# Term II: Residual Term



By applying integration by parts elementwise and noting that $u$ is the analytical solution of the PDE, we get

$$\begin{aligned}\left|\tilde{B}_{\mathrm{d}}(u, w_h) - \tilde{\ell}(u, w_h)\right| &= \Big|\sum_{F \in \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}} \int_F \{\!\{(\nabla u - \mathbf{\Pi}_{L^2}(\nabla u))\}\!\} \cdot [\![w_h]\!] \mathrm{d}s\Big| \\ &\leq \Big(\sum_{F \in \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}} \int_F \sigma^{-1} |\{\!\{(\nabla u - \mathbf{\Pi}_{L^2}(\nabla u))\}\!\}|^2 \mathrm{d}s\Big)^{1/2} \\ &\quad \times |||w_h|||_{\mathrm{DG}}.\end{aligned}$$

Here, we have used the identity

$$\sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \mathbf{q}^+ \cdot \mathbf{n}^+ v^+ \mathrm{d}s = \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mathbf{q}\}\!\} \cdot [\![v]\!] \mathrm{d}s + \sum_{F \in \mathcal{F}_h^{\mathcal{I}}} \int_F [\![\mathbf{q}]\!] \{\!\{v\}\!\} \mathrm{d}s.$$

Adding and subtracting $\tilde{\boldsymbol{\Pi}}_p(\nabla u)$, gives

$$\sum_{F\in\mathcal{F}_h^{\mathcal{I}}\cup\mathcal{F}_h^{\mathcal{B}}}\int_F \sigma^{-1}|\{\!\{(\nabla u-\boldsymbol{\Pi}_{L^2}(\nabla u))\}\!\}|^2\mathrm{d}s$$
$$\le \sum_{F\in\mathcal{F}_h^{\mathcal{I}}\cup\mathcal{F}_h^{\mathcal{B}}}\int_F 2\sigma^{-1}(|\{\!\{(\nabla u-\tilde{\boldsymbol{\Pi}}_p(\nabla u))\}\!\}|^2+|\{\!\{(\boldsymbol{\Pi}_{L^2}(\tilde{\boldsymbol{\Pi}}_p(\nabla u)-\nabla u))\}\!\}|^2)\mathrm{d}s$$
$$\equiv \mathrm{II}_1+\mathrm{II}_2.$$

Exploiting approximation bounds yields

$$\mathrm{II}_1\le C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(s_\kappa-1)}}{p_\kappa^{2(l_\kappa-1)}}\frac{h_\kappa^{-d}}{p_\kappa^{-1}}\left(\sum_{F\subset\partial\kappa}C_m(p_\kappa,\kappa,F)\sigma^{-1}|F|\right)\|\mathfrak{E}u\|^2_{H^{l_\kappa}(\mathcal{K})}.$$

Similarly, employing the inverse inequality, the $L^2$-stability of the projector $\boldsymbol{\Pi}_{L^2}$, and approximation bounds, gives

$$\mathrm{II}_2\le C\sum_{\kappa\in\mathcal{T}_h}\frac{h_\kappa^{2(s_\kappa-1)}}{p_\kappa^{2(l_\kappa-1)}}\frac{|\kappa|^{-1}}{p_\kappa^{-2}}\left(\sum_{F\subset\partial\kappa}C_{\mathrm{INV}}(p_\kappa,\kappa,F)\sigma^{-1}|F|\right)\|\mathfrak{E}u\|^2_{H^{l_\kappa}(\mathcal{K})}.$$

Combining the above above bounds, we deduce:

$$
\begin{aligned}
\sup_{w_h \in V^{\mathbf{p}}(\mathcal{T}_h)\setminus\{0\}} & \frac{|\tilde{B}_{\mathrm{d}}(u,w_h) - \tilde{\ell}(u,w_h)|}{|||w_h|||_{\mathrm{DG}}} \\
& \leq \left(\mathrm{II}_1 + \mathrm{II}_2\right)^{1/2} \\
& \leq C\Bigg( \sum_{\kappa\in\mathcal{T}_h} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \\
& \qquad \times \left( \sum_{F\subset\partial\kappa} \left( C_m(p_\kappa,\kappa,F)\frac{h_\kappa^{-d}}{p_\kappa^{-1}} + C_{\mathrm{INV}}(p_\kappa,\kappa,F)\frac{|\kappa|^{-1}}{p_\kappa^{-2}} \right)\sigma^{-1}|F| \right) \\
& \qquad \times \|\mathfrak{E}u\|^2_{H^{l_\kappa}(\mathcal{K})} \Bigg)^{1/2}.
\end{aligned}
$$

## Theorem I

Given that $\mathcal{T}_h = \{\kappa\}$ satisfies Assumptions 2 & 3. Assuming $u|_\kappa \in H^{l_\kappa}(\kappa)$, $l_\kappa > 1 + d/2$, for each $\kappa \in \mathcal{T}_h$, such that $\mathfrak{E}u|_\mathcal{K} \in H^{l_\kappa}(\mathcal{K})$, where $\mathcal{K} \in \mathcal{T}_h^\sharp$ with $\kappa \subset \mathcal{K}$, then

$$|||u - u_h|||_{\mathrm{DG}}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2(s_\kappa - 1)}}{p_\kappa^{2(l_\kappa - 1)}} \left(1 + \mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa)\right) \|\mathfrak{E}u\|_{H^{l_\kappa}(\mathcal{K})}^2,$$

with $s_\kappa = \min\{p_\kappa + 1, l_\kappa\}$,

$$\begin{aligned} \mathcal{G}_\kappa(F, C_{\mathrm{INV}}, C_m, p_\kappa) \quad := \quad & p_\kappa h_\kappa^{-d} \sum_{F \subset \partial\kappa} C_m(p_\kappa, \kappa, F)\sigma^{-1}|F| \\ & + p_\kappa^2 |\kappa|^{-1} \sum_{F \subset \partial\kappa} C_{\mathrm{INV}}(p_\kappa, \kappa, F)\sigma^{-1}|F| \\ & + h_\kappa^{-d+2} p_\kappa^{-1} \sum_{F \subset \partial\kappa} C_m(p_\kappa, \kappa, F)\sigma|F|, \end{aligned}$$

where $C$ is a positive constant, which depends on the shape-regularity of $\mathcal{T}_h^\sharp$.

# A Priori Error Bound



- For uniform orders $p_\kappa = p \geq 1$, $h = \max_{\kappa \in \mathcal{T}_h} h_\kappa$, $s_\kappa = s$, $s = \min\{p+1, l\}$, $l > 1 + d/2$, and $diam(F) \sim h_\kappa$, $F \subset \partial\kappa$, $\kappa \in \mathcal{T}_h$, we get the bound

$$|||u - u_h|||_{\mathrm{DG}} \leq C \frac{h^{s-1}}{p^{l-3/2}} \|u\|_{H^l(\Omega)},$$

cf. H., Schwab & Süli 2002.

- Optimal in $h$ and suboptimal in $p$ by $p^{1/2}$.

# DGFEMs on Polytopic Meshes I: Elliptic PDEs

## Error Analysis II: Arbitrary number of element faces

# Shape-Regular Polygonal/Polyhedral Meshes

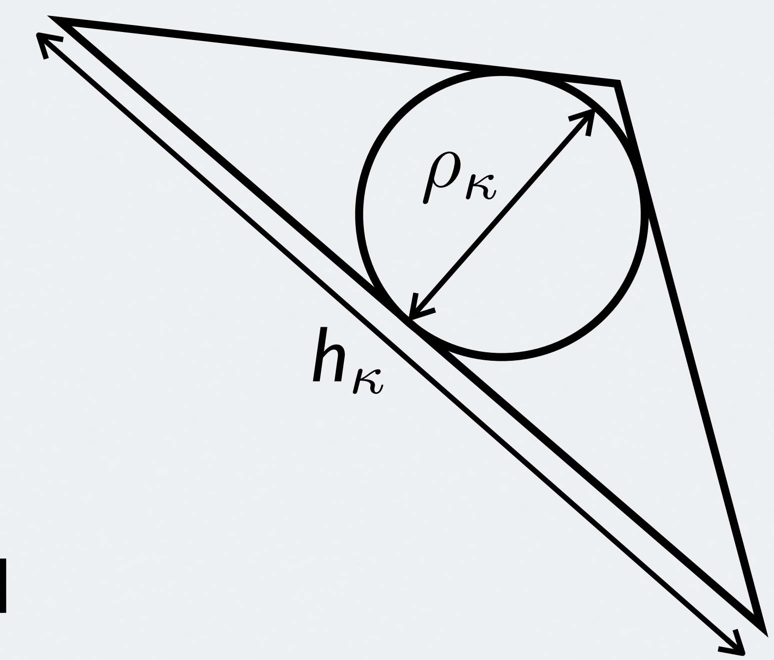
The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

## Recall: Definition 1 (Shape-Regularity)

A subdivision $\mathcal{T}_h$ is said to be shape-regular if there exists a positive constant $C_{\mathrm{r}}$ such that:

$$\forall \kappa \in \mathcal{T}_h, \qquad \frac{h_\kappa}{\rho_\kappa} \leq C_{\mathrm{r}},$$

independently of the mesh parameters, where $\rho_\kappa$ denotes the diameter of the largest ball contained in $\kappa$.




Given $F \subset \partial T$, where $T$ is a $d$-simplex, the volume of $T$ is given by

$$|T| = \frac{1}{d}|F|\mathfrak{h}_T \quad \Rightarrow \quad \mathfrak{h}_T = \frac{|T|}{|F|}d,$$

where $\mathfrak{h}_T$ denotes the height of $T$ above $F$.

## Assumption 4

For any $\kappa \in \mathcal{T}_h$, there exists a set of non-overlapping $d$-dimensional simplices $\{\kappa_\flat^F\}_{F \subset \partial\kappa} \subset \mathcal{F}_\flat^\kappa$ contained in $\kappa$, such that for all $F \subset \partial\kappa$, the following holds

$$h_\kappa \leq C_s \frac{d|\kappa_\flat^F|}{|F|},$$

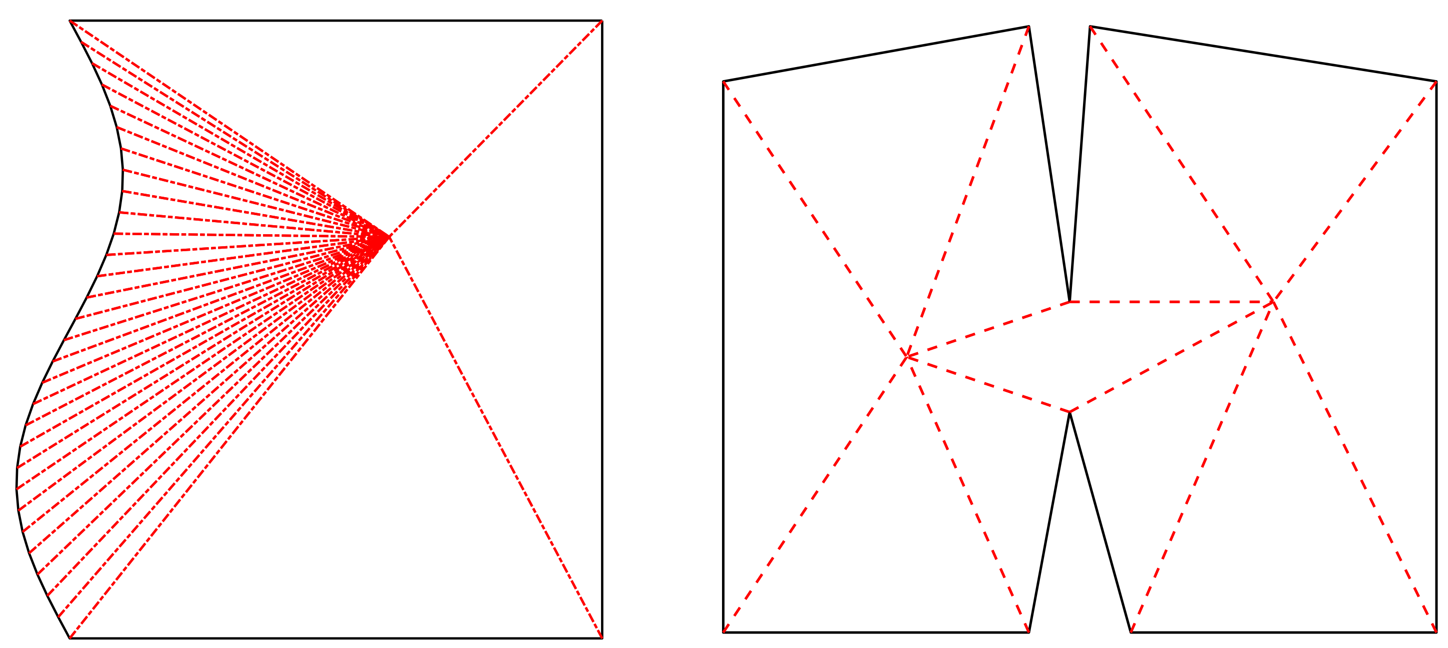
where $C_s$ is a positive constant, which is independent of the discretization parameters, the number of faces that the element possesses, and $|F|$.

Note that

$$\bigcup_{F \subset \partial\kappa} \bar{\kappa}_\flat^F \subseteq \bar{\kappa}.$$

Furthermore,

$$\rho_\kappa := \min_{F \subset \partial\kappa} d|\kappa_\flat^F|/|F|.$$



# Shape-Regular Polygonal/Polyhedral Meshes



## Lemma 11

Let $\kappa \in \mathcal{T}_h$; then assuming Assumption 4 is satisfied, for each $v \in \mathcal{P}_p(\kappa)$, the following inverse inequality holds

$$\|v\|^2_{L^2(\partial\kappa)} \leq C_s C_{\mathrm{inv},1} d \frac{p^2}{h_\kappa} \|v\|^2_{L^2(\kappa)}.$$

Here, $C_s$ is independent of $v$, $|\kappa| / \sup_{\kappa_\flat^F \subset \kappa} |\kappa_\flat^F|$, $|F|$, and $p$

The proof is based on applying Lemma 2 over each simplex $\kappa_\flat^F$ contained within $\kappa$, together with Assumption 4; thereby, we get

$$\begin{aligned}
\|v\|^2_{L^2(\partial\kappa)} &\leq \sum_{F \subset \partial\kappa} C_{\mathrm{inv},1} p^2 \frac{|F|}{|\kappa_\flat^F|} \|v\|^2_{L^2(\kappa_\flat^F)} \leq \sum_{F \subset \partial\kappa} C_s C_{\mathrm{inv},1} d \frac{p^2}{h_\kappa} \|v\|^2_{L^2(\kappa_\flat^F)} \\
&\leq C_s C_{\mathrm{inv},1} d \frac{p^2}{h_\kappa} \|v\|^2_{L^2(\kappa)}.
\end{aligned}$$

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

## Lemma 12

Let $\kappa \in \mathcal{T}_h$ and $\mathcal{K} \in \mathcal{T}_h^\sharp$ the corresponding simplex such that $\kappa \subset \mathcal{K}$. Suppose that $v \in H^1(\Omega)$ is such that $\mathfrak{E}v|_\mathcal{K} \in H^{l_\kappa}(\mathcal{K})$, for some $l_\kappa > 1/2$. Then, given that Assumption 4 is satisfied, we have

$$\|v - \tilde{\Pi}_p v\|_{L^2(\partial\kappa)} \leq C_{\mathrm{I},5} \frac{h_\kappa^{s_\kappa - 1/2}}{p^{l_\kappa - 1/2}} \|\mathcal{E}v\|_{H^{l_\kappa}(\mathcal{K})},$$

where $s_\kappa = \min\{p+1, l_\kappa\}$ and $C_{\mathrm{I},5}$ is a positive constant depending on $C_s$ and the shape-regularity of $\mathcal{K}$, but is independent of $v$, $h_\kappa$, $p$.