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Lemma 7 (Di Pietro & Ern 2012, Carstensen & Funken 2000)

Given a simplex T in $\mathbb{R}^d$, $d = 2, 3$, we write $F \subset \partial T$ to denote one of its faces. Then, given $v \in H^1(T)$, the following inequality holds:

$$\|v\|^2_{L^2(F)} \leq C_t \frac{|F|}{|T|} \left(\|v\|^2_{L^2(T)} + h_T \|v\|_{L^2(T)} \|\nabla v\|_{L^2(T)} \right),$$

where C_t is a positive constant which depends on d, but is independent of v, h_T, $|T|$, $|F|$, and the shape-regularity of T.

We write $\tilde{\Pi}_p v = \Pi_p(\mathfrak{E} v|_{\mathcal{K}})|_\kappa$.

- Π_p: Projector on $\mathcal{K}$ (standard element shape).
- $\mathfrak{E}$: Extension operator.

Lemma 8

Let $\kappa \in \mathcal{T}_h$, $F \subset \partial\kappa$ denote one of its faces, and $\mathcal{K} \in \mathcal{T}_h^\sharp$ be the d-simplex, such that $\kappa \subset \mathcal{K}$. Suppose that $v \in L^2(\Omega)$ is such that $\mathfrak{E}v|_{\mathcal{K}} \in H^{l_\kappa}(\mathcal{K})$, for some $l_\kappa \geq 0$. Then, given Assumption 3 is satisfied, the following bounds hold

$$\|v - \tilde{\Pi}_p v\|_{H^q(\kappa)} \leq C_{\mathrm{I},3} \frac{h_\kappa^{s_\kappa - q}}{p^{l_\kappa - q}} \|\mathfrak{E}v\|_{H^{l_\kappa}(\mathcal{K})}, \quad l_\kappa \geq 0,$$

for $0 \leq q \leq l_\kappa$, and

$$\|v - \tilde{\Pi}_p v\|_{L^2(F)} \leq C_{\mathrm{I},4} |F|^{1/2} \frac{h_\kappa^{s_\kappa - d/2}}{p^{l_\kappa - 1/2}} C_m(p, \kappa, F)^{1/2} \|\mathfrak{E}v\|_{H^{l_\kappa}(\mathcal{K})}, \ l_\kappa > d/2,$$

where

$$C_m(p, \kappa, F) = \min\left\{ \frac{h_\kappa^d}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, p^{d-1} \right\},$$

$s_\kappa = \min\{p+1, l_\kappa\}$ and $C_{\mathrm{I},3}$ and $C_{\mathrm{I},4}$ are positive constants, which depend on the shape-regularity of $\mathcal{K}$, but are independent of v, h_κ, and p.

To prove the first bound, we note that

$$\|v - \tilde{\Pi}_p v\|_{H^q(\kappa)} = \|\mathfrak{E}v - \Pi_p(\mathfrak{E}v)\|_{H^q(\kappa)} \leq \|\mathfrak{E}v - \Pi_p(\mathfrak{E}v)\|_{H^q(\mathcal{K})}.$$

Thereby, upon application of Lemma 6 and noting that Assumption 3 holds, the desired bound follows immediately with $C_{\mathrm{I},3} = C_{\mathrm{I},1} C_{\mathrm{diam}}^{s_\kappa - q}$.

To prove the second bound, we recall that $\kappa_\flat^F \subset \kappa$; then, applying Lemma 7, together with the first bound, we obtain

$$\begin{aligned} \|v - \tilde{\Pi}_p v\|^2_{L^2(F)} \quad &\leq \quad C_{\mathrm{t}} \frac{|F|}{|\kappa_\flat^F|} \Big(\|v - \tilde{\Pi}_p v\|^2_{L^2(\kappa_\flat^F)} \\ &\quad + h_{\kappa_\flat^F} \|v - \tilde{\Pi}_p v\|_{L^2(\kappa_\flat^F)} \|\nabla(v - \tilde{\Pi}_p v)\|_{L^2(\kappa_\flat^F)} \Big) \\ &\leq \quad C_{\mathrm{t}}\, C_{\mathrm{I},1}^2\, C_{\mathrm{diam}}^{2s_\kappa - 1} \frac{|F|}{|\kappa_\flat^F|} \left(C_{\mathrm{diam}} \frac{h_\kappa}{p} + h_{\kappa_\flat^F} \right) \frac{h_\kappa^{2s_\kappa - 1}}{p^{2l_\kappa - 1}} \|\mathfrak{E}v\|^2_{H^{l_\kappa}(\mathcal{K})}. \end{aligned}$$

Given that $h_{\kappa_\flat^F} \leq h_\kappa$ and $\kappa_\flat^F$ is arbitrary, we conclude that

$$\|v - \tilde{\Pi} v\|^2_{L^2(F)} \leq C_t\, C^2_{\mathrm{I},1}\, C^{2s_\kappa - 1}_{\mathrm{diam}} (C_{\mathrm{diam}} + 1) \frac{|F|}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|} \frac{h_\kappa^{2s_\kappa}}{p^{2l_\kappa - 1}} \|\mathfrak{E} v\|^2_{H^{l_\kappa}(\mathcal{K})}.$$

On the other hand, proceeding as in the proof of the inverse inequality, we observe that

$$\|v - \tilde{\Pi}_p v\|^2_{L^2(F)} \leq |F| \|v - \tilde{\Pi}_p v\|^2_{L^\infty(F)}.$$

Hence, employing the definition of $\tilde{\Pi}_p$, together with Lemma 6 and Assumption 3, we deduce that

$$\|v - \tilde{\Pi}_p v\|^2_{L^2(F)} \leq C^2_{\mathrm{I},2}\, C^{2s_\kappa - d}_{\mathrm{diam}}\, |F| \frac{h_\kappa^{2s_\kappa - d}}{p^{2l_\kappa - d}} \|\mathfrak{E} v\|^2_{H^{l_\kappa}(\mathcal{K})}, \qquad \text{for } l_\kappa > d/2.$$

Thereby, taking the minimum of the two bounds, gives the desired result.

DGFEMs on Polytopic Meshes I: Elliptic PDEs

Error Analysis 1: Bounded number of element faces

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Poisson's Equation

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and $f \in L_2(\Omega)$: find u such that

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

- To each element $\kappa \in \mathcal{T}_h$, we associate a local polynomial degree $p_\kappa \geq 1$, and collect the p_κ, $\kappa \in \mathcal{T}_h$, in the vector $\mathbf{p} := (p_\kappa : \kappa \in \mathcal{T}_h)$
- Finite element space:

$$V^{\mathbf{p}}(\mathcal{T}_h) := \{u \in L^2(\Omega) : u|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa), \kappa \in \mathcal{T}_h\},$$

where, $\mathcal{P}_p(\kappa)$ denotes the space of polynomials of total degree p on κ.

- By construction the local elemental polynomial spaces employed within of $V^{\mathbf{p}}(\mathcal{T}_h)$ are defined in the physical space, without the need to map from a given reference or canonical frame

SIP-DGFEM

Find $u_h \in V^{\mathrm{p}}(\mathcal{T}_h)$ such that

$$B_{\mathrm{d}}(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V^{\mathrm{p}}(\mathcal{T}_h),$$

where

$$\begin{aligned} B_{\mathrm{d}}(u, v) \quad &:= \quad \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u \cdot \nabla v \mathrm{d}\mathbf{x} \\ &\quad - \sum_{F \in \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}} \int_F \left(\{\!\{\nabla u\}\!\} \cdot [\![v]\!] + \{\!\{\nabla v\}\!\} \cdot [\![u]\!] - \sigma [\![u]\!] \cdot [\![v]\!] \right) \mathrm{d}s, \end{aligned}$$

and linear functional

$$\ell(v) := \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f v \mathrm{d}\mathbf{x} - \sum_{F \in \mathcal{F}_h^{\mathcal{B}}} \int_F \mathrm{g}(\nabla v \cdot \mathbf{n} - \sigma v) \mathrm{d}s.$$

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- Since the DGFEM is not well-defined for functions in $H^1(\Omega)$, we consider an extension of $B_d(\cdot,\cdot)$ and $\ell(\cdot)$ (inhomogeneous case).

- Define $\mathbf{\Pi}_{L^2} : [L^2(\Omega)]^d \to [V^{\mathbf{p}}(\mathcal{T}_h)]^d$ to denote the orthogonal L^2-projection onto $[V^{\mathbf{p}}(\mathcal{T}_h)]^d$: given $\boldsymbol{v} \in [L^2(\Omega)]^d$ we define $\mathbf{\Pi}_{L^2}\boldsymbol{v}$ by

$$\int_\Omega \mathbf{\Pi}_{L^2}\boldsymbol{v} \cdot \boldsymbol{w} \,\mathrm{d}\mathbf{x} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} \,\mathrm{d}\mathbf{x} \qquad \forall \boldsymbol{w} \in [V^{\mathbf{p}}(\mathcal{T}_h)]^d.$$

- Define

$$\begin{aligned} \tilde{B}_{\mathrm{d}}(w,v) &:= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \nabla w \cdot \nabla v \mathrm{d}\mathbf{x} \\ &\quad - \sum_{F\in\mathcal{F}_h^{\mathcal{I}}\cup\mathcal{F}_h^{\mathcal{B}}} \int_F \left(\{\!\{\mathbf{\Pi}_{L^2}(\nabla w)\}\!\} \cdot [\![v]\!] + \{\!\{\mathbf{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![w]\!] - \sigma [\![w]\!] \cdot [\![v]\!]\right) \mathrm{d}s, \\ \tilde{\ell}(v) &:= \sum_{\kappa\in\mathcal{T}_h} \int_\kappa f v \mathrm{d}\mathbf{x} - \sum_{F\in\mathcal{F}_h^{\mathcal{B}}} \int_F \mathrm{g}(\mathbf{\Pi}_{L^2}(\nabla v)\cdot\mathbf{n} - \sigma v)\mathrm{d}s \end{aligned}$$

for all $w, v \in \mathcal{V} := H^1(\Omega) + V^{\mathbf{p}}(\mathcal{T}_h)$.

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SIP-DGFEM

Find $u_h \in V^{\mathbf{p}}(\mathcal{T}_h)$ such that

$$\tilde{B}_{\mathrm{d}}(u_h, v_h) = \tilde{\ell}(v_h)$$

for all $v_h \in V^{\mathbf{p}}(\mathcal{T}_h)$.

- For $w_h, v_h \in V^{\mathbf{p}}(\mathcal{T}_h)$, we have

$$\tilde{B}_{\mathrm{d}}(u_h, v_h) = B_{\mathrm{d}}(u_h, v_h) \qquad \tilde{\ell}(v_h) = \ell(v_h).$$

(equivalent reformulation of the DGFEM)

- The bilinear form $\tilde{B}_{\mathrm{d}}$ is inconsistent due to the discrete nature of the projection operator $\mathbf{\Pi}_{L^2}$ (Galerkin orthogonality no longer holds).