Alternative: we begin by observing that, since $F \subset \partial\kappa_\flat^F$, we have

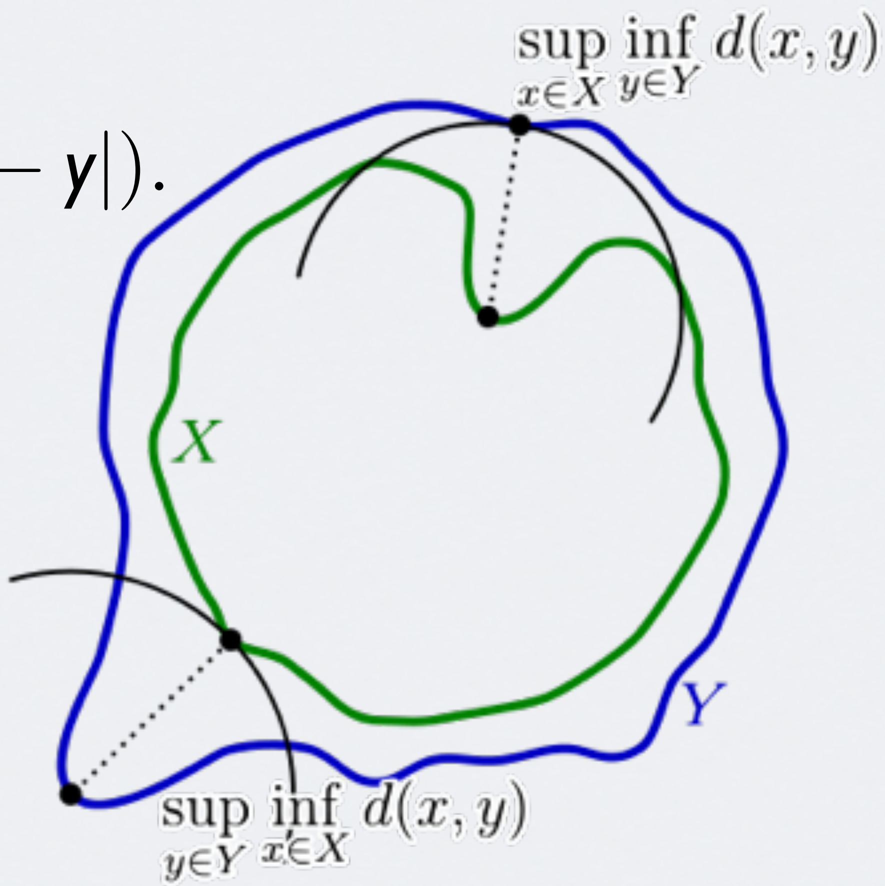
$$\|v\|^2_{L^2(F)} \leq |F|\|v\|^2_{L^\infty(\kappa_\flat^F)}.$$

➡ Additional mesh assumptions will be required.

Definition 2

Given two sets X and Y in $\mathbb{R}^d$, $d \geq 1$, we write $\mathrm{dist}(X, Y)$ to denote the Hausdorff distance between X and Y, defined by

$$\mathrm{dist}(X, Y) := \max(\sup_{x\in X}\inf_{y\in Y}|x-y|, \sup_{y\in Y}\inf_{x\in X}|x-y|).$$

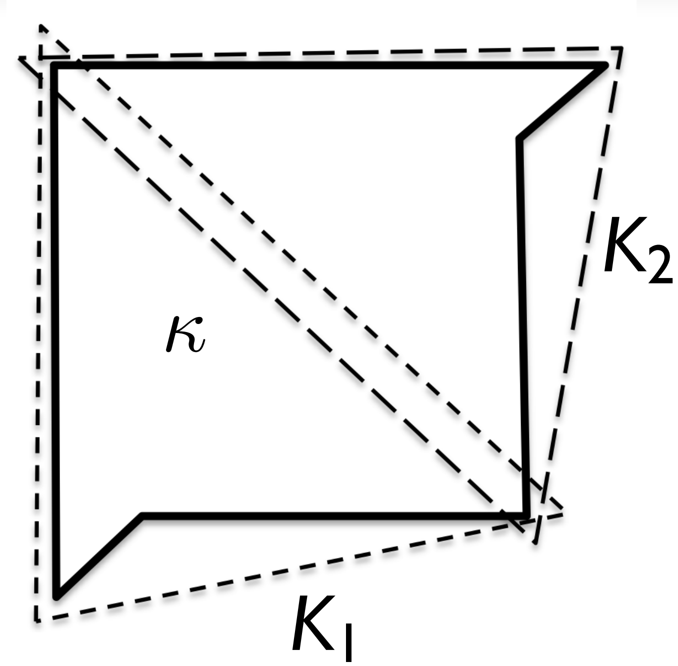


Definition 3

An element $\kappa \in \mathcal{T}_h$ is said to be *p-coverable* with respect to $p \in \mathbb{N}$, if there exists a set of m_κ overlapping shape-regular simplices K_i, $i = 1, \ldots, m_\kappa$, $m_\kappa \in \mathbb{N}$, such that

$$\mathrm{dist}(\kappa, \partial K_i) < C_{as} \frac{\mathrm{diam}(K_i)}{p^2}, \qquad \text{and} \qquad |K_i| \geq c_{as} |\kappa|$$

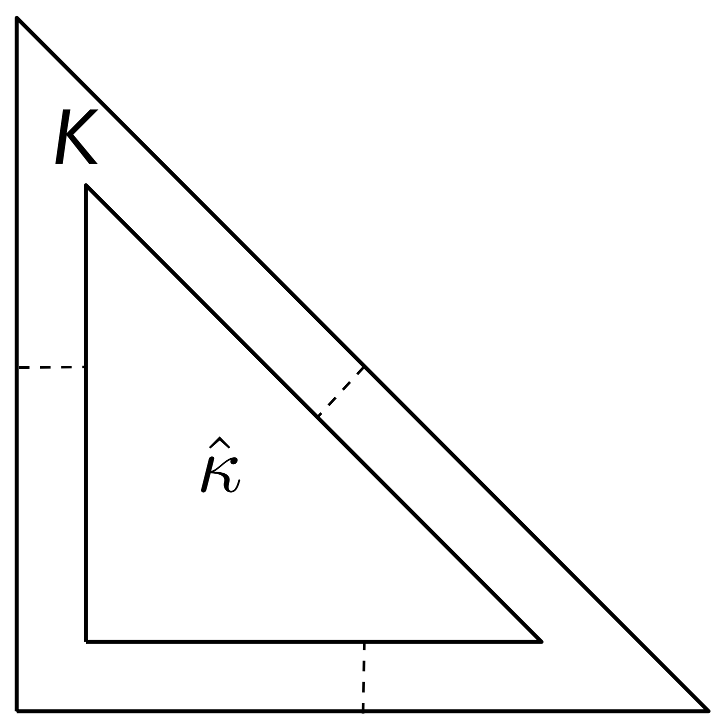
for all $i = 1, \ldots, m_\kappa$, where C_{as} and c_{as} are positive constants, independent of κ and $\mathcal{T}_h$.



Lemma 3 (Georgoulis 2008)

Let K be a shape-regular simplex in $\mathbb{R}^d$, $d = 2, 3$. Then, for each $v \in \mathcal{P}_p(K)$, there exists a simplex $\hat{\kappa} \subset K$, having the same shape as K and faces parallel to the faces of K, with $\mathrm{dist}(\partial\hat{\kappa}, \partial K) > C_{as}\,\mathrm{diam}(K)/p^2$, where C_{as} is a positive constant, independent of v, K, and p, such that

$$\|v\|_{L^2(\hat{\kappa})} \geq \frac{1}{2}\|v\|_{L^2(K)}.$$



Lemma 4

Let $\kappa \in \mathcal{T}_h$, $F \subset \partial\kappa$ denote one of its faces. Then, for each $v \in \mathcal{P}_p(\kappa)$, the following inverse inequality holds

$$\|v\|_{L^2(F)}^2 \leq C_{\mathrm{INV}}(p, \kappa, F) p^2 \frac{|F|}{|\kappa|} \|v\|_{L^2(\kappa)}^2,$$

where

$$C_{\mathrm{INV}}(p, \kappa, F) := \begin{cases} C_{\mathrm{inv},4} \min\left\{ \dfrac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, p^{2(d-1)} \right\}, & \text{if } \kappa \text{ is } p\text{-coverable} \\ C_{\mathrm{inv},1} \dfrac{|\kappa|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, & \text{otherwise,} \end{cases}$$

and $C_{\mathrm{inv},1}$ and $C_{\mathrm{inv},4}$ are positive constants which are independent of $|\kappa| / \sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|$, $|F|$, p, and v.

Case 1: If κ, $\kappa \in \mathcal{T}_h$, is not p-coverable then the bound follows immediately.

Case 2: Assuming is p-coverable we note that

$$\kappa_\flat^F \subset \kappa \subset \cup_{i=1}^{m_\kappa} K_i,$$

with $|K_i| \geq c_{as}|\kappa|$, $i = 1, \ldots, m_\kappa$.

Recall:

$$\|v\|_{L^2(F)}^2 \leq |F| \|v\|_{L^\infty(\kappa_\flat^F)}^2.$$

Furthermore,

$$\begin{aligned}
\|v\|_{L^\infty(\kappa_\flat^F)}^2 &\leq \max_{i=1,\ldots,m_\kappa} \|v\|_{L^\infty(K_i)}^2 \\
&\leq C_{\mathrm{inv},2} p^{2d} \max_{i=1,\ldots,m_\kappa} \frac{\|v\|_{L^2(K_i)}^2}{|K_i|} \\
&\leq \frac{C_{\mathrm{inv},2}}{c_{as}} \frac{p^{2d}}{|\kappa|} \max_{i=1,\ldots,m_\kappa} \|v\|_{L^2(K_i)}^2.
\end{aligned}$$

We now define $\hat{\kappa}_i \subset K_i$ to denote the simplex relative to K_i; by construction:

$$\hat{\kappa}_i \subset \kappa \cap K_i \subset K_i \qquad \text{and} \qquad K_i \cap \kappa \subset \kappa, \quad i = 1, \dots, m_\kappa.$$

Thereby,

$$\frac{1}{4}\|v\|^2_{L^2(K_i)} \leq \|v\|^2_{L^2(\hat{\kappa}_i)} \leq \|v\|^2_{L^2(K_i \cap \kappa)} \leq \|v\|^2_{L^2(\kappa)}.$$

Hence,

$$\begin{aligned} \|v\|^2_{L^\infty(\kappa_\flat^F)} &\leq \frac{C_{\mathrm{inv},2}}{c_{as}} \frac{p^{2d}}{|\kappa|} \max_{i=1,\dots,m_\kappa} \|v\|^2_{L^2(K_i)} \\ &\leq \frac{4\, C_{\mathrm{inv},2}}{c_{as}} \frac{p^{2d}}{|\kappa|} \|v\|^2_{L^2(\kappa)}. \end{aligned}$$

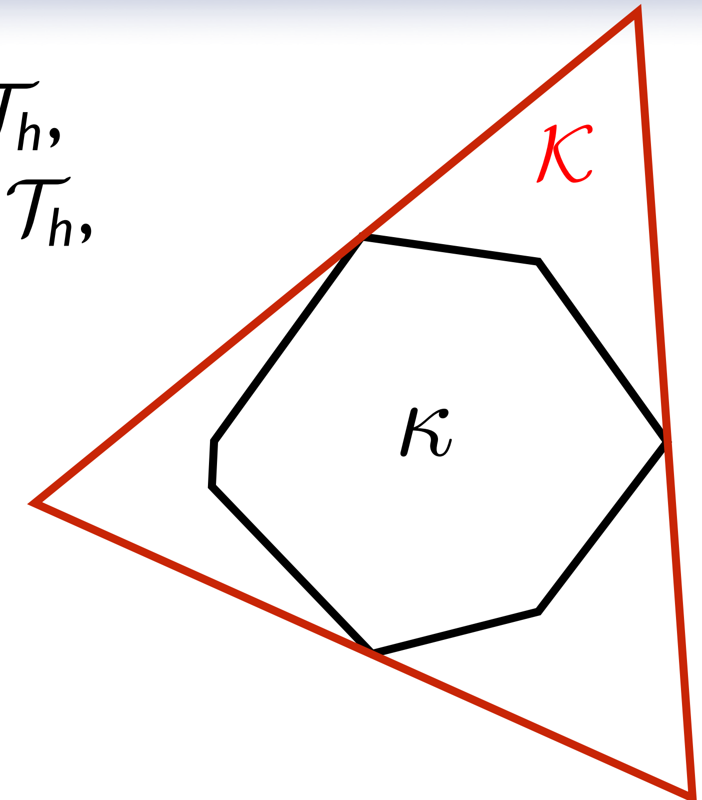
$$\Rightarrow \|v\|^2_{L^2(F)} \leq \frac{4\, C_{\mathrm{inv},2}}{c_{as}} \frac{|F|}{|\kappa|} p^{2d} \|v\|^2_{L^2(\kappa)}.$$

Taking the minimum between the two bounds, gives the desired result.

Polytopic Meshes
Approximation Theory

Let $\mathcal{T}_h^\sharp = \{\mathcal{K}\}$ denote a shape-regular covering of $\mathcal{T}_h$, consisting of d-simplices $\mathcal{K}$, such that, for each $\kappa \in \mathcal{T}_h$, there exists a $\mathcal{K} \in \mathcal{T}_h^\sharp$, such that $\kappa \subset \mathcal{K}$.

Given $\mathcal{T}_h^\sharp$, we denote by $\Omega_\sharp$ the *covering domain* given by $\bar{\Omega}_\sharp := \cup_{\mathcal{K} \in \mathcal{T}_h^\sharp} \bar{\mathcal{K}}$.



Lemma 5 (Stein 1970)

Let Ω be a domain with a Lipschitz boundary. Then there exists a linear extension operator $\mathfrak{E} : H^s(\Omega) \mapsto H^s(\mathbb{R}^d)$, $s \in \mathbb{N}_0$, such that $\mathfrak{E}v|_\Omega = v$ and

$$\|\mathfrak{E}v\|_{H^s(\mathbb{R}^d)} \leq C_\mathfrak{E} \|v\|_{H^s(\Omega)},$$

where $C_\mathfrak{E}$ is a positive constant depending only on s and Ω.

$\Rightarrow$ Used to extend the analytical solution globally from Ω to $\Omega_\sharp$.

Assumption 3

We assume that there exists a covering $\mathcal{T}_h^\sharp$ of $\mathcal{T}_h$ and a positive constant $\mathcal{O}_\Omega$, independent of the mesh parameters, such that

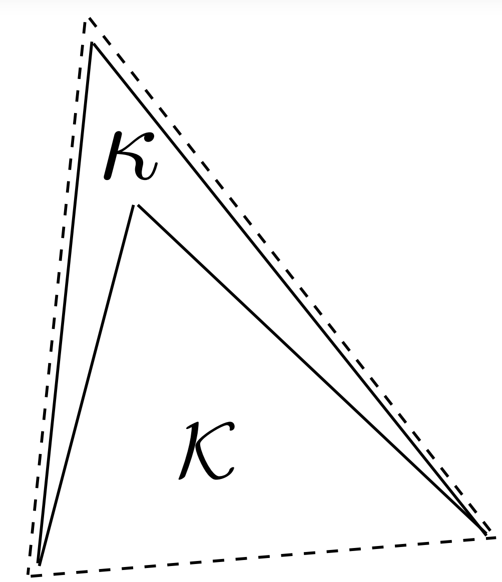
$$\max_{\kappa\in\mathcal{T}_h} \text{card}\left\{\kappa' \in \mathcal{T}_h : \kappa' \cap \mathcal{K} \neq \emptyset,\ \mathcal{K} \in \mathcal{T}_h^\sharp \text{ such that } \kappa \subset \mathcal{K}\right\} \leq \mathcal{O}_\Omega,$$

and

$$h_{\mathcal{K}} := \text{diam}(\mathcal{K}) \leq C_{\text{diam}} h_\kappa,$$

for each pair $\kappa \in \mathcal{T}_h$, $\mathcal{K} \in \mathcal{T}_h^\sharp$, with $\kappa \subset \mathcal{K}$, for a constant $C_{\text{diam}} > 0$, uniformly with respect to the mesh size.

Assumption 3 requires shape-regularity of the mesh covering $\mathcal{T}_h^\sharp$, but not shape-regularity of $\mathcal{T}_h$.



Lemma 6 (Babuska & Suri 1987; see also Schwab 1998)

Let T be a d-simplex $d = 2, 3$, with diameter h_T. Suppose further that $v|_T \in H^l(T)$, for some $l \geq 0$. Then, for $p \in \mathbb{N}$, there exists $\Pi_p v \in \mathcal{P}_p(T)$, such that

$$\|v - \Pi_p v\|_{H^q(T)} \leq C_{\mathrm{I},1} \frac{h_T^{s-q}}{p^{l-q}} \|v\|_{H^l(T)}, \quad l \geq 0,$$

for $0 \leq q \leq l$, and

$$\|v - \Pi_p v\|_{L^\infty(T)} \leq C_{\mathrm{I},2} \frac{h_T^{s-d/2}}{p^{l-d/2}} \|v\|_{H^l(T)}, \quad l > d/2.$$

Here, $s = \min\{p+1, l\}$ and $C_{\mathrm{I},1}$ and $C_{\mathrm{I},2}$ are positive constants which depend on the shape-regularity of T, but are independent of v, h_T, and p.