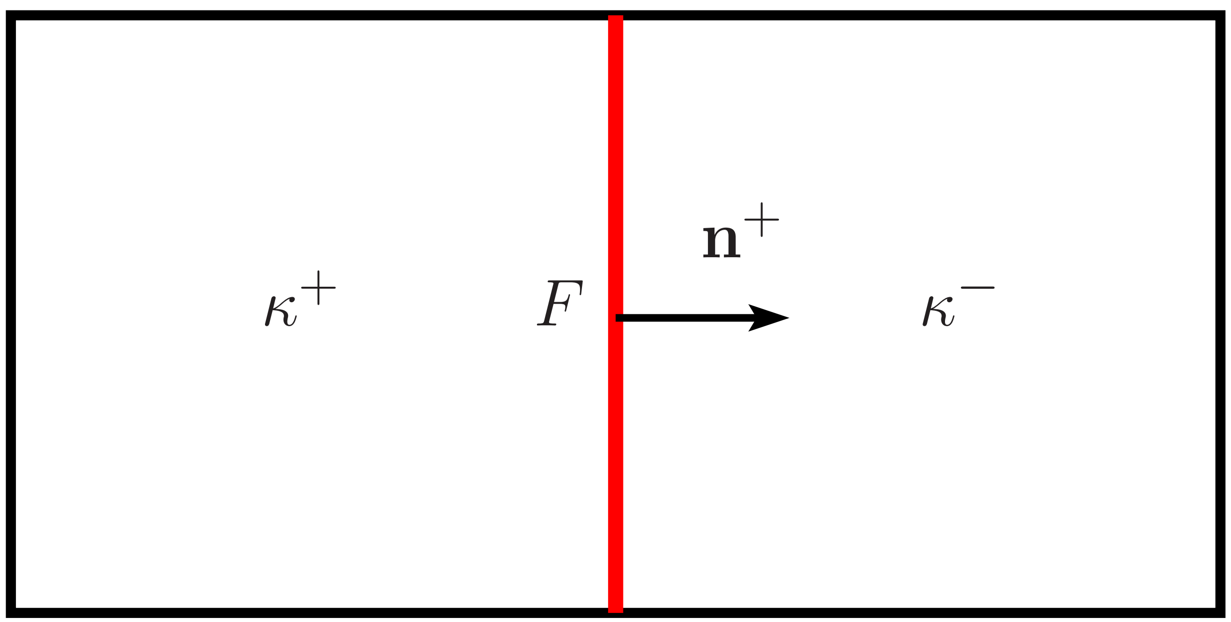
$$\int_\Omega \nabla_h u_h \cdot \nabla v_h \mathrm{d}\mathbf{x} - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} (u_h^+ - \hat{u}) \nabla_h v_h^+ \cdot \mathbf{n}_\kappa \mathrm{d}s$$
$$- \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{\mathbf{s}} \cdot \mathbf{n}_\kappa v_h^+ \mathrm{d}s = \int_\Omega f v_h \mathrm{d}\mathbf{x}$$

for all $v_h \in V_h$.

Notation

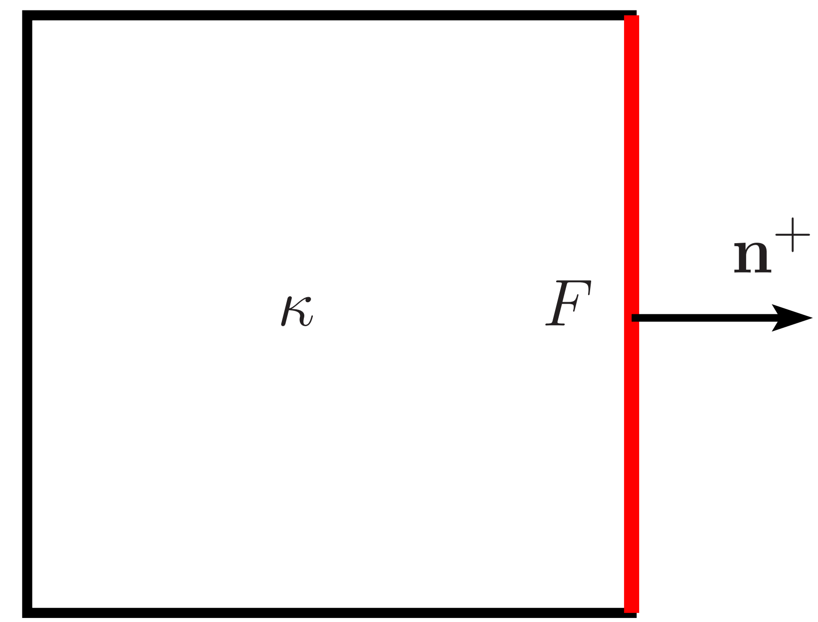
- Let $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$ denote the set of all faces in the mesh $\mathcal{T}_h$.
- Notation:

$F \subset \mathcal{F}_h^{\mathcal{I}}$



$$
\begin{aligned}
[\![v]\!] &= v^+\mathbf{n}^+ + v^-\mathbf{n}^- \\
[\![\mathbf{q}]\!] &= \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^- \\
\{\!\{v\}\!\} &= (v^+ + v^-)/2
\end{aligned}
$$

$F \subset \mathcal{F}_h^{\mathcal{B}}$



$$
\begin{aligned}
[\![v]\!] &= v\mathbf{n} \\
[\![\mathbf{q}]\!] &= \mathbf{q} \cdot \mathbf{n} \\
\{\!\{v\}\!\} &= v
\end{aligned}
$$

- The following identity holds:

$$
\sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \mathbf{q}^+ \cdot \mathbf{n}^+ v^+ \mathrm{d}s = \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mathbf{q}\}\!\} \cdot [\![v]\!] \mathrm{d}s + \sum_{F \in \mathcal{F}_h^{\mathcal{I}}} \int_F [\![\mathbf{q}]\!] \{\!\{v\}\!\} \mathrm{d}s.
$$