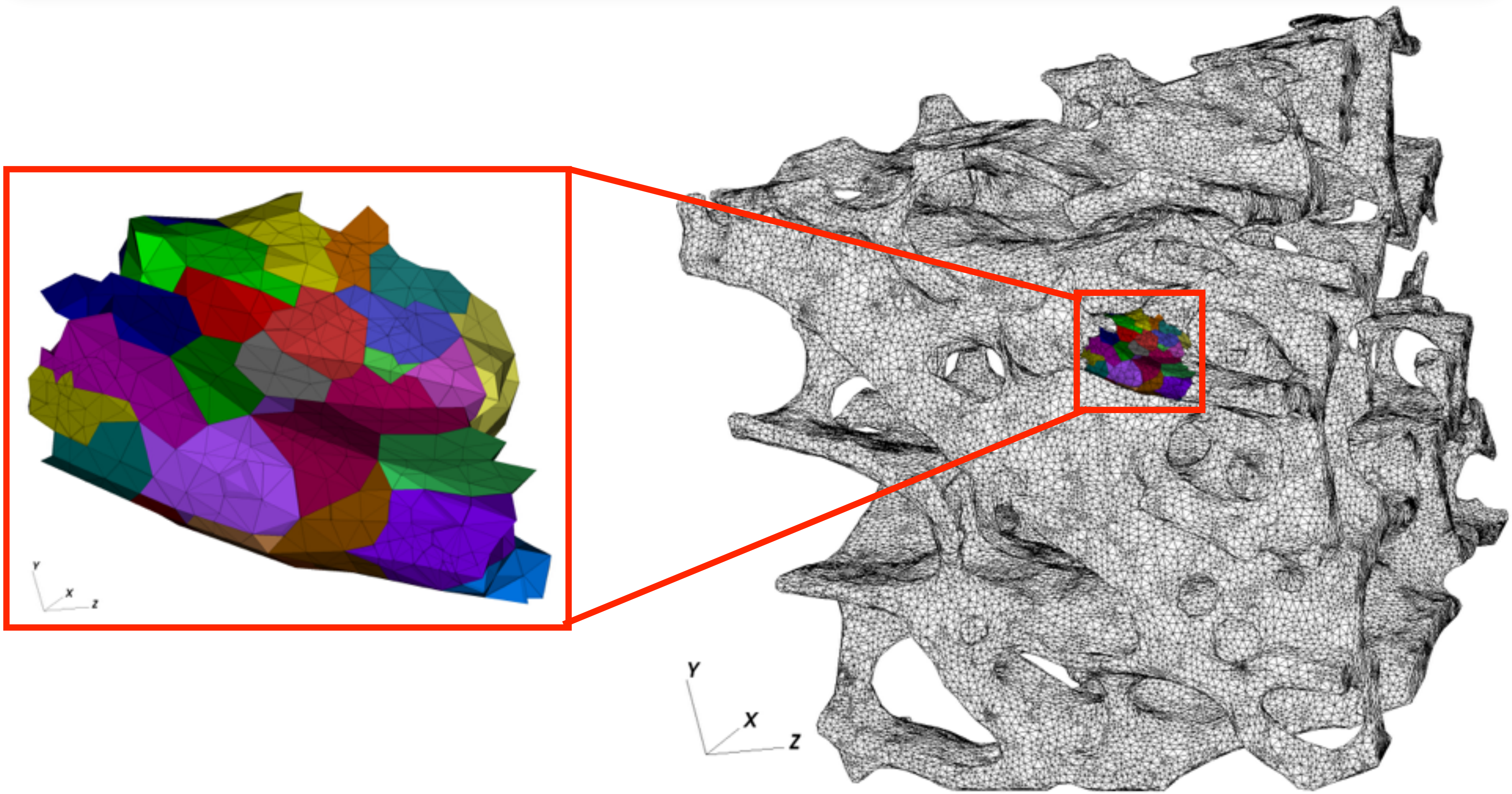
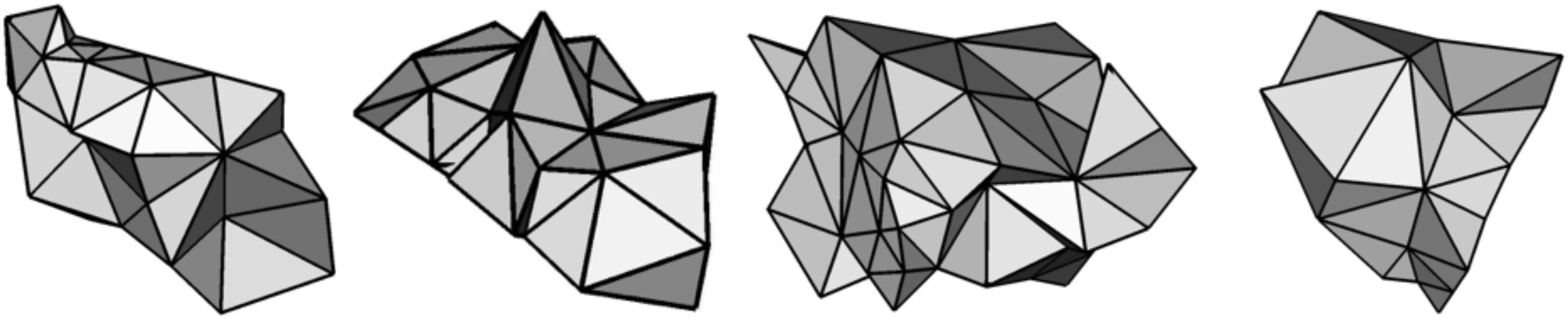
Modelling Trabecular Bone

Fine mesh consists of 1.2M elements; Agglomerated mesh with 8K elements.

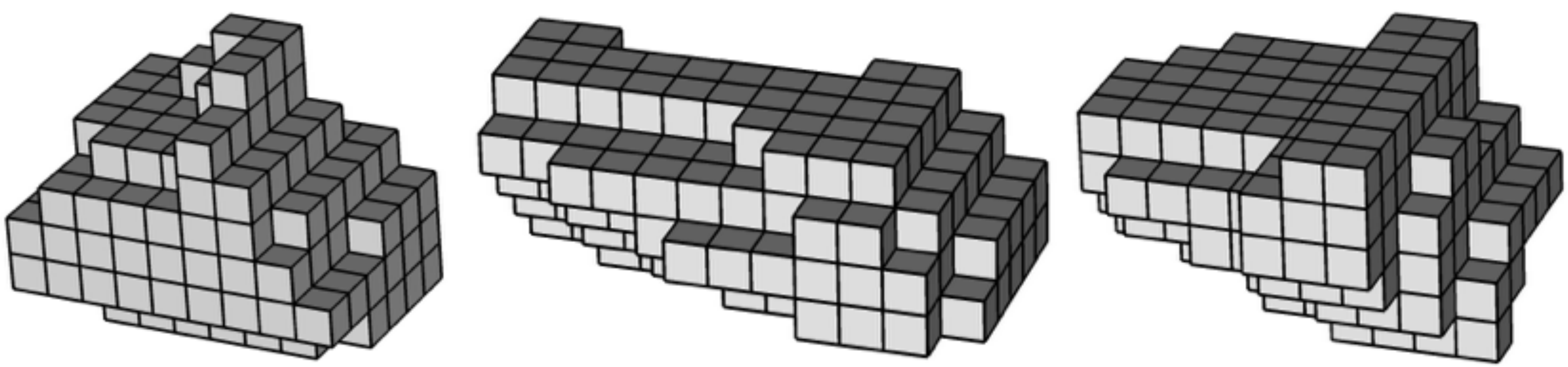


Verhoosel, van Zwieten, van Rietbergen & de Borst 2015, Collis & H 2016

Graph Partitioning Techniques: Element Shapes



Agglomeration of Tetrahedral Mesh



Agglomeration of Voxel Mesh

Polytopic Meshes

Mesh Assumptions

General Polygonal/Polyhedral Meshes

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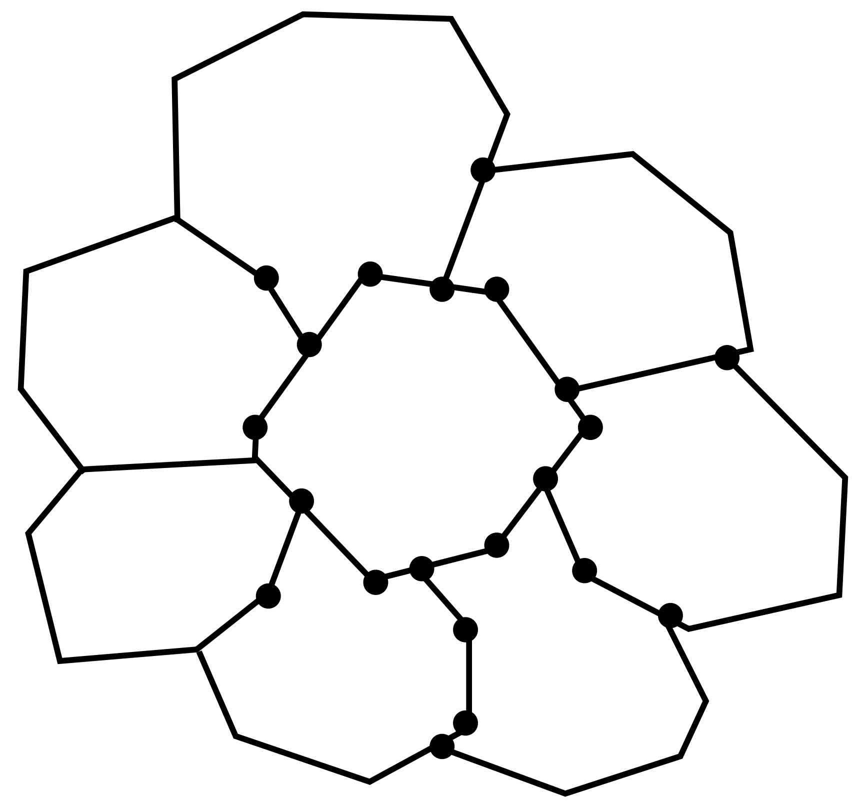
- Mesh: $\mathcal{T}_h = \{\kappa\}$ is a polytopic subdivision of Ω.
- $\mathcal{T}_h = \{\kappa\}$ may contain hanging nodes.
- Interfaces of $\mathcal{T}_h$: intersections of $(d-1)$-dimensional facets of neighbouring elements.



Assumption 1

Each interface of each $\kappa \in \mathcal{T}_h$ may be subdivided into a set of $(d-1)$-dimensional simplices.

- Assumption 1: Naturally covered in 2D; for $d = 3$ each interface of each $\kappa \in \mathcal{T}_h$ must be subdivided into a set of co-planar triangles.

General Polygonal/Polyhedral Meshes

- Faces $\mathcal{F}_h$ of $\mathcal{T}_h$: $(d-1)$-dimensional simplices which whose union form the interfaces of $\mathcal{T}_h$.
- We assume that the sub-tessellation of element interfaces into $(d-1)$-dimensional simplices is given; for example, from an agglomeration of a fine tetrahedral mesh.
- $\mathcal{F}_h^{\mathcal{I}}/\mathcal{F}_h^{\mathcal{B}}$: Interior/Boundary faces, respectively, such that $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$.

Assumption 2: Bounded Number of Element Faces

For each element $\kappa \in \mathcal{T}_h$, we define

$$C_\kappa = \text{card}\Big\{F \in \mathcal{F}_h : F \subset \partial\kappa\Big\}.$$

We assume there exists a positive constant C_F, independent of the mesh parameters, such that

$$\max_{\kappa\in\mathcal{T}_h} C_\kappa \leq C_F.$$

General Polygonal/Polyhedral Meshes

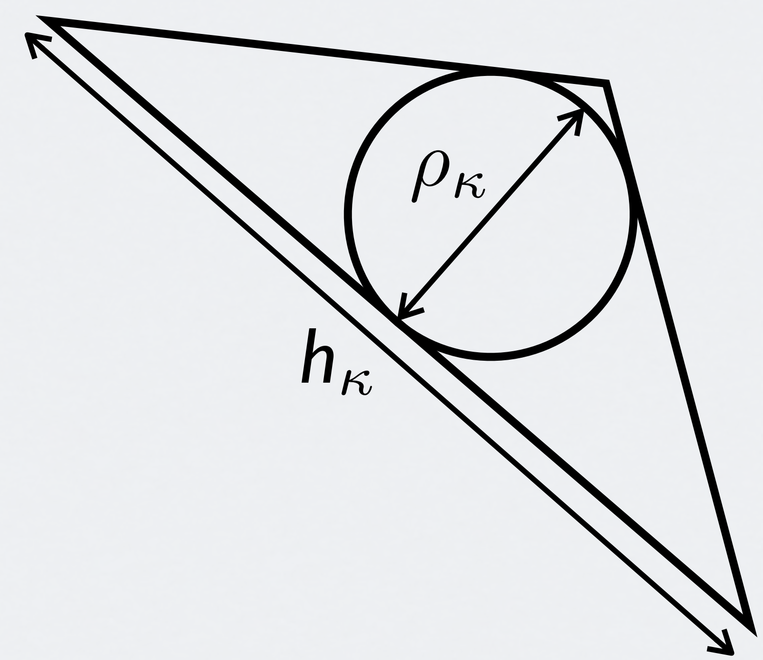
- Assumption 2 may be violated on sequences of agglomerated meshes.
- The analysis will also be pursued under an alternative assumption which assumes that a generalised shape-regularity condition is satisfied.
- Recall:

Definition 1 (Shape-Regularity)

A subdivision $\mathcal{T}_h$ is said to be shape-regular if there exists a positive constant $C_{\rm r}$ such that:

$$\forall \kappa \in \mathcal{T}_h, \qquad \frac{h_\kappa}{\rho_\kappa} \leq C_{\rm r},$$

independently of the mesh parameters, where ρ_κ denotes the diameter of the largest ball contained in κ.



➡ Given that Assumption 2 holds, we do not require a shape-regularity condition to hold on the underlying polytopic mesh, cf. below.

Polytopic Meshes

Inverse Inequalities

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Lemma 2

Given a simplex T in $\mathbb{R}^d$, $d = 2, 3$, we write $F \subset \partial T$ to denote one of its faces. Then, for $v \in \mathcal{P}_p(T)$, the following inverse inequalities hold

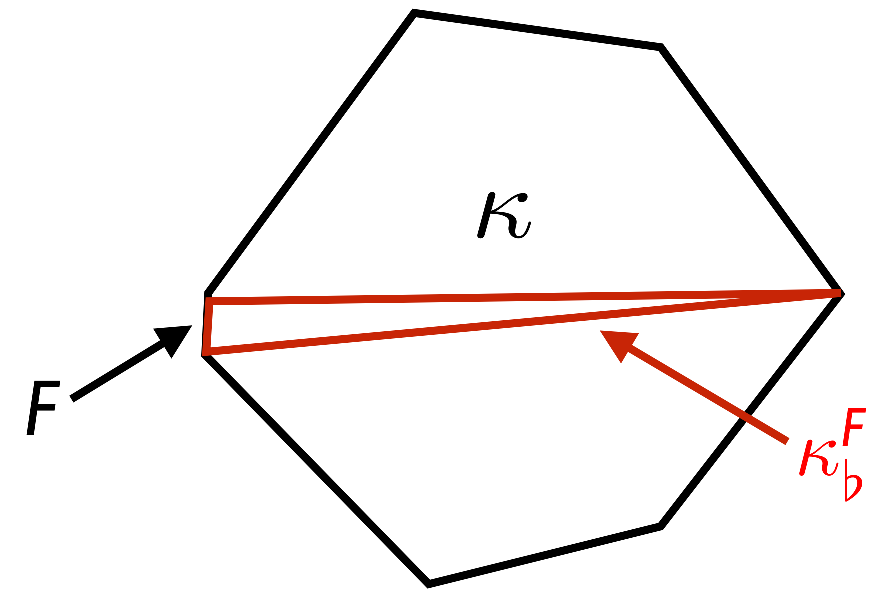
$$\begin{aligned} \|v\|^2_{L^2(F)} &\leq C_{\mathrm{inv},1} p^2 \frac{|F|}{|T|} \|v\|^2_{L^2(T)}, \\ \|v\|^2_{L^\infty(T)} &\leq C_{\mathrm{inv},2} \frac{p^{2d}}{|T|} \|v\|^2_{L^2(T)}, \\ \|\nabla v\|^2_{L^2(T)} &\leq C_{\mathrm{inv},3} \frac{p^4}{h_T^2} \|v\|^2_{L^2(T)}, \end{aligned}$$

with $C_{\mathrm{inv},i}$, $i = 1, 2, 3$, positive constants, which are independent of v, p, and h_T. In particular, $C_{\mathrm{inv},3}$ depends on the shape-regularity of T

Proof: See Schwab 1998, Warburton & Hesthaven 2003.

For each element $\kappa \in \mathcal{T}_h$, and each face $F \subset \partial\kappa$, $\kappa_\flat^F$ represents any *d*-dimensional simplex such that

- $\kappa_\flat^F \subset \kappa$,
- $F \subset \partial\kappa_\flat^F$.



For $v \in \mathcal{P}_p(\kappa)$, applying the classical inverse inequality on $\kappa_\flat^F$ gives

$$\|v\|_{L^2(F)}^2 \le C_{\mathrm{inv},1} p^2 \frac{|F|}{|\kappa_\flat^F|} \|v\|_{L^2(\kappa_\flat^F)}^2 \le C_{\mathrm{inv},1} p^2 \frac{|F|}{|\kappa_\flat^F|} \|v\|_{L^2(\kappa)}^2.$$

Given that the choice of $\kappa_\flat^F$ is not unique, we may select $\kappa_\flat^F$ to have the largest possible measure, i.e.,

$$\|v\|_{L^2(F)}^2 \le C_{\mathrm{inv},1} p^2 \frac{|F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \|v\|_{L^2(\kappa)}^2.$$

Inverse Inequalities: Extension to Polytopic Elements

Recall:

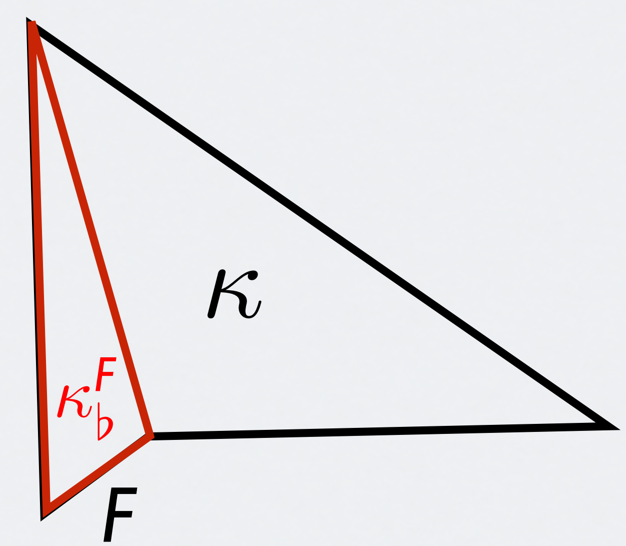
$$\|v\|^2_{L^2(F)} \leq C_{\mathrm{inv},1} p^2 \frac{|F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \|v\|^2_{L^2(\kappa)}.$$

This estimate is not sharp with respect to $(d-k)$-dimensional facet degeneration, $k = 1, \ldots, d-1$; i.e., it is not sensitive to $|F|$ relative to $|\kappa|$.

Example

For $\epsilon > 0$, let

$$\begin{aligned} \kappa \quad := \quad & \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\} \\ & \cup \{(x,y) \in \mathbb{R}^2 : x > 0, y \leq 0, x - y < \epsilon\}. \end{aligned}$$



Hence,

$$\|v\|^2_{L^2(F)} \leq C_{\mathrm{inv},1} \frac{\sqrt{2} p^2 \epsilon}{|\kappa_\flat^F|} \|v\|^2_{L^2(\kappa)},$$

where

$$\kappa_\flat^\kappa := \{(x,y) \in \mathbb{R}^2 : x > 0,\ x + \epsilon y < \epsilon,\ x - y < \epsilon\}.$$

Recall:

$$\|v\|^2_{L^2(F)} \leq C_{\text{inv},1} p^2 \frac{|F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \|v\|^2_{L^2(\kappa)}.$$

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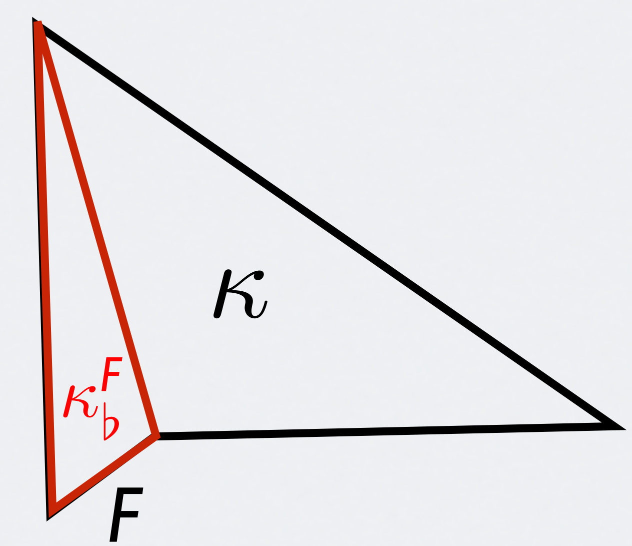

Noting that $|\kappa_\flat^F| = \epsilon(1+\epsilon)/2$, gives

$$\|v\|^2_{L^2(F)} \leq C_{\text{inv},1} \frac{2\sqrt{2}p^2}{1+\epsilon} \|v\|^2_{L^2(\kappa)}.$$