Bondary Element Methods: Derivation, Analysis, and Implementation

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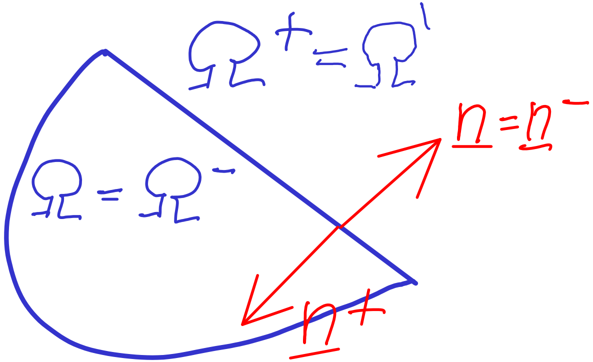
I. Representation Formulas

I.1. (Model) Boundary Value Problems (BVPs)



$\Omega \subset \mathbb{R}^d$ is

- a bounded Lipschitz polyhedron
- or the complement of a bounded Lipschitz polyhedron
- or $\Omega = \mathbb{R}^3$

I. Representation Formulas

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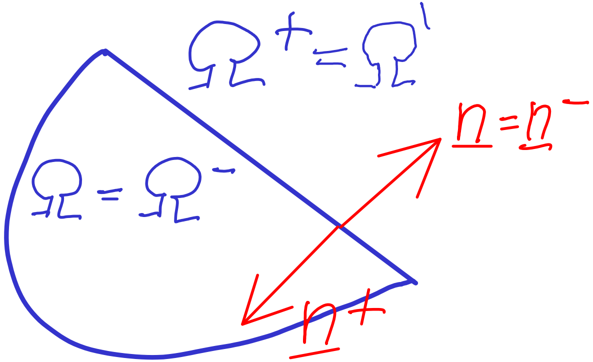

$\Omega \subset \mathbb{R}^d$ is

- a bounded Lipschitz polyhedron
- or the complement of a bounded Lipschitz polyhedron
- or $\Omega = \mathbb{R}^3$

(Ω^- bounded, $\Omega^+ := \mathbb{R}^d \setminus \Omega^-$ unbounded)

[$\underline{n} : \Gamma \to \mathbb{R}^d$ ≙ exterior unit normal vector field, $\Gamma := \partial\Omega = \partial\Omega^\pm$]

I.1.1. Frequency-Domain Acoustic Scattering

Complex pressure amplitude $u : \Omega \longrightarrow \mathbb{C}$

governed by Helmholtz equation

(1.1.1.A) $$-\Delta u - k^2 n(x) u = f \quad \text{in } \Omega$$

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source

wave number $k \geq 0$

refractive index $n(x) > n_0 > 0$ a.e. in Ω

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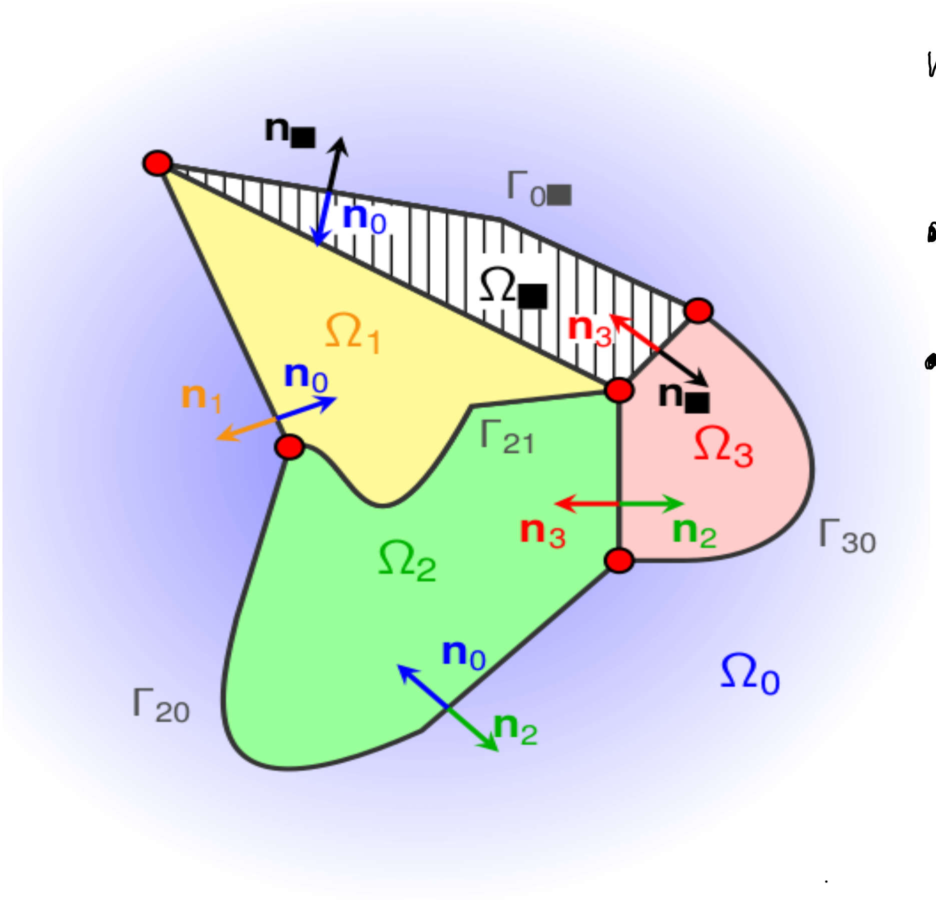
Boundary conditions on Γ :

Sound soft : $u_{|\Gamma} = g$ (→ Dirichlet BVP)

Sound hard : $\nabla u \cdot n = h$ (→ Neumann BVP)

I.1.1. Frequency-Domain Acoustic Scattering



Most general geometric setting for boundary element methods

- Partition $\mathbb{R}^d = \Omega_\blacksquare \cup \Omega_0 \cup \dots \cup \Omega_N$
- $\Omega_j \hat{=}$ curvilinear Lipschitz polyhedra

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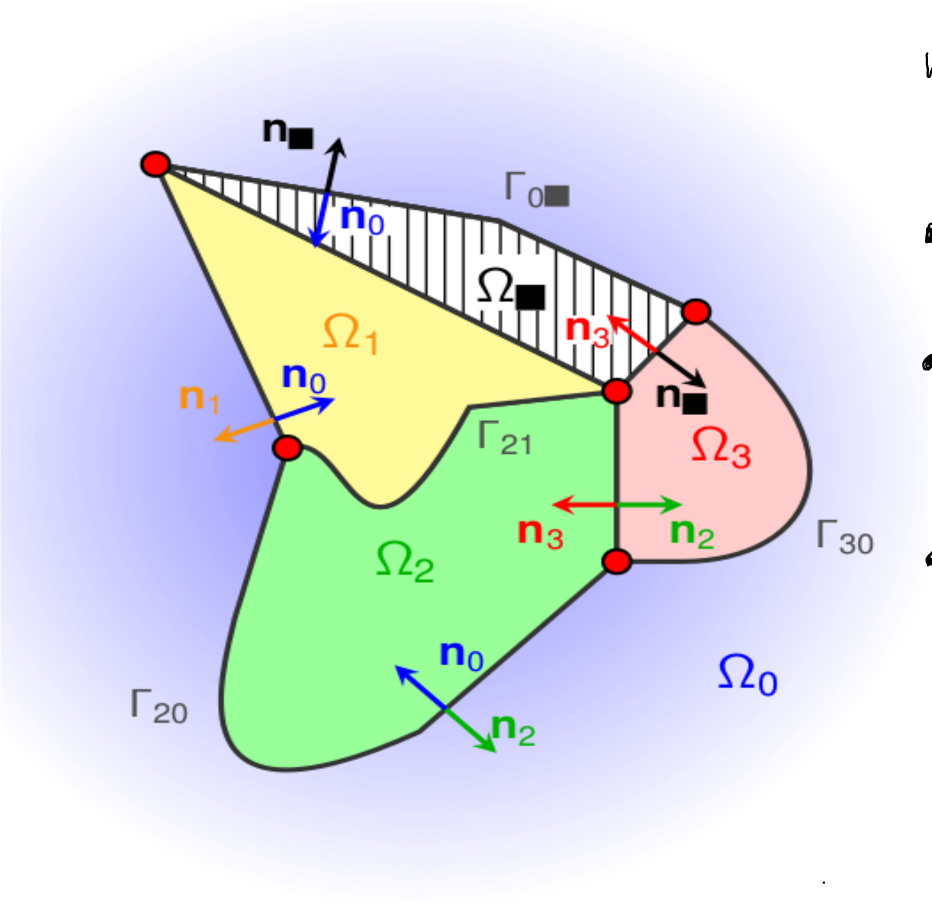

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- $n(x)$ piecewise constant

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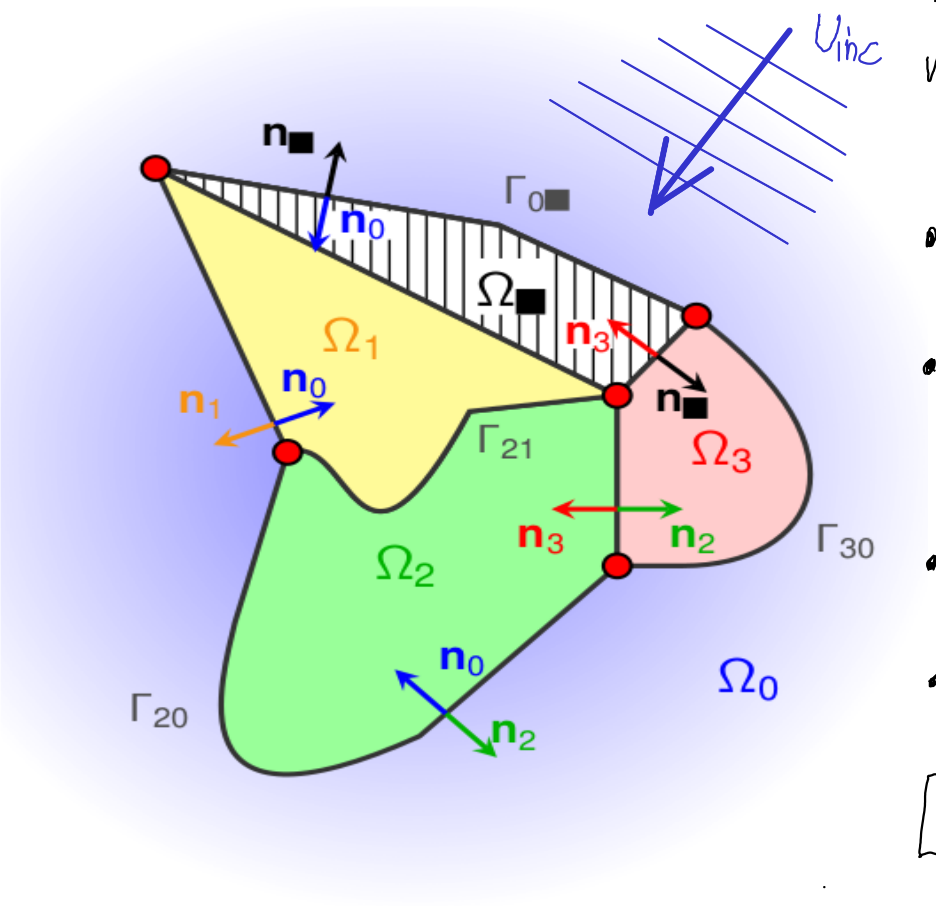

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- $\Omega_j \hat{=}$ curvilinear Lipschitz polyhedra
- $n(x)$ piecewise constant
- Dirichlet/Neumann b.c. on $\partial\Omega_\blacksquare$

[• Excitation by incident wave]

I.1.1. Frequency-Domain Acoustic Scattering

Sommerfeld Radiation conditions "at ∞"

(if Ω unbounded)

$$\lim_{r \to \infty} \int_{|x|=r} \left| \nabla u \cdot \frac{x}{\|x\|} - iku \right|^2 dS \to 0$$

(we assume $n(x) = 1$ for large $\|x\|$)

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Sommerfeld Radiation conditions "at ∞" (if Ω unbounded)

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(we assume $n(x) = 1$ for large $\|x\|$)

Thm: Ω unbounded or Ω bounded and $k \notin$ discrete set of resonant frequencies

$\triangleright$ $-\Delta u - k^2 u = 0$ in Ω
$u = g$ or $\nabla u \cdot \underline{n} = h$ on $\partial\Omega$
has unique weak solution $u \in H^1(\Delta, \Omega)$

I. 1.2. Frequency-Domain Electromagnetic Scattering

Complex amplitude of electric field (magnetic field / vector potential)

$$\underline{u} : \Omega \longrightarrow \mathbb{C}^3$$

solves frequency-domain Maxwell's equations

$$\nabla \times (\nabla \times \underline{u}) - k^2 \underline{\underline{n}}(x) \underline{u} = f \quad \text{in } \Omega \subset \mathbb{R}^3$$

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source

wave number $k > 0$

refractive index $\underline{n} : \Omega \to \mathbb{R}^{3,3}$
(uniformly positive definite)

I. 1.2. Frequency-Domain Electromagnetic Scattering

Boundary conditions on Γ:

PEC : $\underline{u} \times \underline{n} = g$ (→ "Dirichlet problem")

PMC : $(\nabla \times \underline{u}) \times \underline{n} = \underline{h}$ (→ "Neumann problem")

I. 1.2. Frequency-Domain Electromagnetic Scattering

Boundary conditions on Γ :

PEC : $\underline{u} \times \underline{n} = \underline{g}$ (→ "Dirichlet problem")

PMC : $(\nabla \times \underline{u}) \times \underline{n} = \underline{h}$ (→ "Neumann problem")

Silver-Müller radiation conditions (at ∞)
(if Ω unbounded)

$$\lim_{r\to\infty} \int_{|x|=r} \left\| (\nabla \times \underline{u}) \times \frac{x}{\|x\|} - ik\underline{u} \right\|^2 dS = 0$$

(assuming $\underline{\underline{n}}(x) \equiv \underline{\underline{I}}$ for large $\|x\|$)

I. 1.2. Frequency-Domain Electromagnetic Scattering

- PEC / PMC boundary conditions on $\partial\Omega$
- Silver-Müller boundary conditions at ∞
- $\nabla \times (\nabla \times \underline{u}) - k^2 \underline{u} = 0$ in Ω

Thm: Ω unbounded or Ω bounded and $k \notin$ discrete set of resonant frequencies

$\Rightarrow$ Existence and uniqueness of (weak) solutions $\in H_{loc}(curl^2; \Omega)$

I. 1.2. Frequency-Domain Electromagnetic Scattering

?

Connection : acoustic scattering ↔ electromagnetic scattering

$$-\nabla\cdot\nabla u - k^2 n(x) u = 0 \quad \longleftrightarrow \quad \nabla\times(\nabla\times\underline{u}) - k^2 n(x)\underline{u} = 0$$

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Connection: acoustic scattering ↔ electromagnetic scattering ?

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Framework (differential geometry) exterior calculus (calculus of differential forms)

$\Lambda^\ell(\Omega) \hat{=}$ ℓ-forms on Ω, $0 \leq \ell \leq d$

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Framework (differential geometry) exterior calculus (calculus of differential forms)

$\Lambda^l(\Omega) \mathrel{\hat{=}}$ l-forms on Ω , $0 \leq l \leq d$

Euclidean vector proxy model for $\Omega \subset \mathbb{R}^3$:

$l = 0, 3$: $\omega \in \Lambda^l(\Omega) \longleftrightarrow$ scalar function $u : \Omega \to \mathbb{C}$

$l = 1, 2$: $\omega \in \Lambda^l(\Omega) \longleftrightarrow$ vector field $\underline{u} : \Omega \to \mathbb{C}^3$

I. 1.2. Frequency-Domain Electromagnetic Scattering

Connection : acoustic scattering ⟷ electromagnetic scattering ?

$$-\nabla\cdot\nabla \underset{\uparrow}{u} - k^2 n(x)\underset{\uparrow}{u} = 0 \quad\longleftrightarrow\quad \nabla\times(\nabla\times \underset{\uparrow}{\underline{u}}) - k^2 n(x)\underset{\uparrow}{\underline{u}} = 0$$

vector proxies of differential forms

What about the differential operators ?

I. 1.2. Frequency-Domain Electromagnetic Scattering

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$$-\nabla\cdot\nabla u - k^2 n(x) u = 0 \quad\longleftrightarrow\quad \nabla\times(\nabla\times\underline{u}) - k^2 n(x)\underline{u} = 0$$

Linear 2nd-order BVP for l-form $\omega \in \Lambda^l(\Omega)$

$$(-1)^{l+1}\, d * d\omega - k^2 *_n \omega = \varphi \quad \text{in } \Omega$$

exterior derivative   Hodge operators

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Linear 2nd-order BVP for ℓ-form $\omega\in\Lambda^\ell(\Omega)$

$$(-1)^{\ell+1}\, d * d\omega - k^2 *_n \omega = \varphi \quad \text{in } \Omega$$

exterior derivative

Hodge operators

↳ • material tensors

ℓ	d
0	∇
1	$\nabla\times$
2	$\nabla\cdot$

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Linear 2nd-order BVP for ℓ-form $\omega\in\Lambda^\ell(\Omega)$

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exterior derivative ↗ d Hodge operators ↗ $*$, $*_n$

In Euclidean vector proxies in $\mathbb{R}^3$: $\ell=0 \leftrightarrow$ acoustics
$\ell=1 \leftrightarrow$ electromagnetics

$$\left.\begin{array}{ll}\text{Dirichlet b.c.:} & \iota^*\omega = \gamma \\ \text{Neumann b.c.:} & \iota^*(d*\omega) = \eta\end{array}\right\}\ \text{on } \Gamma$$

I.2. Layer Potential Representation

I.2.1. Green's Formulas and Traces

$$\int_\Omega \underline{j} \cdot \nabla v + (\nabla \cdot \underline{j}) v \, dx = \int_\Gamma (\underline{j} \cdot \underline{n}) v \, dS \quad \forall \underline{j}, v \in C^\infty(\overline{\Omega})$$

I.2. Layer Potential Representation

I.2.1. Green's Formulas and Traces

$$\int_\Omega g \cdot \nabla v + (\nabla \cdot g) v \, dx = \int_\Gamma (g \cdot \underline{n}) v \, dS \quad \forall g, v \in C^\infty(\overline{\Omega})$$

extend to $v \in \overset{\downarrow}{H^1}(\Omega)$

$g \in H(div, \Omega)$

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$$\int_{\Omega} \mathbf{j}\cdot\nabla v + (\nabla\cdot\mathbf{j})\,v \,dx = \int_{\Gamma} (\mathbf{j}\cdot\mathbf{n})\,v \,dS \quad \forall\, \mathbf{j}, v \in C^\infty(\overline{\Omega})$$

extend to $v \in H^1(\Omega)$, $\mathbf{j} \in H(\mathrm{div},\Omega)$ ▷ defined on

$$\underbrace{\underset{\overset{\|}{H^{1/2}(\Gamma)}}{H^1(\Omega)/H^1_0(\Omega)} \times \underset{\overset{\|}{H^{-1/2}(\Gamma)}}{H(\mathrm{div},\Omega)/H_0(\mathrm{div},\Omega)}}_{\text{trace spaces}}$$

Trace spaces = quotient spaces

I.2. Layer Potential Representation

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$$\int_\Omega \mathbf{j}\cdot\nabla v + (\nabla\cdot\mathbf{j})\,v \,dx = \int_\Gamma (\mathbf{j}\cdot\underline{\mathbf{n}})\,v \,dS \quad \forall\, \mathbf{j}, v \in C^\infty(\overline{\Omega})$$

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$$\underbrace{\begin{array}{c} H^1(\Omega)/H^1_0(\Omega) \\ \Vert \\ H^{1/2}(\Gamma) \end{array} \times \begin{array}{c} H(\mathrm{div},\Omega)/H_0(\mathrm{div},\Omega) \\ \Vert \\ H^{-1/2}(\Gamma) \end{array}}_{\text{trace spaces}}$$

$\downarrow$

defines "L^2-type" pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$

I.2.1. Green's Formulas and Traces

Thm : $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are dual to each other w.r.t. the L^2-type pairing

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From quotient space to function space :

"Trace theorem" : On $C^\infty(\Omega)_{|\Gamma}$ the norms of $H^1(\Omega)/H^1_0(\Omega)$ and $[L^2(\Gamma), H^1(\Gamma)]_{1/2}$ are equivalent

↳ interpolation space

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$$\int_{\Omega} \underline{j} \cdot \nabla v + (\nabla \cdot \underline{j})\, v \, dx = \int_{\Gamma} (\underline{j} \cdot \underline{n})\, v \, dS \qquad \forall\, \underline{j}, v \in C^{\infty}(\overline{\Omega})$$

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$$\Downarrow$$

$$\int_\Omega \mathbf{j}\cdot\nabla v + (\nabla\cdot\mathbf{j})\, v \, dx = \langle \gamma_n \mathbf{j}, \gamma v \rangle_\Gamma \qquad \forall\, \mathbf{j}, v \in C^\infty(\overline{\Omega})$$

normal component trace $(\gamma_n \mathbf{j})(x) := \mathbf{j}(x)\cdot \mathbf{n}(x)$, point trace: $(\gamma v)(x) = v(x)$, $x \in \Gamma$

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▷ $$\left.\begin{array}{l} \gamma_n : H(\mathrm{div}, \Omega) \longrightarrow H^{-1/2}(\Gamma) \\ \gamma : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma) \end{array}\right\} \text{continuous \& surjective}$$

I.2.1. Green's Formulas and Traces

$$\int_\Omega (\nabla \times \underline{v})\cdot \underline{u} - (\nabla \times \underline{u})\cdot \underline{v}\,dx = \int_\Gamma (\underline{u}\times n)\cdot \underline{v}\,dS \quad \forall \underline{u},\underline{v} \in C^\infty(\overline{\Omega})$$

$$\downarrow$$

extend to $H(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega)$

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$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

extend to $H(\mathrm{curl},\Omega)\times H(\mathrm{curl},\Omega)$, on $H(\mathrm{curl},\Omega)/H_0(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega)/H_0(\mathrm{curl},\Omega)$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

trace spaces : $H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \times H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

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$$\parallel \qquad\qquad\qquad \parallel$$

trace spaces : $H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \times H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

Tangential trace : $(\gamma_t\underline{v})(x) = v(x)\times n(x)$, $x\in\Gamma$

▷ $\displaystyle\int_\Omega (\nabla\times\underline{v})\cdot\underline{u} - (\nabla\times\underline{u})\cdot\underline{v}\,dx = \langle \gamma_t\underline{u}, \gamma_t\underline{v}\rangle_{t,\Gamma}$ ← bilinear pairing

I.2.1. Green's Formulas and Traces

$$\int_{\Omega} (\nabla \times \underline{v})\cdot\underline{u} - (\nabla\times\underline{u})\cdot\underline{v}\,dx = \int_{\Gamma} (\underline{u}\times n)\cdot\underline{v}\,dS \quad \forall \underline{u},\underline{v}\in C^{\infty}(\overline{\Omega})$$

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▷ $$\langle \gamma_t \underline{u}, \gamma_t \underline{v}\rangle_{t,\Gamma} = \int_{\Gamma} (\underline{u}\times n)\cdot\underline{v}\,dS = \int_{\Gamma} \gamma_t\underline{u}\cdot(n\times\gamma_t\underline{v})\,dS$$

for $\gamma_t\underline{u}, \gamma_t\underline{v} \in L^2_t(\Gamma)$

I.2.1. Green's Formulas and Traces

$$\int_{\Omega} (\nabla \times \underline{v})\cdot \underline{u} - (\nabla \times \underline{u})\cdot \underline{v}\, dx = \langle \gamma_t \underline{u}, \gamma_t \underline{v} \rangle_{t,\Gamma}$$

Thm : $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is self-dual w.r.t. the anti-symmetric bilinear pairing $\langle \cdot , \cdot \rangle_{t,\Gamma}$

cf. Thm : $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are dual to each other w.r.t. the L^2-type pairing

I.2.1. Green's Formulas and Traces

Exterior calculus background:

$$\omega \in H(d, \Lambda^{\ell}(\Omega)), \quad \eta \in H(d, \Lambda^{d-\ell-1}(\Omega))$$

$$\int_{\Omega} d\omega \wedge \eta + (-1)^{\ell}(\omega \wedge d\eta) = \int_{\Gamma} \imath^{*}\omega \wedge \imath^{*}\eta$$

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$\ell = 0$: $$\int_{\Omega} j \cdot \nabla v + (\nabla \cdot j)\, v \, dx = \int_{\Gamma} (j \cdot \underline{n})\, v \; dS$$

$\ell = 1$: $$\int_{\Omega} (\nabla \times \underline{v}) \cdot \underline{u} - (\nabla \times \underline{u}) \cdot \underline{v} \, dx = \int_{\Gamma} (\underline{u} \times n) \cdot \underline{v} \, dS$$

I.2.1. Green's Formulas and Traces

Exterior calculus background:

$\omega \in H(d, \Lambda^{\ell}(\Omega))$, $\eta \in H(d, \Lambda^{d-\ell-1}(\Omega))$

$$\int_{\Omega} d\omega \wedge \eta + (-1)^{\ell}(\omega \wedge d\eta) = \underbrace{\int_{\Gamma} \imath^{*}\omega \wedge \imath^{*}\eta}_{\text{pairing of traces}}$$

$$\omega \wedge \eta = (-1)^{\ell(d-\ell-1)}(\eta \wedge \omega) \Rightarrow$$

- symmetric for $\ell = 0$
- antisymmetric for $\ell = 1, d = 3$

I.2.1. Green's Formulas and Traces

De Rham co-homology & traces

$$H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} H(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega)$$

I.2.1. Green's Formulas and Traces

De Rham co-homology & traces

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla"] \arrow[d, "\gamma"'] & H(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\gamma_t"'] & H(\mathrm{div},\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, "\gamma_n"'] & L^2(\Omega) \\
H^{1/2}(\Gamma) & H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) & H^{-1/2}(\Gamma) &
\end{tikzcd}$$

← continuous & surjective traces

I.2.1. Green's Formulas and Traces

De Rham co-homology & traces ⇒ surface differential operators

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla"] \arrow[d, "\gamma"'] & H(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\gamma_t"'] & H(\mathrm{div},\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, "\gamma_n"'] & L^2(\Omega) \\
H^{1/2}(\Gamma) \arrow[r, "\nabla_\Gamma\times"] & H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \arrow[r, "\nabla_\Gamma\cdot"] & H^{-1/2}(\Gamma) &
\end{tikzcd}$$

← continuous & surjective traces

↑ surface curl

↑ surface divergence

Diagram commutes ⇒ $\nabla_\Gamma\cdot(\nabla_\Gamma\times) = 0$

I.2.1. Green's Formulas and Traces

An equivalent norm on $H^{-1/2}(\mathrm{div}_T, \Gamma) := H(\mathrm{curl}, \Omega) / H_0(\mathrm{curl}, \Omega)$

$$\boxed{\|\underline{u}\|^2_{H^{-1/2}(\mathrm{div}_T, \Gamma)} \approx \|\underline{u}\|^2_{\underline{H}^{-1/2}(\Gamma)} + \|\nabla_T \cdot \underline{u}\|^2_{H^{-1/2}(\Gamma)}}$$

$\uparrow$ L^2-dual of $H_t^{1/2}(\Gamma) := \gamma_t \underline{H}^1(\Omega)$

I.2.1. Green's Formulas and Traces

An equivalent norm on $H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := H(\mathrm{curl},\Omega) / H_0(\mathrm{curl},\Omega)$

$$\boxed{\|\underline{u}\|^2_{H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \approx \|\underline{u}\|^2_{\underline{H}^{-1/2}(\Gamma)} + \|\nabla_\Gamma \cdot \underline{u}\|^2_{H^{-1/2}(\Gamma)}}$$

$\uparrow$

L^2-dual of $H_t^{1/2}(\Gamma) := \gamma_t \underline{H}^1(\Omega)$

Tool for proof: stable regular decomposition

$$H(\mathrm{curl},\Omega) = \underline{H}^1(\Omega) + \nabla H^1(\Omega)$$

I.2.2. Fundamental Solutions

$\mathcal{L} : (C^\infty(\mathbb{R}^d))^m \longrightarrow (C^\infty(\mathbb{R}^d))^m \;\widehat{=}$ linear partial differential operator

Def: $\underline{\underline{G}} : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^{m \times m}$ is a **fundamental solution** for $\mathcal{L}$, if

$$\mathcal{L}_y^* \underline{\underline{G}}(x,y) = \delta_x(y)\cdot \underline{I}, \quad \mathcal{L}_x \underline{\underline{G}}(x,y) = \delta_x(y)\cdot \underline{I} \text{ in } \mathcal{D}(\mathbb{R}^d)'$$

and satisfies radiation/decay conditions.

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and satisfies radiation/decay conditions.

- $\mathcal{L}_{x/y} G(x,y) = 0$ on $\mathbb{R}^d \setminus \{x = y\}$

I.2.2. Fundamental Solutions

$\mathcal{L} : (C^\infty(\mathbb{R}^d))^m \longrightarrow (C^\infty(\mathbb{R}^d))^m \mathrel{\hat{=}}$ linear partial differential operator

Def: $\underline{\underline{G}} : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^{m\times m}$ is a fundamental solution for $\mathcal{L}$, if

$$\mathcal{L}_y^* \underline{\underline{G}}(x,y) = \delta_x(y)\cdot\underline{I}\,,\quad \mathcal{L}_x \underline{\underline{G}}(x,y) = \delta_x(y)\cdot\underline{I} \quad \text{in } \mathcal{D}(\mathbb{R}^d)'$$

and satisfies radiation/decay conditions.

- $\mathcal{L}_{x/y} G(x,y) = 0$ on $\mathbb{R}^d \setminus \{x = y\}$
- $\mathcal{L}$ translation invariant $\Rightarrow$ $\underline{\underline{G}}(x,y) = \underline{\underline{G}}(x-y)$

I.2.2. Fundamental Solutions

$\mathcal{L} : (C^\infty(\mathbb{R}^d))^m \longrightarrow (C^\infty(\mathbb{R}^d))^m \hat{=}$ linear partial differential operator

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- $\mathcal{L}$ translation & rotation invariant $\Rightarrow$ $\underline{G}(x,y) = G(\|x-y\|)$

I.2.2. Fundamental Solutions

Example (m = 1) : $\mathcal{L}u = -\nabla\cdot(\underline{\underline{A}}\nabla u) + 2\underline{b}\cdot\nabla u + cu$

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$$G(\underline{z}) = \begin{cases} -\dfrac{e^{\underline{b}^T\underline{\underline{A}}\,\underline{z}}}{4\cdot\pi\sqrt{\det\underline{\underline{A}}}}\log \underline{z}^T\underline{\underline{A}}\,\underline{z} & , d=2,\ \lambda=0 \\ \dfrac{e^{\underline{b}^T\underline{\underline{A}}\,\underline{z}}}{4\sqrt{\det\underline{\underline{A}}}}\, iH_0^{(1)}\left(i\lambda\sqrt{\underline{z}^T\underline{\underline{A}}\,\underline{z}}\right) & , d=2,\ \lambda\neq 0 \\ \dfrac{1}{4\pi\sqrt{\det\underline{\underline{A}}}}\cdot\dfrac{e^{\underline{b}^T\underline{\underline{A}}-\lambda\sqrt{\underline{z}^T\underline{\underline{A}}\,\underline{z}}}}{\sqrt{\underline{z}^T\underline{\underline{A}}\,\underline{z}}} & , d=3 \end{cases}$$

$$[\ \vartheta := c + \underline{b}^T\underline{\underline{A}}\,\underline{b}\ ,\quad \lambda = \begin{cases} \sqrt{\vartheta} & , \text{if } \vartheta \geq 0 \\ i\sqrt{|\vartheta|} & , \text{if } \vartheta < 0 \end{cases}$$

I.2.2. Fundamental Solutions

$$\mathcal{L}u = -\Delta u - k^2 u \qquad \triangleright \qquad G(\underline{z}) = \frac{e^{ik\underline{\underline{z}}}}{\|\underline{z}\|}$$

$(m = 1, d = 3)$

$\uparrow$

wave generated by a point source

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↑ wave generated by a point source

- $G : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}$ analytic

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$\uparrow$ wave generated by a point source

- $G : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}$ analytic
- $G(\underline{z}) = O(\|\underline{z}\|^{-1})$ for $\|\underline{z}\| \longrightarrow 0$

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$$\mathcal{L}u = -\Delta u - k^2 u \quad \triangleright \quad G(\underline{z}) = \frac{e^{ik\underline{z}}}{\|\underline{z}\|}$$

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$\uparrow$

wave generated by a point source

- $G : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}$ analytic
- $G(z) = O(\|z\|^{-1})$ for $\|z\| \to 0$
- $G \in L^1(\mathbb{R}^d)$

I.2.2. Fundamental Solutions

Volume potential representation:

[$m = 1$, $\mathcal{L} \overset{1}{=}$ translation invariant 2nd-order PD operator]

Formal: $\mathcal{L}_x^{*} G(x,y) = \delta_x(y)$

$$\Leftrightarrow \quad \int_{\mathbb{R}^d} G(x,y)\, \mathcal{L} u(y)\, dy = u(x) \quad \forall u \in \mathcal{D}(\mathbb{R}^d)$$

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Volume potential representation:

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$$\Rightarrow \quad \mathcal{G} \circ \mathcal{L} = \mathrm{Id} \quad \text{on} \quad \mathcal{D}(\mathbb{R}^d)$$

[$(\mathcal{G}f)(x) := \int_{\mathbb{R}^d} G(x,y)f(y)\,dy$, $\mathcal{G} : \mathcal{D}(\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{R}^d)$]

I.2.2. Fundamental Solutions

$$\mathcal{L}_x G(x,y) = \delta_x(y) \cdot I$$

$$f(x) = \int_{\mathbb{R}^d} \delta_x(y) f(y)\,dy = \int_{\mathbb{R}^d} \mathcal{L}_x G(x,y) f(y)\,dy = \mathcal{L}_x \int_{\mathbb{R}^d} G(x,y) f(y)\,dy$$

I.2.2. Fundamental Solutions

$$\mathcal{L}_x G(x,y) = \delta_x(y)\cdot I$$

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▷ $\mathcal{L}\circ\mathcal{G} = \mathrm{Id}$ on $\mathcal{D}(\Omega)$

$\left[\ (\mathcal{G}f)(x) := \int_{\mathbb{R}^d} G(x,y) f(y)\,dy\ ,\ \mathcal{G} : \mathcal{D}(\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{R}^d)\ \right]$

I.2.2. Fundamental Solutions

Volume potential : $(\mathcal{G}f)(x) := \int_{\mathbb{R}^d} G(x,y)f(y)\,dy\,,\ x \in \mathbb{R}^d$

[$m = 1$, $\mathcal{L} \hat{=}$ 2nd-order PD-Op.]

I.2.2. Fundamental Solutions

Volume potential : $(\mathcal{G} f)(x) := \int_{\mathbb{R}^d} G(x,y) f(y) \, dy$

[m = 1, $\mathcal{L} \;\hat{=}\;$ 2nd-order PD-Op.]

- $\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathrm{Id}$ on $\mathcal{D}(\mathbb{R}^d)$

I.2.2. Fundamental Solutions

Volume potential: $(\mathcal{G}f)(x) := \int_{\mathbb{R}^d} G(x,y) f(y)\,dy$

[m = 1, $\mathcal{L} \hat{=}$ 2nd-order PD-Op.]

- $\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathrm{Id}$ on $\mathcal{D}(\mathbb{R}^d)$
- $\mathcal{G} : \mathcal{D}(\Omega) \longrightarrow C^\infty(\Omega)$ can be extended to a continuous mapping

$$\mathcal{G} : \widetilde{H}^s(\mathbb{R}^d) \longrightarrow H^{s+2}(\mathbb{R}^d),\ s \in \mathbb{R}$$

I.2.2. Fundamental Solutions

Volume potential: $(\mathcal{G}f)(x) := \int_{\mathbb{R}^d} G(x,y) f(y)\, dy$

[$m = 1$, $\mathcal{L} \hat{=}$ 2nd-order PD-Op.]

- $$\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathrm{Id} \quad \text{on} \quad \mathcal{D}(\mathbb{R}^d)$$
- $\mathcal{G} : \mathcal{D}(\Omega) \longrightarrow C^\infty(\Omega)$ can be extended to a continuous mapping
$$\mathcal{G} : \tilde{H}^s(\mathbb{R}^d) \longrightarrow H^{s+2}_{loc}(\mathbb{R}^d), \; s \in \mathbb{R}$$
- $\mathcal{L}$ translation-invariant $\Rightarrow$ $\mathcal{G} \hat{=}$ convolution operator.

I.2. Layer Potential Representation

I.2.3. Layer Representation Formula: Scalar Case

Green's 2nd formula: [Neumann trace: $\gamma_N u := \nabla u \cdot n_{|\Gamma}$]

$$\int_\Omega u \Delta v - v \Delta u \, dx = \int_\Gamma \gamma u \cdot \gamma_N v - \gamma_N u \cdot \gamma v \, dS$$

$$\forall u, v \in H^1(\Delta, \Omega)$$

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Green's 2nd formula: [Neumann trace: $\gamma_N u := \nabla u \cdot n_{|\Gamma}$]

$$\int_\Omega u \Delta v - v \Delta u \, dx = \int_\Gamma \gamma u \cdot \gamma_N v - \gamma_N u \cdot \gamma v \, dS$$

$$\forall u, v \in H^1(\Delta, \Omega)$$

If $-\Delta u = f \in L^2(\mathbb{R}^d)$ in $\mathbb{R}^d \setminus \Gamma$, $v \in \mathcal{D}(\mathbb{R}^3)$

$$-\langle u, \Delta v \rangle_\Omega = \langle f, v \rangle_\Omega + \langle \gamma_N v, [\gamma u]_\Gamma \rangle_\Gamma - \langle [\gamma_N u]_\Gamma, \gamma v \rangle_\Gamma$$

↳ jump $\gamma' u - \gamma u$

I.2.3. Layer Representation Formula: Scalar Case

Now: $\mathcal{L} = -\Delta - k^2$

$$\langle u, \mathcal{L} v \rangle = \langle f, v \rangle + \langle \gamma_N v, [\gamma u]_\Gamma \rangle_\Gamma - \langle [\gamma_N u], \gamma v \rangle_\Gamma$$

I.2.3. Layer Representation Formula: Scalar Case

Now: $\mathcal{L} = -\Delta - k^2$

$$\langle u, \mathcal{L} v \rangle = \langle f, v \rangle + \langle \gamma_N v, [\gamma u]_\Gamma \rangle_\Gamma - \langle [\gamma_N u], \gamma v \rangle_\Gamma$$

[$\mathcal{L} u = f \in L^2(\mathbb{R}^d)$ in $\mathbb{R}^d \setminus \Gamma$, u satisfies radiation conditions
↳ compactly supported $u \in C^0(\mathbb{R}^d \setminus \Gamma)$]

I.2.3. Layer Representation Formula: Scalar Case

Now: $\mathcal{L} = -\Delta - k^2$,

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[$\mathcal{L}u = f \in L^2(\mathbb{R}^d)$ in $\mathbb{R}^d \setminus \Gamma$, u satisfies radiation conditions
↳ compactly supported, $u \in C^0(\mathbb{R}^d \setminus \Gamma)$]

Formal: $v \longrightarrow G_k(x,\cdot)$, $x \notin \Gamma$

$$u(x) = (\mathcal{G}_k f)(x) + \langle \gamma_N G_k(x,\cdot), [\gamma u]_\Gamma \rangle_\Gamma - \langle [\gamma_N u], \gamma G_k(x,\cdot) \rangle_\Gamma$$

I.2.3. Layer Representation Formula: Scalar Case

Now: $\mathcal{L} = -\Delta - k^2$,

$$\langle u, \mathcal{L}v \rangle = \langle f, v \rangle + \langle \gamma_N v, [\gamma u]_\Gamma \rangle_\Gamma - \langle [\gamma_N u], \gamma v \rangle_\Gamma$$

[$\mathcal{L}u = f \in L^2(\mathbb{R}^d)$ in $\mathbb{R}^d \setminus \Gamma$, u satisfies radiation conditions
↳ compactly supported $\quad u \in C^0(\mathbb{R}^d \setminus \Gamma)$]

Formal: $v \longrightarrow G_k(x, \cdot)$, $x \notin \Gamma$

$$\underset{\substack{\downarrow \\ = \langle u, \delta_x \rangle}}{u(x)} = \underset{\substack{\downarrow \\ \text{volume potential}}}{(\mathcal{G}_k f)(x)} + \underset{\substack{\downarrow \\ \text{double layer potential}}}{\langle \gamma_N G_k(x, \cdot), [\gamma u]_\Gamma \rangle_\Gamma} - \underset{\substack{\downarrow \\ \text{single layer potential}}}{\langle [\gamma_N u], \gamma G_k(x, \cdot) \rangle_\Gamma}$$

I.2.3. Layer Representation Formula: Scalar Case

single layer potential : $(\Psi_{SL}^{\kappa}\varphi)(x) := \langle \varphi , \gamma G_{\kappa}(x,\cdot)\rangle_{\Gamma}, \; x \notin \Gamma$

I.2.3. Layer Representation Formula: Scalar Case

single layer potential : $(\Psi_{SL}^{\kappa}\varphi)(x) := \langle \varphi , \gamma G_{\kappa}(x,\cdot) \rangle_{\Gamma} , x \notin \Gamma$

$$\left[\gamma_N : H^1(\Delta,\Omega) \to H^{-1/2}(\Gamma) \Rightarrow \varphi \in H^{-1/2}(\Gamma) \right]$$

I.2.3. Layer Representation Formula: Scalar Case

single layer potential : $(\Psi^{\kappa}_{SL}\varphi)(x) := \langle \varphi , \gamma G_{\kappa}(x,\cdot) \rangle_{\Gamma} \, , \; x \notin \Gamma$

$$\left[\gamma_N : H^1(\Delta,\Omega) \longrightarrow H^{-1/2}(\Gamma) \quad \Rightarrow \quad \varphi \in H^{-1/2}(\Gamma) \right]$$

$\varphi \in L^1(\Gamma)$: $(\Psi^{\kappa}_{SL}\varphi)(x) = \int_{\Gamma} G_{\kappa}(x-y)\,\varphi(y)\,dS(y) \, , \; x \notin \Gamma$

I.2.3. Layer Representation Formula : Scalar Case

single layer potential : $(\Psi_{SL}^{\kappa}\varphi)(x) := \langle \varphi, \gamma G_{\kappa}(x,\cdot) \rangle_{\Gamma}, x \notin \Gamma$

$[\gamma_N : H^1(\Delta, \Omega) \to H^{-1/2}(\Gamma) \Rightarrow \varphi \in H^{-1/2}(\Gamma)]$

$\varphi \in L^1(\Gamma) : (\Psi_{SL}^{\kappa}\varphi)(x) = \int_{\Gamma} G_{\kappa}(x-y)\,\varphi(y)\,dS(y), x \notin \Gamma$

double layer potential : $(\Psi_{DL}^{\kappa} v)(x) := \langle \gamma_N G_{\kappa}(x,\cdot), v \rangle_{\Gamma}, x \notin \Gamma$

$[\gamma_N G(x,\cdot) \in H^{-1/2}(\Gamma) \Rightarrow v \in H^{1/2}(\Gamma)].$

$v \in L^1(\Gamma) : (\Psi_{DL}^{\kappa} v)(x) = -\int_{\Gamma} (\nabla G_{\kappa})(x-y) \cdot n(y)\, v(y)$

I.2.3. Layer Representation Formula: Scalar Case

$$u(x) = (\Psi_\kappa f)(x) + \langle \gamma_N G_\kappa(x,\cdot), [\gamma u]_\Gamma \rangle_\Gamma - \langle [\gamma_N u], \gamma G_\kappa(x,\cdot) \rangle_\Gamma$$

single layer potential : $(\Psi^\kappa_{SL} \varphi)(x) := \langle \varphi, \gamma G_\kappa(x,\cdot) \rangle_\Gamma , \; x \notin \Gamma$

double layer potential : $(\Psi^\kappa_{DL} v)(x) := \langle \gamma_N G_\kappa(x,\cdot), v \rangle_\Gamma , \; x \notin \Gamma$

I.2.3. Layer Representation Formula: Scalar Case

$$u(x) = (\mathcal{G}_\kappa f)(x) + \langle \gamma_n G_\kappa(x,\cdot), [\gamma u]_\Gamma \rangle_\Gamma - \langle [\gamma_n u], \gamma G_\kappa(x,\cdot) \rangle_\Gamma$$

single layer potential : $(\Psi_{SL}^{\kappa}\varphi)(x) := \langle \varphi, \gamma G_\kappa(x,\cdot) \rangle_\Gamma , x \notin \Gamma$

double layer potential : $(\Psi_{DL}^{\kappa} v)(x) := \langle \gamma_n G_\kappa(x,\cdot), v \rangle_\Gamma , x \notin \Gamma$

$$\triangleright \quad u = \mathcal{G}_\kappa(\Delta_{\mathbb{R}^d\setminus\Gamma} u) - \Psi_{SL}^{\kappa}([\gamma_n u]_\Gamma) + \Psi_{DL}^{\kappa}([\gamma u]_\Gamma)$$

[$u \in H^1_{loc}(\Delta, \mathbb{R}^d \setminus \Gamma)$, $\Delta_{\mathbb{R}^d\setminus\Gamma} u$ compactly supported]

I.2.4. Layer Representation Formula: Maxwell

- $\underline{u} \in H_{loc}(\mathrm{curl}^2, \mathbb{R}^3 \setminus \Gamma)$ + Silver-Müller radiation conditions
- $\nabla \times (\nabla \times \underline{u}) - k^2 \underline{u} = 0$ in $\mathbb{R}^3 \setminus \Gamma$ $\Rightarrow$ $\nabla \cdot \underline{u} = 0$ in $\mathbb{R}^3 \setminus \Gamma$

I.2.4. Layer Representation Formula: Maxwell

- $\underline{u} \in H_{loc}(\text{curl}^2, \mathbb{R}^3 \setminus \Gamma)$ + Silver-Müller radiation conditions
- $\nabla \times (\nabla \times \underline{u}) - k^2 \underline{u} = 0$ in $\mathbb{R}^3 \setminus \Gamma \Rightarrow \nabla \cdot \underline{u} = 0$ in $\mathbb{R}^3 \setminus \Gamma$

Green's formula:

$$\int_\Omega \underline{u} \cdot \nabla \times (\nabla \times \underline{v}) - \nabla \times (\nabla \times u) \cdot \underline{v} \, dx = \int_\Gamma ((\nabla \times u) \times n) \cdot \underline{v} - u \cdot ((\nabla \times v) \times n) \, dS$$

I.2.4. Layer Representation Formula: Maxwell

- $\underline{U} \in H_{loc}(\text{curl}^2, \mathbb{R}^3 \setminus \Gamma)$ + Silver-Müller radiation conditions
- $\nabla\times(\nabla\times\underline{U}) - k^2\underline{U} = 0$ in $\mathbb{R}^3\setminus\Gamma$ $\Rightarrow$ $\nabla\cdot\underline{U} = 0$ in $\mathbb{R}^3\setminus\Gamma$

Green's formula:

$$\int_\Omega \underline{U}\cdot\nabla\times(\nabla\times\underline{V}) - \nabla\times(\nabla\times U)\cdot\underline{V}\,dx = \int_\Gamma ((\nabla\times U)\times n)\cdot\underline{V} - U\cdot((\nabla\times V)\times n)\,dS$$

$$\Downarrow$$

$$\int_\Omega \underline{U}\cdot\nabla\times(\nabla\times\underline{V}) - \nabla\times(\nabla\times U)\cdot\underline{V}\,dx = k\langle \gamma_t U, \gamma_n V\rangle_{t,\Gamma} - k\langle \gamma_t V, \gamma_n U\rangle_{t,\Gamma}$$

[Magnetic trace: $\gamma_n\underline{V} = \frac{1}{k}(\nabla\times\underline{V})\times n_{|\Gamma}$]

I.2.4. Layer Representation Formula: Maxwell

$$\int_{\mathbb{R}^3} \underline{u} \left(-\underline{\Delta} - k^2 \right) \underline{\Phi} \, dy = \int_{\mathbb{R}^3} \underline{u} \left\{ \nabla \times \left(\nabla \times \underline{\Phi} \right) - \nabla \left(\nabla \cdot \Phi \right) - k^2 \Phi \right\} dy$$

I.2.4. Layer Representation Formula: Maxwell

$$\int_{\mathbb{R}^3} \underline{U}\,(-\underline{\Delta}-k^2)\,\underline{\Phi}\,dy = \int_{\mathbb{R}^3} \underline{U}\,\{\nabla\times(\nabla\times\underline{\Phi}) - \nabla(\nabla\cdot\Phi) - k^2\Phi\}\,dy$$

normal component trace

$$= -k\langle [\gamma_t \underline{U}]_\Gamma \, \gamma_N \underline{\Phi} \rangle_{t,\Gamma} + k\langle \gamma_t \Phi , [\gamma_N \underline{U}]_\Gamma \rangle_{t,\Gamma} + \langle [\gamma_n \underline{U}]_\Gamma , \nabla\cdot\underline{\Phi} \rangle_\Gamma$$

$$\forall\, \underline{\Phi} \in H^1_{loc}(\Delta, \mathbb{R}^3)$$

I.2.4. Layer Representation Formula: Maxwell

$$\int_{\mathbb{R}^3} \underline{U}\,(-\underline{\Delta}-k^2)\,\underline{\Phi}\,dy = \int_{\mathbb{R}^3} \underline{U}\,\{\nabla\times(\nabla\times\underline{\Phi}) - \nabla(\nabla\cdot\Phi) - k^2\Phi\}\,dy$$

normal component trace

$$= -k\langle [\gamma_t U]_\Gamma, \gamma_N \Phi\rangle_{t,\Gamma} + k\langle \gamma_t \Phi, [\gamma_N U]_\Gamma\rangle_{t,\Gamma} + \langle [\gamma_n \underline{U}]_\Gamma, \nabla\cdot\Phi\rangle_\Gamma$$

$$\forall\, \Phi \in H^1_{loc}(\Delta, \mathbb{R}^3)$$

Replace: $\Phi \longrightarrow G_k(x,\cdot)\vec{e}_j\,,\quad x \notin \Gamma$

I.2.4. Layer Representation Formula: Maxwell

$$\int_{\mathbb{R}^3} \underline{U}(-\underline{\Delta}-k^2)\underline{\Phi}\,dy = \int_{\mathbb{R}^3} \underline{U}\{\nabla\times(\nabla\times\underline{\Phi})-\nabla(\nabla\cdot\Phi)-k^2\Phi\}\,dy$$

normal component trace

$$= -k\langle[\gamma_t U]_\Gamma, \gamma_N\underline{\Phi}\rangle_{t,\Gamma} + k\langle\gamma_t\Phi, [\gamma_N U]_\Gamma\rangle_{t,\Gamma} + \langle[\gamma_n \underline{U}]_\Gamma, \nabla\cdot\underline{\Phi}\rangle_\Gamma$$

$$\forall\, \underline{\Phi}\in H^1_{loc}(\Delta, \mathbb{R}^3)$$

Replace: $\underline{\Phi} \longrightarrow G_k(x,\cdot)\vec{e}_j\,,\quad x\notin\Gamma$

$$\triangleright\ U_j(x) = \int_{\mathbb{R}^3}\underline{U}\cdot\delta_x\vec{e}_j\,dy = \qquad [j=1,2,3]$$

$$= -k\langle[\gamma_t U]_\Gamma, \gamma_N\{G_k(x,\cdot)\vec{e}_j\}\rangle + k\langle\gamma_t G_k(x,\cdot), [\gamma_N U]_\Gamma\rangle_{t,\Gamma}$$

$$+\ \langle[\gamma_n U]_\Gamma, \nabla\cdot\{G_k(x,\cdot)\vec{e}_j\}\rangle_\Gamma\,, \quad x\notin\Gamma$$

I.2.4. Layer Representation Formula: Maxwell

$$U_j(x) = -k\left< [\gamma_t U]_\Gamma , \gamma_M \{G_k(x,\cdot) e_j\} \right> + k \left< \gamma_t G_k(x,\cdot)\vec{e}_j , [\gamma_M U]_\Gamma \right>_{t,\Gamma}$$

$$+ \left< [\gamma_n U]_\Gamma , \nabla\cdot\{G_k(x,\cdot) e_j\} \right>_\Gamma \qquad , x \notin \Gamma$$

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$$U_j(x) = -k\langle [\gamma_t U]_\Gamma , \gamma_n \{G_k(x,\cdot) e_j\} \rangle + k \langle \gamma_t G_k(x,\cdot)\vec{e}_j , [\gamma_n U]_\Gamma \rangle_{t,\Gamma}$$
$$+ \langle [\gamma_n U]_\Gamma , \nabla \cdot \{G_k(x,\cdot) e_j\} \rangle_\Gamma \qquad , x \notin \Gamma$$

+ vector-analytic identities

I.2.4. Layer Representation Formula: Maxwell

$$\mathcal{U}_j(x) = -k\langle [\gamma_t \mathcal{U}]_\Gamma, \gamma_n\{G_k(x,\cdot)e_j\}\rangle + k\langle \gamma_t G_k(x,\cdot)\vec{e}_j, [\gamma_n \mathcal{U}]_\Gamma\rangle_{t,\Gamma}$$

$$+ \langle [\gamma_n \mathcal{U}]_\Gamma, \nabla\cdot\{G_k(x,\cdot)e_j\}\rangle_\Gamma \quad , x \notin \Gamma$$

\+ vector-analytic identities

$$+ \quad \gamma_n \underline{\mathcal{U}} = \frac{1}{k^2}\nabla\times(\nabla\times\underline{\mathcal{U}})\cdot n = \frac{1}{k_2}\nabla_\Gamma\cdot((\nabla\times\underline{\mathcal{U}})\times n) = \frac{1}{k}\nabla_\Gamma\cdot\gamma_n\underline{\mathcal{U}}$$

I.2.4. Layer Representation Formula: Maxwell

$$u_j(x) = -k\langle [\gamma_t u]_\Gamma, \gamma_n\{G_k(x,\cdot)e_j\}\rangle + k\langle \gamma_t G_k(x,\cdot)\vec{e}_j, [\gamma_n u]_\Gamma\rangle_{t,\Gamma}$$
$$+ \langle [\gamma_n u]_\Gamma, \nabla\cdot\{G_k(x,\cdot)e_j\}\rangle_\Gamma \quad , x \notin \Gamma$$

\+ vector-analytic identities

\+ $\gamma_n \underline{u} = \frac{1}{k^2}\nabla\times(\nabla\times\underline{u})\cdot n = \frac{1}{k_2}\nabla_\Gamma\cdot((\nabla\times\underline{u})\times n) = \frac{1}{k}\nabla_\Gamma\cdot\gamma_n\underline{u}$

▷ Stratton-Chu representation formula

$$\underline{u}(x) = -\nabla\times\int_\Gamma G_k(x-y)[\gamma_t \underline{u}]_\Gamma(y)\,dS(y) - k\int_\Gamma G_k(x-y)[\gamma_n \underline{u}]_\Gamma(y)\,dS(y)$$
$$-\frac{1}{k}\nabla\int_\Gamma G_k(x-y)\nabla_\Gamma\cdot[\gamma_n \underline{u}]_\Gamma(y)\,dS(y)$$

I.2.4. Layer Representation Formula: Maxwell

▷ Stratton-Chu jump representation formula

$$\underline{u}(x) = -\nabla_x \times \int_\Gamma G_k(x-y)[\gamma_t \underline{u}]_\Gamma(y)\,dS(y) - k\int_\Gamma G_k(x-y)[\gamma_N \underline{u}]_\Gamma(y)\,dS(y)$$

$$-\frac{1}{k}\nabla\int_\Gamma G_k(x-y)\,\nabla_\Gamma\cdot[\gamma_N \underline{u}]_\Gamma(y)\,dS(y)$$

I.2.4. Layer Representation Formula: Maxwell

▷ Stratton-Chu jump representation formula

$$\underline{u}(x) = -\nabla \times \int_\Gamma G_\kappa(x-y)[\gamma_t \underline{u}]_\Gamma(y)\,dS(y) - \kappa\int_\Gamma G_\kappa(x-y)[\gamma_N \underline{u}]_\Gamma(y)\,dS(y)$$
$$-\frac{1}{\kappa}\nabla\int_\Gamma G_\kappa(x-y)\,\nabla_\Gamma\cdot[\gamma_N \underline{u}]_\Gamma(y)\,dS(y)$$

▷ $\underline{u} = -\Psi_M([\gamma_t \underline{u}]_\Gamma) - \Psi_E([\gamma_N \underline{u}]_\Gamma)$

$$\Updownarrow$$

$$u = -\Psi^\kappa_{SL}([\gamma_N u]_\Gamma) + \Psi^\kappa_{DL}([\gamma u]_\Gamma)$$

I.2.4. Layer Representation Formula: Maxwell

▷ Stratton-Chu jump representation formula

$$\underline{u}(x) = -\nabla\times\int_\Gamma G_k(x-y)[\gamma_t\underline{u}]_\Gamma(y)\,dS(y) - k\int_\Gamma G_k(x-y)[\gamma_N\underline{u}]_\Gamma(y)\,dS(y)$$
$$-\frac{1}{k}\nabla\int_\Gamma G_k(x-y)\,\nabla_\Gamma\cdot[\gamma_N\underline{u}]_\Gamma(y)\,dS(y)$$

▷ $\underline{u} = -\Psi_M([\gamma_t\underline{u}]_\Gamma) - \Psi_E([\gamma_N\underline{u}]_\Gamma)$

Maxw. DLP: $\Psi_M(\underline{v})(x) := \nabla\times\int_\Gamma G_k(x-y)\,\underline{v}(y)\,dS(y)$

I.2.4. Layer Representation Formula: Maxwell

▷ Stratton-Chu jump representation formula

$$\underline{u}(x) = -\nabla\times\int_\Gamma G_k(x-y)[\gamma_t\underline{u}]_\Gamma(y)\,dS(y) - k\int_\Gamma G_k(x-y)[\gamma_N\underline{u}]_\Gamma(y)\,dS(y)$$
$$-\frac{1}{k}\nabla\int_\Gamma G_k(x-y)\nabla_\Gamma\cdot[\gamma_N\underline{u}]_\Gamma(y)\,dS(y)$$

▷ $\underline{u} = -\Psi_M([\gamma_t\underline{u}]_\Gamma) - \Psi_E([\gamma_N\underline{u}]_\Gamma)$

Maxw. DLP : $\Psi_M(\underline{v})(x) := \nabla\times\int_\Gamma G_k(x-y)\,\underline{v}(y)\,dS(y)$

Maxw. SLP [$x\notin\Gamma$] : $\Psi_E(\underline{\eta})(x) := k\int_\Gamma G_k(x-y)\,\underline{\eta}(y)\,dS(y) + \frac{1}{k}\nabla\int_\Gamma G_k(x-y)(\nabla_\Gamma\cdot\underline{\eta})(y)\,dS(y)$

I.3. Properties of Layer Potentials

I.3.1. Continuity

Scalar case: $\Psi_{SL}^{\kappa}(\varphi)(x) = \langle \gamma G_{\kappa}(x,\cdot), \varphi \rangle_{\Gamma} = \int_{\Gamma} G_{\kappa}(x-y)\varphi(y)\,dS(y)$

$\triangleright \; \langle \Psi_{SL}^{\kappa}(\varphi), \Phi \rangle_{\Omega} = \int_{\Omega}\int_{\Gamma} G_{\kappa}(x-y)\,\varphi(y)\,\Phi(x)\,dS(y)\,dx = \langle \gamma \mathcal{G}_{\kappa}\Phi, \varphi \rangle_{\Gamma}$

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$\triangleright$ $\Psi_{SL} = \mathcal{G}_{\kappa} \circ \gamma'$ in $\mathcal{D}(\Omega)'$

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▷ $\Psi_{SL} = \mathcal{G}_{\kappa} \circ \gamma'$ in $\mathcal{D}(\Omega)'$

$[\gamma : H^1_{loc}(\Omega) \longrightarrow H^{1/2}(\Gamma) \quad , \quad \mathcal{G}_{\kappa} : \tilde{H}^{-1}(\Omega) \longrightarrow H^1_{loc}(\mathbb{R}^d)]$

I.3. Properties of Layer Potentials

I.3.1. Continuity

Scalar case: $\Psi_{SL}^{\kappa}(\varphi)(x) = \langle \gamma G_{\kappa}(x,\cdot), \varphi \rangle_{\Gamma} = \int_{\Gamma} G_{\kappa}(x-y)\varphi(y)\,dS(y)$

▷ $\langle \Psi_{SL}^{\kappa}(\varphi), \Phi \rangle_{\Omega} = \int_{\Omega}\int_{\Gamma} G_{\kappa}(x-y)\,\varphi(y)\,\Phi(x)\,dS(y)\,dx = \langle \gamma \mathcal{G}_{\kappa}\Phi, \varphi \rangle_{\Gamma}$

▷ $\Psi_{SL} = \mathcal{G}_{\kappa} \circ \gamma'$ in $\mathcal{D}(\Omega)'$

$[\ \gamma : H^{1}_{loc}(\Omega) \longrightarrow H^{1/2}(\Gamma)\ ,\ \mathcal{G}_{\kappa} : \tilde{H}^{-1}(\Omega) \longrightarrow H^{1}_{loc}(\mathbb{R}^{d})\ ]$

▷ $\Psi_{SL} : H^{-1/2}(\Gamma) \longrightarrow H^{1}_{loc}(\mathbb{R}^{d})$

$[\leftrightarrow$ RF: $u = \Psi_{SL}(\gamma_{n} u) - \Psi_{DL}(\gamma u)$, $\gamma_{n} : H^{1}(\Delta, \Omega) \to H^{-1/2}(\Gamma)\,]$

I.3.1. Continuity

Scalar case: $\Psi^{k}_{DL}(v)(x) = \langle \gamma_N G(x,\cdot), v \rangle_{\Gamma} = \int_{\Gamma} \gamma_{N,y} G(x-y)\, v(y)\, dS(y)$

$\triangleright \ \langle \Psi_{DL}(\varphi), \Phi \rangle_{\Omega} = \int_{\Omega} \int_{\Gamma} \gamma_{N,y} G(x-y)\, v(y)\, dS(y)\, \Phi(x)\, dx = \langle \gamma_N \mathcal{G}\Phi, v \rangle_{\Gamma}$

I.3.1. Continuity

Scalar case: $\Psi_{DL}^{K}(v)(x) = \langle \gamma_N G(x,\cdot), v \rangle_\Gamma = \int_\Gamma \gamma_{N,y} G(x-y)\, v(y)\, dS(y)$

▷ $\langle \Psi_{DL}(\varphi), \Phi \rangle_\Omega = \int_\Omega \int_\Gamma \gamma_{N,y} G(x-y)\, v(y)\, dS(y)\, \phi(x)\, dx = \langle \gamma_N \mathcal{G}\phi, v \rangle_\Gamma$

▷ $\Psi_{DL} = \mathcal{G}_K \circ \gamma_N'$ in $\mathcal{D}(\Omega)'$

I.3.1. Continuity

Scalar case: $\Psi^{k}_{DL}(v)(x) = \langle \gamma_N G_k(x,\cdot), v \rangle_\Gamma = \int_\Gamma \gamma_{N,y} G_k(x-y)\, v(y)\, dS(y)$

▷ $\langle \Psi_{DL}(\varphi), \Phi \rangle_\Omega = \int_\Omega \int_\Gamma \gamma_{N,y} G_k(x-y)\, v(y)\, dS(y)\, \Phi(x)\, dx = \langle \gamma_N \mathcal{G}\Phi, v \rangle_\Gamma$

▷ $\Psi_{DL} = \mathcal{G}_K \circ \gamma_N'$ in $\mathcal{D}(\Omega)'$

[$\gamma_N : H^1_{loc}(\Delta, \mathbb{R}^d|\Gamma) \longrightarrow H^{-1/2}(\Gamma)$, $\mathcal{G}_K : \widetilde{H}^{-1}(\Delta, \mathbb{R}^d|\Gamma) \to H^1_{loc}(\mathbb{R}^d|\Gamma)$]

▷ $\Psi_{DL} : H^{1/2}(\Gamma) \longrightarrow H^1_{loc}(\mathbb{R}^d|\Gamma)$

[$\leftrightarrow$ RF: $u = \Psi_{SL}(\gamma_N u) - \Psi_{DL}(\gamma u)$, $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$]

I.3.1. Continuity

$\mathcal{L}_K = -\Delta - k^2$: $\quad \mathcal{L}_K \Psi_{SL}(\varphi) = 0 \quad \forall \varphi \in H^{-1/2}(\Gamma)$

$\mathcal{L}_K \Psi_{DL}(v) = 0 \quad \forall v \in H^{1/2}(\Gamma)$

I.3.1. Continuity

$\mathcal{L}_k = -\Delta - k^2$: $\quad \mathcal{L}_k \Psi_{SL}(\varphi) = 0 \quad \forall \varphi \in H^{-1/2}(\Gamma)$

$\qquad\qquad\qquad\qquad \mathcal{L}_k \Psi_{DL}(v) = 0 \quad \forall v \in H^{1/2}(\Gamma)$

▷ $\Psi_{SL} : H^{-1/2}(\Gamma) \rightarrow H^1_{loc}(\mathbb{R}^d) \cap H^1_{loc}(\Delta, \mathbb{R}^d | \Gamma)$

$\quad \Psi_{DL} : H^{1/2}(\Gamma) \longrightarrow H^1_{loc}(\Delta, \mathbb{R}^d | \Gamma)$

↳ satisfy radiation conditions

(Both potentials provide radiating Helmholtz solutions)

I.3.1. Continuity

Maxwell case: $\underline{\Psi}_E^\kappa(\underline{\eta})(x) := \int_\Gamma G_\kappa(x-y)\,\underline{\eta}(y)\,dS(y) + \frac{1}{\kappa}\nabla\int_\Gamma G_\kappa(x-y)(\nabla_\Gamma\cdot\underline{\eta})(y)\,dS(y)$

▷ $\underline{\Psi}_E^k(\underline{\eta}) = k\,\underline{\Psi}_V^k(\underline{\eta}) + \frac{1}{k}\nabla\Psi_{SL}^k(\nabla_\Gamma\cdot\underline{\eta})$ $\underline{\eta}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

Vectorial SLP: $\underline{\Psi}_V^\kappa(\underline{\eta})(x) := \int_\Gamma G_\kappa(x-y)\,\underline{\eta}(y)\,dS(y)$

Scalar SLP

I.3.1. Continuity

Maxwell case: $\underline{\Psi}_E^{\kappa}(\underline{\eta})(x) := \int_\Gamma G_\kappa(x-y)\underline{\eta}(y)\,dS(y) + \frac{1}{\kappa}\nabla\int_\Gamma G_\kappa(x-y)(\nabla_\Gamma\cdot\underline{\eta})(y)\,dS(y)$

$\underline{\eta}\in \underline{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

$\triangleright$ $\underline{\Psi}_E^{k}(\underline{\eta}) = k\,\underline{\Psi}_V^{k}(\underline{\eta}) + \frac{1}{k}\nabla\Psi_{SL}^{k}(\nabla_\Gamma\cdot\underline{\eta})$

Vectorial SLP : $\underline{\Psi}_V^{\kappa}(\underline{\eta})(x) := \int_\Gamma G_\kappa(x-y)\underline{\eta}(y)\,dS(y)$ Scalar SLP

$\underline{\Psi}_V^{\kappa} : \underline{H}^{-1/2}(\Gamma) \to \underline{H}^1_{loc}(\mathbb{R}^3)$, $\Psi_{SL}^{\kappa} : H^{-1/2}(\Gamma) \to H^1_{loc}(\mathbb{R}^3)$

I.3.1. Continuity

Maxwell case: $\underline{\Psi}^{\kappa}_{E}(\underline{\eta})(x) := \int_{\Gamma} G_{\kappa}(x-y)\,\underline{\eta}(y)\,dS(y) + \frac{1}{\kappa}\nabla\int_{\Gamma} G_{\kappa}(x-y)(\nabla_{\Gamma}\cdot\underline{\eta})(y)\,dS(y)$

▷ $\underline{\Psi}^{k}_{E}(\underline{\eta}) = k\,\underline{\Psi}^{k}_{V}(\underline{\eta}) + \frac{1}{k}\nabla\Psi^{k}_{SL}(\nabla_{\Gamma}\cdot\underline{\eta})$ $\quad\underline{\eta}\in H^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)$

$\nearrow \qquad\qquad \nwarrow$

Vectorial SLP: $\underline{\Psi}^{\kappa}_{V}(\underline{\eta})(x) := \int_{\Gamma} G_{\kappa}(x-y)\,\underline{\eta}(y)\,dS(y)$ Scalar SLP

$\underline{\Psi}^{\kappa}_{V} : \underline{H}^{-1/2}(\Gamma) \to \underline{H}^{1}_{loc}(\mathbb{R}^3)$, $\Psi^{\kappa}_{SL} : H^{-1/2}(\Gamma) \to H^{1}_{loc}(\mathbb{R}^3)$

- $\underline{\mathcal{L}}_{\kappa}\Psi^{\kappa}_{E}(\eta) := \nabla\times(\nabla\times\Psi^{\kappa}_{E}(\eta)) - \kappa^2\Psi^{\kappa}_{E}(\eta) = 0$ in $\mathbb{R}^3\setminus\Gamma$

▷ $\Psi^{\kappa}_{E} : H^{-1/2}(\mathrm{div}_{\Gamma},\Gamma) \longrightarrow H(\mathrm{curl},\mathbb{R}^3)\cap H(\mathrm{curl}^2,\mathbb{R}^3\setminus\Gamma)\cap H(\mathrm{div}\,0,\mathbb{R}^3)$

I.3.1. Continuity

Maxwell case: $\Psi_M^\kappa(\underline{v})(x) = \nabla\times\int_\Gamma G_\kappa(x-y)\,\underline{v}(y)\,dS(y)$

$$\nabla\times\Psi_M^\kappa(\underline{v}) = \nabla\times\left(\nabla\times\Psi_V^\kappa(\underline{v})\right) = \left(-\Delta+\nabla\nabla\cdot\right)\Psi_V^\kappa(\underline{v})$$

$$= \kappa^2\Psi_V^\kappa(v) + \nabla\Psi_{SL}^\kappa\left(\nabla_\Gamma\cdot\underline{v}\right) \in H_{loc}(\mathrm{curl},\mathbb{R}^3)$$

I.3.1. Continuity

Maxwell case: $\Psi_M^k(\underline{v})(x) = \nabla\times\int_\Gamma G_k(x-y)\,\underline{v}(y)\,dS(y)$

$$\nabla\times\Psi_M^k(\underline{v}) = \nabla\times\left(\nabla\times\Psi_V^k(\underline{v})\right) = (-\Delta+\nabla\nabla\cdot)\,\Psi_V^k(\underline{v})$$

$$= k^2\Psi_V^k(v) + \nabla\,\Psi_{SL}^k(\nabla_\Gamma\cdot\underline{v}) \in H_{loc}(\mathbf{curl},\mathbb{R}^3)$$

$$\left[\ \Psi_V^k : \underline{H}^{-1/2}(\Gamma) \longrightarrow \underline{H}^1_{loc}(\mathbb{R}^3),\quad \Psi_{SL}^k : H^{-1/2}(\Gamma) \longrightarrow H^1_{loc}(\mathbb{R}^3)\right]$$

I.3.1. Continuity

Maxwell case: $\Psi_M^\kappa(\underline{v})(x) = \nabla\times\int_\Gamma G_\kappa(x-y)\,\underline{v}(y)\,dS(y)$

$$\nabla\times\Psi_M^\kappa(\underline{v}) = \nabla\times(\nabla\times\Psi_V^\kappa(\underline{v})) = (-\Delta+\nabla\nabla\cdot)\,\Psi_V^\kappa(\underline{v})$$

$$= \kappa^2\Psi_V^\kappa(v) + \nabla\,\Psi_{SL}^\kappa(\nabla_\Gamma\cdot\underline{v}) \in H_{loc}(\mathbf{curl},\mathbb{R}^3)$$

$$\left[\ \Psi_V^\kappa : \underline{H}^{-1/2}(\Gamma) \longrightarrow \underline{H}^1_{loc}(\mathbb{R}^3)\ ,\quad \Psi_{SL}^\kappa : H^{-1/2}(\Gamma) \longrightarrow H^1_{loc}(\mathbb{R}^3)\right]$$

▷ $\Psi_M^\kappa : H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \longrightarrow H_{loc}(\mathbf{curl}^2,\mathbb{R}^3\setminus\Gamma)\cap H_{loc}(\mathrm{div}0,\mathbb{R}^3)$

Ψ_M^κ & Ψ_E^κ provide radiating Maxwell solutions

I.3.2 Jump Relations

Scalar case: $\mathcal{L}_\kappa u := (-\Delta - \kappa^2) u = 0$ in $\mathbb{R}^d \setminus \Gamma$ & radiating

"Dirichlet jumps":

$$u = -\Psi_{SL}^{\kappa}([\gamma_N u]_\Gamma) + \Psi_{DL}^{\kappa}([\gamma u]_\Gamma)$$

I.3.2 Jump Relations

Scalar case: $\mathcal{L}_k u := (-\Delta - k^2) u = 0$ in $\mathbb{R}^d \setminus \Gamma$ & radiating

"Dirichlet jumps":

$$u = \underbrace{-\Psi^k_{SL}([\gamma_N u]_\Gamma)}_{\in H^1_{loc}(\mathbb{R}^3)} + \Psi^k_{DL}([\gamma u]_\Gamma)$$

$$\Rightarrow \quad [\gamma \Psi^k_{SL}(\varphi)]_\Gamma = 0 \quad \forall \varphi \in H^{-1/2}(\Gamma)$$

I.3.2 Jump Relations

Scalar case: $\mathcal{L}_\kappa u := (-\Delta - \kappa^2) u = 0$ in $\mathbb{R}^d \setminus \Gamma$ & radiating

"Dirichlet jumps":

$$u = \underbrace{-\Psi^{\kappa}_{SL}([\gamma_N u]_\Gamma)}_{\in H^1_{loc}(\mathbb{R}^3)} + \Psi^{\kappa}_{DL}([\gamma u]_\Gamma)$$

$$\Rightarrow [\gamma \Psi^{\kappa}_{SL}(\varphi)]_\Gamma = 0 \quad \forall \varphi \in H^{-1/2}(\Gamma)$$

$$\to [\gamma u]_\Gamma = \Psi^{\kappa}_{DL}([\gamma u]_\Gamma) \Rightarrow [\gamma \Psi^{\kappa}_{DL}(v)]_\Gamma = v \quad \forall v \in H^{1/2}(\Gamma)$$

I.3.2 Jump Relations

Scalar case : "Neumann jumps"

$$\mathcal{L}_\kappa \underbrace{\mathcal{G}_\kappa(f)}_{\in H^1_{loc}(\mathbb{R}^d)} = 0 \Rightarrow \int_{\mathbb{R}^d} \nabla \mathcal{G}_\kappa(f) \cdot \nabla \phi \, dx = \int_{\mathbb{R}^d} f \phi \, dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)$$

$$f \in \widetilde{H}^{-1}(\mathbb{R}^d)$$

I.3.2 Jump Relations

Scalar case : "Neumann jumps"

$$\underbrace{\mathcal{L}_K \mathcal{G}_K(f)}_{\in H^1_{loc}(\mathbb{R}^d)} = 0 \Rightarrow \int_{\mathbb{R}^d} \nabla \mathcal{G}_K(f) \cdot \nabla \phi \, dx = \int_{\mathbb{R}^d} f \phi \, dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)$$

$f \in \tilde{H}^{-1}(\mathbb{R}^d)$

$$\Psi^K_{SL}(\varphi) = \mathcal{G}(\varphi \cdot \delta_\Gamma) \Rightarrow \int_{\mathbb{R}^d} \nabla \Psi^K_{SL}(\varphi) \cdot \nabla \phi + k^2 \Psi^K_{SL}(\varphi) \phi \, dx = \int_\Gamma \varphi \phi \, dS \quad \forall \phi$$

I.3.2 Jump Relations

Scalar case : "Neumann jumps"

$$\mathcal{L}_\kappa \underbrace{\mathcal{G}_\kappa(f)}_{\in H^1_{loc}(\mathbb{R}^d)} = 0 \Rightarrow \int_{\mathbb{R}^d} \nabla \mathcal{G}_\kappa(f) \cdot \nabla \phi \, dx = \int_{\mathbb{R}^d} f \phi \, dx \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d)$$

$$f \in \widetilde{H}^{-1}(\mathbb{R}^d)$$

$$\Psi^\kappa_{SL}(\varphi) = \mathcal{G}(\varphi \cdot \delta_\Gamma) \Rightarrow \int_{\mathbb{R}^d} \nabla \Psi^\kappa_{SL}(\varphi) \cdot \nabla \phi + \kappa^2 \Psi^\kappa_{SL}(\varphi) \phi \, dx = \int_\Gamma \varphi \phi \, dS \quad \forall \phi$$

$$[\mathcal{L}_\kappa \Psi^\kappa_{SL}(\varphi) = 0 \text{ in } \mathbb{R}^d \setminus \Gamma] \Rightarrow [\gamma_N \Psi^\kappa_{SL}(\varphi)]_\Gamma = -\varphi \quad \forall \varphi \in H^{-1/2}(\Gamma)$$

I.3.2 Jump Relations

Scalar case : "Neumann jumps"

$$\mathcal{L}_\kappa \underbrace{\mathcal{G}_\kappa(f)}_{\in H^1_{loc}(\mathbb{R}^d)} = 0 \quad \Rightarrow \int_{\mathbb{R}^d} \nabla \mathcal{G}_\kappa(f)\cdot\nabla\phi\,dx = \int_{\mathbb{R}^d} f\phi\,dx \quad \forall \phi\in\mathcal{D}(\mathbb{R}^d)$$

$$f\in\widetilde{H}^{-1}(\mathbb{R}^d)$$

$$\Psi^\kappa_{SL}(\varphi) = \mathcal{G}(\varphi\cdot\delta_\Gamma) \Rightarrow \int_{\mathbb{R}^d} \nabla\Psi^\kappa_{SL}(\varphi)\cdot\nabla\phi + \kappa^2\Psi^\kappa_{SL}(\varphi)\,\phi\,dx = \int_\Gamma \varphi\,\phi\,dS \quad \forall\phi$$

$$[\mathcal{L}_\kappa\Psi^\kappa_{SL}(\varphi)=0 \text{ in } \mathbb{R}^d\setminus\Gamma] \Rightarrow \quad [\gamma_N\Psi^\kappa_{SL}(\varphi)]_\Gamma = -\varphi \quad \forall\varphi\in H^{-1/2}(\Gamma)$$

$$+\ \text{RF}: \quad U = -\Psi^\kappa_{SL}([\gamma_N U]_\Gamma) + \Psi^\kappa_{DL}([\gamma U]_\Gamma)$$

$$\Rightarrow \qquad [\gamma_N\Psi^\kappa_{DL}(v)]_\Gamma = 0 \quad \forall v\in H^{1/2}(\Gamma)$$

I.3.2 Jump Relations

Maxwell case, $\underline{\mathcal{L}}_k \underline{u} := \nabla \times (\nabla \times \underline{u}) - k^2 \underline{u}$

I.3.2 Jump Relations

Maxwell case, $\underline{\mathcal{L}}_\kappa \underline{u} := \nabla \times (\nabla \times \underline{u}) - \kappa^2 \underline{u}$

Tangential jumps:

RF: $$\underline{u} = \underbrace{-\,\underline{\Psi}_E^\kappa([\gamma_N \underline{u}]_\Gamma)}_{\in H_{loc}(\mathrm{curl}, \mathbb{R}^3)} - \underline{\Psi}_M^\kappa([\gamma_t \underline{u}]_\Gamma)$$

$$\Rightarrow [\gamma_t \underline{\Psi}_F^\kappa(\underset{\sim}{\eta})]_\Gamma = 0 \quad \forall \underset{\sim}{\eta} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

I.3.2 Jump Relations

Maxwell case, $\underline{\mathcal{L}}_k \underline{u} := \nabla\times(\nabla\times\underline{u}) - k^2\underline{u}$

Tangential jumps:

RF: $$\underline{u} = \underbrace{-\underline{\Psi}_E^k([\gamma_n \underline{u}]_\Gamma)}_{} - \underline{\Psi}_M^k([\gamma_t \underline{u}]_\Gamma)$$

$$\in H_{loc}(curl, \mathbb{R}^3) \Rightarrow [\gamma_t \underline{\Psi}_E^k(\underline{\eta})]_\Gamma = 0 \quad \forall \underline{\eta} \in H^{-1/2}(div_\Gamma, \Gamma)$$

$$\Rightarrow [\gamma_t \underline{u}]_\Gamma = -[\underline{\Psi}_M^k([\gamma_t \underline{u}]_\Gamma)]_\Gamma \Rightarrow [\gamma_t \underline{\Psi}_M^k(\underline{v})]_\Gamma = -\underline{v} \quad \forall \underline{v} \in H^{-1/2}(div_\Gamma, \Gamma)$$

I.3.2 Jump Relations

Maxwell case : jumps of the magnetic trace

- $\nabla \times \Psi_E^{\kappa}(\underline{z}) = k\, \nabla \times \Psi_V^{\kappa}(\underline{z}) = k\, \Psi_M^{\kappa}(\underline{z})$
- $\gamma_N \underline{v} = \frac{1}{k} \gamma_t (\nabla \times \underline{v})$

I.3.2 Jump Relations

Maxwell case: jumps of the magnetic trace

- $\nabla \times \Psi_E^k(\underline{\eta}) = k \, \nabla \times \Psi_V^k(\underline{\eta}) = k \, \Psi_M^k(\underline{\eta})$
- $\gamma_N \underline{v} = \frac{1}{k} \gamma_t (\nabla \times \underline{v})$

▷ $[\gamma_N \underline{\Psi}_E^k(\underline{\eta})]_\Gamma = [\gamma_t \underline{\Psi}^k(\underline{\eta})]_\Gamma = -\underline{\eta} \quad \forall \underline{\eta} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

I.3.2 Jump Relations

Maxwell case : jumps of the magnetic trace

- $\nabla \times \Psi_E^{\kappa}(\underline{\eta}) = k \nabla \times \Psi_V^{\kappa}(\underline{\eta}) = k \Psi_M^{\kappa}(\underline{\eta})$
- $\gamma_M \underline{v} = \frac{1}{k} \gamma_t (\nabla \times \underline{v})$

▷ $[\gamma_M \underline{\Psi}_E^{\kappa}(\underline{\eta})]_\Gamma = [\gamma_t \underline{\Psi}^{\kappa}(\underline{\eta})]_\Gamma = -\underline{\eta} \quad \forall \underline{\eta} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- $\nabla \times \underline{\Psi}_M^{\kappa}(\underline{v}) = (-\Delta + \nabla \nabla \cdot) \Psi_V^{\kappa}(\underline{v}) = k^2 \Psi_V^{\kappa}(\underline{v}) + \nabla \Psi_{SL}^{\kappa}(\nabla_\Gamma \cdot \underline{v}) = k \Psi_E^{\kappa}(\underline{v})$

▷ $[\gamma_M \Psi_M^{\kappa}(\underline{v})]_\Gamma = [\gamma_t \Psi_E^{\kappa}(\underline{v})]_\Gamma = 0 \quad \forall \underline{v} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Supplement: Compact notations

Cauchy trace:

Scalar case: $\gamma : H^1_{loc}(\Delta,\Omega) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$, $\gamma(u) := \begin{bmatrix}\gamma u \\ \gamma_n u\end{bmatrix}$

Maxwell case: $\gamma : H(\mathrm{curl}^2,\Omega) \to \underbrace{H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}_{}$, $\gamma(u) := \begin{bmatrix}\gamma_t u \\ \gamma_N u\end{bmatrix}$

Abstract: $\gamma : H(\mathcal{L},\Omega) \to \mathcal{T}(\Gamma)$ [$\hat{=}$ Cauchy trace space]

Supplement: Compact notations

Single domain representation formulas

Scalar case: [$\mathcal{L}_\kappa u := (-\Delta - k^2) u = 0$ in $\mathbb{R}^d \setminus \Gamma$ & radiating]

$$u = -\Psi^{\kappa}_{SL}([\gamma_N u]_\Gamma) + \Psi^{\kappa}_{DL}([\gamma u]_\Gamma) \quad \text{in } \mathbb{R}^d \setminus \Gamma$$

Supplement : Compact notations

Single domain representation formulas

Scalar case : [$\mathcal{L}_\kappa \mathcal{U} := (-\Delta - \kappa^2)\mathcal{U} = 0$ in $\mathbb{R}^d \setminus \Gamma$ & radiating]

$$\mathcal{U} = -\Psi^\kappa_{SL}([\gamma_N \mathcal{U}]_\Gamma) + \Psi^\kappa_{DL}([\gamma \mathcal{U}]_\Gamma) \quad \text{in } \mathbb{R}^d \setminus \Gamma$$

Maxwell case : [$\underline{\mathcal{L}}_\kappa \underline{\mathcal{U}} := \nabla \times (\nabla \times \underline{\mathcal{U}}) - \kappa^2 \underline{\mathcal{U}} = 0$ in $\mathbb{R}^3 \setminus \Gamma$ & radiating]

$$\underline{\mathcal{U}} = -\Psi^\kappa_{E}([\gamma_N \underline{\mathcal{U}}]_\Gamma) - \Psi^\kappa_{M}([\gamma_t \underline{\mathcal{U}}]_\Gamma) \quad \text{in } \mathbb{R}^3 \setminus \Gamma$$

Supplement: Compact notations

Single domain representation formulas

Scalar case: [$\mathcal{L}_\kappa \mathcal{U} := (-\Delta - \kappa^2)\mathcal{U} = 0$ in $\mathbb{R}^d \setminus \Gamma$ & radiating]

$$\mathcal{U} = -\Psi^\kappa_{SL}([\gamma_N \mathcal{U}]_\Gamma) + \Psi^\kappa_{DL}([\gamma \mathcal{U}]_\Gamma) \quad \text{in } \mathbb{R}^d \setminus \Gamma$$

Maxwell case: [$\underline{\mathcal{L}}_\kappa \underline{\mathcal{U}} := \nabla \times (\nabla \times \underline{\mathcal{U}}) - \kappa^2 \underline{\mathcal{U}} = 0$ in $\mathbb{R}^3 \setminus \Gamma$ & radiating]

$$\underbrace{\underline{\mathcal{U}} = -\Psi^\kappa_{E}([\gamma_N \underline{\mathcal{U}}]_\Gamma) - \Psi^\kappa_{M}([\gamma_t \underline{\mathcal{U}}]_\Gamma) \quad \text{in } \mathbb{R}^3 \setminus \Gamma}$$

Abstract:

$$\boxed{\mathcal{U} = \Psi^\kappa([\gamma \mathcal{U}]) \quad \text{in } \mathbb{R}^d \setminus \Gamma}$$

Supplement: Compact notations

Jump relations:

Scalar case: $[\gamma \Psi^k_{SL}]_\Gamma = 0$, $[\gamma \Psi^k_{DL}]_\Gamma = \mathrm{Id}$

$[\gamma_N \Psi^k_{SL}]_\Gamma = -\mathrm{Id}$, $[\gamma_N \Psi^k_{DL}]_\Gamma = 0$

Maxwell case: $[\gamma_t \Psi^k_E]_\Gamma = 0$, $[\gamma_t \Psi^k_M]_\Gamma = -\mathrm{Id}$

$[\gamma_M \Psi^k_E]_\Gamma = -\mathrm{Id}$, $[\gamma_M \Psi^k_M]_\Gamma = 0$

▷ $$\boxed{[\boldsymbol{\gamma} \Psi^k]_\Gamma = \mathrm{Id}}$$