Stability of the DGFEM

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DGFEM Norm

$$|||v|||_{\mathrm{DG}}^2 := \sum_{\kappa \in \mathcal{T}_h} \|\nabla v\|_{L^2(\kappa)}^2 + \sum_{F \in \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}} \int_F \sigma |[\![v]\!]|^2 \mathrm{d}s$$

Arithmetic-Geometric Mean Inequality

For any $a, b \in \mathbb{R}$ and $\epsilon > 0$, we have that

$$ab \leq a^2\epsilon + \frac{b^2}{4\epsilon}.$$

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Interior Penalty Parameter

$$\sigma|_F := \begin{cases} C_\sigma \max_{\kappa\in\{\kappa^+,\kappa^-\}} \left\{ C_{\mathrm{INV}}(p_\kappa,\kappa,F)\, \dfrac{p_\kappa^2 |F|}{|\kappa|} \right\}, & F \in \mathcal{F}_h^{\mathcal{I}},\ F \subset \partial\kappa^+ \cap \partial\kappa^-, \\ C_\sigma C_{\mathrm{INV}}(p_\kappa,\kappa,F)\, \dfrac{p_\kappa^2 |F|}{|\kappa|}, & F \in \mathcal{F}_h^{\mathcal{B}},\ F \subset \partial\kappa. \end{cases}$$

Lemma 9 (Coercivity & Continuity)

For $C_\sigma > C_\sigma^{\min}$ we have

$$\tilde{B}_{\mathrm{d}}(v,v) \geq C_{\mathrm{coer}} |||v|||_{\mathrm{DG}}^2 \quad \text{for all} \quad v \in \mathcal{V},$$

and

$$\tilde{B}_{\mathrm{d}}(w,v) \leq C_{\mathrm{cont}} |||w|||_{\mathrm{DG}}\, |||v|||_{\mathrm{DG}} \quad \text{for all} \quad w, v \in \mathcal{V}.$$

For $v \in \mathcal{V}$, we have the following identity

$$\tilde{B}_{\mathrm{d}}(v,v) \quad = \quad |||v|||_{\mathrm{DG}}^2 - 2 \sum_{F \in \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}} \int_F \{\!\{\mathbf{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \mathrm{d}s.$$

(Recall: $|||v|||_{\mathrm{DG}}^2 := \sum_{\kappa \in \mathcal{T}_h} \|\nabla v\|_{L^2(\kappa)}^2 + \sum_{F \in \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}} \int_F \sigma |[\![v]\!]|^2 \mathrm{d}s$)

For $F \subset \partial\kappa^+ \cap \partial\kappa^-$, $\kappa^\pm \in \mathcal{T}_h$, upon application of the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we deduce that

$$\begin{aligned}
\int_F \{\!\{\mathbf{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \mathrm{d}s \quad &\le \quad \frac{1}{2} \left(\|\frac{1}{\sqrt{\sigma}} \mathbf{\Pi}_{L^2}(\nabla v^+)\|_{L^2(F)} + \|\frac{1}{\sqrt{\sigma}} \mathbf{\Pi}_{L^2}(\nabla v^-)\|_{L^2(F)} \right) \\
&\quad \times \|\sqrt{\sigma}[\![v]\!]\|_{L^2(F)} \\
&\le \quad \epsilon \left(\|\frac{1}{\sqrt{\sigma}} \mathbf{\Pi}_{L^2}(\nabla v^+)\|_{L^2(F)}^2 + \|\frac{1}{\sqrt{\sigma}} \mathbf{\Pi}_{L^2}(\nabla v^-)\|_{L^2(F)}^2 \right) \\
&\quad + \frac{1}{8\epsilon} \|\sqrt{\sigma}[\![v]\!]\|_{L^2(F)}^2.
\end{aligned}$$

Proof: Coercivity

Stability of the L^2-projector $\mathbf{\Pi}_{L^2}$ in the L^2-norm: for $\mathbf{v} \in [\mathcal{V}]^d$, $\kappa \in \mathcal{T}_h$,

$$\|\mathbf{\Pi}_{L^2}\mathbf{v}\|_{L^2(\kappa)} \leq \|\mathbf{v}\|_{L^2(\kappa)}.$$

Employing the inverse inequality, the definition of σ, and the above inequality gives

$$\begin{aligned}
&\int_F \{\!\{\mathbf{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \mathrm{d}s \\
&\leq \epsilon \left(C_{\mathrm{INV}}(p_{\kappa^+}, \kappa^+, F) \frac{p_{\kappa^+}^2 |F|}{|\kappa^+|} \|\frac{1}{\sqrt{\sigma}} \mathbf{\Pi}_{L^2}(\nabla v)\|_{L^2(\kappa^+)}^2 \right. \\
&\quad \left. + C_{\mathrm{INV}}(p_{\kappa^-}, \kappa^-, F) \frac{p_{\kappa^-}^2 |F|}{|\kappa^-|} \|\frac{1}{\sqrt{\sigma}} \mathbf{\Pi}_{L^2}(\nabla v)\|_{L^2(\kappa^-)}^2 \right) + \frac{1}{8\epsilon} \|\sqrt{\sigma}[\![v]\!]\|_{L^2(F)}^2 \\
&\leq \frac{\epsilon}{C_\sigma} \left(\|\nabla v\|_{L^2(\kappa^+)}^2 + \|\nabla v\|_{L^2(\kappa^-)}^2 \right) + \frac{1}{8\epsilon} \|\sqrt{\sigma}[\![v]\!]\|_{L^2(F)}^2.
\end{aligned}$$

Proof: Coercivity

Similarly, for $F \in \mathcal{F}_h^{\mathcal{B}}$, where $F \subset \partial\kappa$, $\kappa \in \mathcal{T}_h$, we get

$$\int_F \{\!\{\mathbf{\Pi}_{L^2}(\nabla v)\}\!\} \cdot [\![v]\!] \mathrm{d}s \le \frac{\epsilon}{C_\sigma} \|\nabla v\|^2_{L^2(\kappa)} + \frac{1}{4\epsilon} \|\sqrt{\sigma}[\![v]\!]\|^2_{L^2(F)}.$$

Hence, we deduce that

$$\tilde{B}_{\mathrm{d}}(v,v) \quad \ge \quad \left(1 - \frac{2C_F}{C_\sigma}\epsilon\right) \sum_{\kappa \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(\kappa)} + \left(1 - \frac{1}{2\epsilon}\right) \sum_{F \in \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}} \|\sqrt{\sigma}[\![v]\!]\|^2_{L^2(F)}.$$

Given Assumption 2 holds, i.e., that the number of elemental faces is uniformly bounded, we deduce that if

$$C_\sigma > 2C_F\epsilon \quad \text{for some} \quad \epsilon > 1/2$$

then the bilinear form $\tilde{B}_{\mathrm{d}}(\cdot,\cdot)$ is coercive over $\mathcal{V} \times \mathcal{V}$.

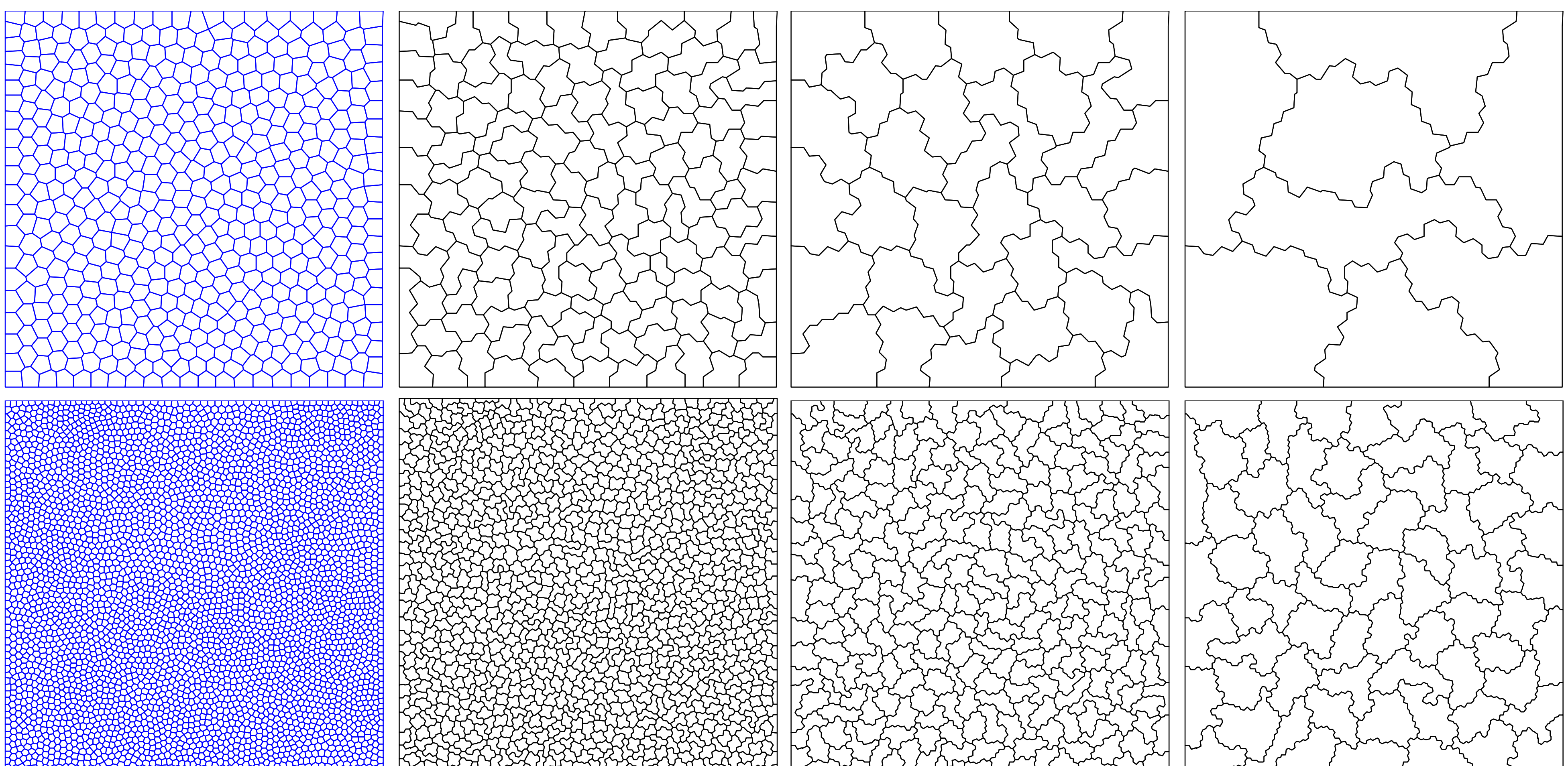
The proof of continuity follows analogously: left as an exercise.

Stability of the DGFEM

	Set 1	Set 2	Set 3	Set 4
Mesh 1	0.7385	0.7375	0.7370	0.7364
Mesh 2	0.7624	0.7564	0.7559	0.7545
Mesh 3	0.7827	0.7818	0.7720	0.7611
Mesh 4	0.8153	0.8054	0.8001	0.7827

Set 1

Set 4



Lemma 10 (Strang's Second Lemma)

Let $u \in H^1(\Omega)$ be the weak solution of our PDE problem and $u_h \in V^{\mathbf{p}}(\mathcal{T}_h)$ its DGFEM approximation. Given Assumption 1 holds, we have that

$$\begin{aligned} |||u - u_h|||_{\mathrm{DG}} \quad &\leq \quad \left(1 + \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}}\right) \inf_{v_h \in V^{\mathbf{p}}(\mathcal{T}_h)} |||u - v_h|||_{\mathrm{DG}} \\ &+ \frac{1}{C_{\mathrm{coer}}} \sup_{w_h \in V^{\mathbf{p}}(\mathcal{T}_h) \setminus \{0\}} \frac{|\tilde{B}_{\mathrm{d}}(u, w_h) - \tilde{\ell}(w_h)|}{|||w_h|||_{\mathrm{DG}}}. \end{aligned}$$

Proof: Employing the triangle inequality gives

$$|||u - u_h|||_{\mathrm{DG}} \leq |||u - v_h|||_{\mathrm{DG}} + |||v_h - u_h|||_{\mathrm{DG}} \quad \forall v_h \in V^{\mathbf{p}}(\mathcal{T}_h).$$

Exploiting coercivity and continuity, we obtain

$$\begin{aligned}
|||v_h - u_h|||_{\mathrm{DG}}^2 &\leq \frac{1}{C_{\mathrm{coer}}}\tilde{B}_{\mathrm{d}}(v_h - u_h, v_h - u_h) \\
&= \frac{1}{C_{\mathrm{coer}}}(\tilde{B}_{\mathrm{d}}(v_h - u, v_h - u_h) + \tilde{B}_{\mathrm{d}}(u - u_h, v_h - u_h)) \\
&\leq \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}}|||u - v_h|||_{\mathrm{DG}}|||v_h - u_h|||_{\mathrm{DG}} \\
&\quad + \frac{1}{C_{\mathrm{coer}}}(\tilde{B}_{\mathrm{d}}(u, v_h - u_h) - \tilde{\ell}(v_h - u_h)).
\end{aligned}$$

Hence

$$\begin{aligned}
|||v_h - u_h|||_{\mathrm{DG}} &\leq \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}}|||u - v_h|||_{\mathrm{DG}} + \frac{1}{C_{\mathrm{coer}}}\frac{\tilde{B}_{\mathrm{d}}(u, v_h - u_h) - \tilde{\ell}(v_h - u_h)}{|||v_h - u_h|||_{\mathrm{DG}}} \\
&\leq \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}}|||u - v_h|||_{\mathrm{DG}} + \frac{1}{C_{\mathrm{coer}}}\sup_{w_h \in V^{\mathbf{p}}(\mathcal{T}_h)\setminus\{0\}}\frac{|\tilde{B}_{\mathrm{d}}(u, w_h) - \tilde{\ell}(w_h)|}{|||w_h|||_{\mathrm{DG}}}
\end{aligned}$$

Term I: Best Approximation

By defining $\tilde{\Pi}_{\mathbf{p}}$ by $\tilde{\Pi}_{\mathbf{p}}|_\kappa := \tilde{\Pi}_{p_\kappa}$, for $\kappa \in \mathcal{T}_h$, given that Assumption 2 holds, we deduce that

$$\inf_{v_h \in V^{\mathbf{p}}(\mathcal{T}_h)} |||u - v_h|||_{\mathrm{DG}}^2 \leq |||u - \tilde{\Pi}_{\mathbf{p}} u|||_{\mathrm{DG}}^2$$

$$\leq \sum_{\kappa \in \mathcal{T}_h} \left(\|\nabla(u - \tilde{\Pi}_{p_\kappa} u)\|_{L^2(\kappa)}^2 + 2 \sum_{F \subset \partial\kappa} \sigma \|(u - \tilde{\Pi}_{p_\kappa} u)|_\kappa\|_{L^2(F)}^2 \right)$$

$$\leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \left(1 + \frac{h_\kappa^{-d+2}}{p_\kappa} \sum_{F \subset \partial\kappa} C_m(p_\kappa, \kappa, F) \sigma |F| \right) \|\mathfrak{E} u\|_{H^{l_\kappa}(\mathcal{K})}^2,$$

with $s_\kappa = \min\{p_\kappa + 1, l_\kappa\}$ and C a positive constant, which depends on the shape-regularity of the covering $\mathcal{T}_h^\sharp$. Here,

$$C_m(p, \kappa, F) = \min\left\{ \frac{h_\kappa^d}{\sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|}, \frac{1}{p^{1-d}} \right\}.$$