Bondary Element Methods: Derivation, Analysis, and Implementation

Ralf Hiptmair

Seminar for Applied Mathematics, ETH Zürich

Spring School 2018
Fundamentals and practice of finite elements
Roscoff, France April 16-20, 2018

Chapter II : Boundary Integral Equations

II.1. Boundary Integral Operators (BIOs)

$\boxed{\text{Trace operator}} \circ \boxed{\text{Layer Potentials}} \triangleright$ Boundary Integral Operators (BIOs)

Trace operator: functions in $H(\mathcal{L}, \mathbb{R}^d \setminus \Gamma)$ $\longrightarrow$ functions on Γ

Layer Potentials: functions on Γ $\longrightarrow$ functions in $H(\mathcal{L}, \mathbb{R}^d \setminus \Gamma)$

II.1. Boundary Integral Operators (BIOs)

$\boxed{\text{Trace operator}} \circ \boxed{\text{Layer Potentials}} \triangleright$ Boundary Integral Operators (BIOs)

Trace operator: functions in $H(\mathcal{L}, \mathbb{R}^d \setminus \Gamma)$ $\to$ functions on Γ

Layer Potentials: functions on Γ $\to$ functions in $H(\mathcal{L}, \mathbb{R}^d \setminus \Gamma)$

- Scalar case: Continuous BIOs [$\{\cdot\}_\Gamma \hat{=}$ average of trace]

$$V_k := \{\gamma \Psi^k_{SL}\}_\Gamma : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \qquad \text{[single layer BIO]}$$

$$K_k := \{\gamma \Psi^k_{DL}\}_\Gamma : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \qquad \text{[double layer BIO]}$$

$$K'_k := \{\gamma_N \Psi^k_{SL}\}_\Gamma : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \qquad \text{[adjoint DL-BIO]}$$

$$W_k := -\{\gamma_N \Psi^k_{DL}\}_\Gamma : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \qquad \text{[hypersingular BIO]}$$

II.1. Boundary Integral Operators (BIOs)

$\boxed{\text{Trace operator}} \circ \boxed{\text{Layer Potentials}} \triangleright$ Boundary Integral Operators (BIOs)

$\downarrow$ functions in $H(\mathcal{L}, \mathbb{R}^d \setminus \Gamma)$ $\longrightarrow$ functions on Γ

$\downarrow$ functions on Γ $\longrightarrow$ functions in $H(\mathcal{L}, \mathbb{R}^d \setminus \Gamma)$

- Scalar case: Continuous BIOs [$\{\cdot\}_\Gamma \hat{=}$ average of trace]

$$V_k := \{\gamma \Psi^k_{SL}\}_\Gamma : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \quad \text{[single layer BIO]}$$

$$K_k := \{\gamma \Psi^k_{DL}\}_\Gamma : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \quad \text{[double layer BIO]}$$

$$K'_k := \{\gamma_N \Psi^k_{SL}\}_\Gamma : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \quad \text{[adjoint DL-BIO]}$$

$$W_k := -\{\gamma_N \Psi^k_{DL}\}_\Gamma : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \quad \text{[hypersingular BIO]}$$

$$\left[V_k = \gamma \circ \mathcal{G}_k \circ \gamma', \; K_k = \gamma \circ \mathcal{G}_k \circ \gamma_N' \leftrightarrow K'_k = \gamma_N \circ \mathcal{G}_k \circ \gamma', \; W_k = \gamma_N \circ \mathcal{G}_k \circ \gamma_N' \right]$$

II.1. Boundary Integral Operators (BIOs)

- Maxwell case: Continuous BIOs [$\{\cdot\}_\Gamma \hat{=}$ average of trace]

$$S_k := \{\gamma_t \Psi_E^k\} = \{\gamma_M \Psi_M^k\} : H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

[Electric field BIO]

$$C_k := \{\gamma_t \Psi_M^k\} = \{\gamma_M \Psi_E^k\} : H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

[Magnetic field BIO]

II.1. Boundary Integral Operators (BIOs)

- Maxwell case: Continuous BIOs [$\{\cdot\}_\Gamma \hat{=}$ average of trace]

$$S_k := \{\gamma_t \Psi_E^k\} = \{\gamma_M \Psi_M^k\} : H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \rightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

[Electric field BIO]

$$C_k := \{\gamma_t \Psi_M^k\} = \{\gamma_M \Psi_E^k\} : H^{-1/2}(\mathrm{div}, \Gamma) \rightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

[Magnetic field BIO]

Structure of S_k

[$\Psi_E^k(\eta) = k\Psi_V^k(\eta) + \frac{1}{k}\nabla\Psi_{SL}^k(\nabla_\Gamma\cdot\eta)$, $\Psi_V^k := g_k \circ \gamma_t'$]

$V_k := \gamma_t \circ \Psi_V^k$.

$$\triangleright \qquad S_k(\eta) = k\,V_k(\eta) + \frac{1}{k}\nabla_\Gamma \times V(\nabla_\Gamma\cdot\eta)$$

II.1. Boundary Integral Operators (BIOs)

[Compact Notation]

	scalar case	Maxwell case
$\mathcal{J}(\Gamma)$		
γ		
ψ		

II.1. Boundary Integral Operators (BIOs)
[Compact Notation]

	scalar case	Maxwell case
$\mathcal{J}(\Gamma)$	$H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$	$H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
γ	$\gamma := \begin{bmatrix} \gamma \\ \gamma_N \end{bmatrix}$	$\gamma := \begin{bmatrix} \gamma_t \\ \gamma_M \end{bmatrix}$
Ψ	$\Psi^k := -\Psi^k_{SL} + \Psi^k_{DL}$	$\Psi^k := -\Psi^k_E - \Psi^k_M$
$\mathbb{A}_k := \{\gamma\}\Psi^k$		

II.1. Boundary Integral Operators (BIOs)

[Compact Notation]

	scalar case	Maxwell case
$\mathbb{J}(\Gamma)$	$H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$	$H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
γ	$\gamma := \begin{bmatrix} \gamma \\ \gamma_N \end{bmatrix}$	$\gamma := \begin{bmatrix} \gamma_t \\ \gamma_M \end{bmatrix}$
Ψ	$\Psi^k := -\Psi^k_{SL} + \Psi^k_{DL}$	$\Psi^k := -\Psi^k_E - \Psi^k_M$
$A_k := \{\gamma\}\Psi^k$	$A_k = \begin{bmatrix} K_k & -V_k \\ -W_k & -K'_k \end{bmatrix}$	$A_k = \begin{bmatrix} -C_k & -S_k \\ -S_k & -C_k \end{bmatrix}$

$\hookrightarrow A_k : \mathbb{J}(\Gamma) \to \mathbb{J}(\Gamma)$ continuous [compound BIO]

II.1. Boundary Integral Operators (BIOs)

[Boundary Integral Representations]

Scalar case: [$*$ $\hat{=}$ non-trivial]

$$V_k(\varphi)(x) = \{\gamma \Psi_{SL}^k(\varphi)\}_\Gamma(x) = \int_\Gamma G_k(x-y)\,\varphi(y)\,dS(y)\,,\quad \varphi \in L^\infty(\Gamma)$$

kernel integrable on Γ

II.1. Boundary Integral Operators (BIOs)

[Boundary Integral Representations]

Scalar case: [$\ast \stackrel{\wedge}{=}$ non-trivial]

$$V_k(\varphi)(x) = \{ \gamma \Psi^k_{SL}(\varphi) \}_\Gamma (x) = \int_\Gamma G_k(x-y)\, \varphi(y)\, dS(y), \quad \varphi \in L^\infty(\Gamma)$$

kernel integrable on Γ

$$K_k(v)(x) \stackrel{\ast}{=} \{ \gamma \Psi^k_{DL}(v) \}_\Gamma (x) = \int_\Gamma \frac{(x-y)\cdot n(y)}{2^{d-1}\pi \|x-y\|^d}\, v(y)\, dS(y), \quad v \in L^\infty(\Gamma)$$

strongly singular, Cauchy principal value

II.1. Boundary Integral Operators (BIOs)

[Boundary Integral Representations]

Scalar case: [$\ast \stackrel{\wedge}{=}$ non-trivial]

$$V_k(\varphi)(x) = \{\gamma \Psi_{SL}^k(\varphi)\}_\Gamma(x) = \int_\Gamma G_k(x-y)\,\varphi(y)\,dS(y)\,, \quad \varphi \in L^\infty(\Gamma)$$

kernel integrable on Γ

$$K_k(v)(x) \stackrel{\ast}{=} \{\gamma \Psi_{DL}^k(v)\}_\Gamma(x) = \int_\Gamma \frac{(x-y)\cdot n(y)}{2^{d-1}\pi\,\|x-y\|^d}\, v(y)\,dS(y)\,, \quad v \in L^\infty(\Gamma)$$

strongly singular, Cauchy principal value

$$K_k'(\varphi)(x) \stackrel{\ast}{=} \{\gamma_N \Psi_{SL}^k(\varphi)\}_\Gamma(x) = \int_\Gamma \frac{(x-y)\cdot n(x)}{2^{d-1}\pi\,\|x-y\|^d}\,\varphi(y)\,dS(y)\,, \quad \varphi \in L^\infty(\Gamma)$$

II.1. Boundary Integral Operators (BIOs)

[Boundary Integral Representations]

Scalar case: [$\overset{*}{\triangleq}$ non-trivial]

$$V_k(\varphi)(x) = \{\gamma \Psi_{SL}^k(\varphi)\}_\Gamma(x) = \int_\Gamma G_k(x-y)\,\varphi(y)\,dS(y)\,,\quad \varphi \in L^\infty(\Gamma)$$

kernel integrable on Γ

$$K_k(v)(x) \overset{*}{=} \{\gamma \Psi_{DL}^k(v)\}_\Gamma(x) = \int_\Gamma \frac{(x-y)\cdot n(y)}{2^{d-1}\pi\,\|x-y\|^d}\,v(y)\,dS(y)\,,\quad v \in L^\infty(\Gamma)$$

strongly singular, Cauchy principal value

$$K_k'(\varphi)(x) \overset{*}{=} \{\gamma_N \Psi_{SL}^k(\varphi)\}_\Gamma(x) = \int_\Gamma \frac{(x-y)\cdot n(x)}{2^{d-1}\pi\,\|x-y\|^d}\,\varphi(y)\,dS(y)\,,\quad \varphi \in L^\infty(\Gamma)$$

$$W_k(v)(x) = \{\gamma_N \Psi_{DL}^k(v)\}_\Gamma(x) = \{\text{ finite-part integral }\}$$

Maxwell case: [similar]

II.1. Boundary Integral Operators (BIOs)

[Variational Forms]

$$\left.\begin{aligned} V_k &: H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma) \\ W_k &: H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma) \\ S_k &: H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \end{aligned}\right\} \Longrightarrow \text{induce bilinear forms}$$

in L^2-duality

II.1. Boundary Integral Operators (BIOs)

[Variational Forms]

$$\left.\begin{aligned} V_k &: H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma) \\ W_k &: H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma) \\ S_k &: H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \end{aligned}\right\} \Rightarrow$$

induce bilinear forms

in L^2-duality

▷ Single layer BIO:

$$a_V^k(\varphi, \varphi') := \langle \varphi, V_k(\varphi) \rangle_\Gamma : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \longrightarrow \mathbb{C}$$

$$\left[\; a_V^k(\varphi, \varphi') = \iint_{\Gamma\,\Gamma} G_k(x-y)\,\varphi(y)\,\varphi'(x)\,dS(y)\,dS(x) \;,\quad \varphi, \varphi' \in L^\infty(\Gamma) \;\right]$$

II.1. Boundary Integral Operators (BIOs)

[Variational Forms]

$$
\left.
\begin{aligned}
V_k &: H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma) \\
W_k &: H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma) \\
S_k &: H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \longrightarrow H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)
\end{aligned}
\right\} \Rightarrow \text{induce bilinear forms}
$$

$\uparrow$ in L^2-duality $\uparrow$

▷ Single Layer BIO:

$$a_V^k(\varphi,\varphi') := \langle \varphi, V_k(\varphi)\rangle_\Gamma : H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma) \longrightarrow \mathbb{C}$$

$$\left[\; a_V^k(\varphi,\varphi') = \int_\Gamma\int_\Gamma G_k(x-y)\,\varphi(y)\,\varphi'(x)\,dS(y)\,dS(x) \quad , \quad \varphi,\varphi' \in L^\infty(\Gamma) \;\right]$$

▷ Electric Field BIO:

$$a_S^k(\underline{\eta},\underline{\eta}') := \langle \underline{\eta}', S_\kappa(\underline{\eta})\rangle_{t,\Gamma} : H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\mathrm{div},\Gamma) \longrightarrow \mathbb{C}$$

$$\left[\, a_S^k(\underline{\eta},\underline{\eta}') = k\int_\Gamma G_\kappa(x-y)\,\underline{\eta}(y)\,\underline{\eta}(x)\,dS(y,x) - \frac{1}{k}\int_\Gamma G_\kappa(x-y)\,\nabla_\Gamma\cdot\underline{\eta}(y)\,\nabla_\Gamma\cdot\underline{\eta}'(x)\,dS(y,x)\,\right]$$

II.1. Boundary Integral Operators (BIOs)
[Variational Forms]

Hypersingular BIO:

$$a_W^k(v, v') := \langle W_k(v), v' \rangle_\Gamma : H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \longrightarrow \mathbb{C}$$

II.1. Boundary Integral Operators (BIOs) [Variational Forms]

Hyperingular BIO:

$$a_W^k(v,v') := \langle W_k(v), v' \rangle_\Gamma : H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \longrightarrow \mathbb{C}$$

<u>Thm</u>: [Variational hypersingular BIO for $d=3$]

$$a_W^k(v,v') = a_{\underline{V}}^{\kappa}(\nabla_\Gamma \times v, \nabla_\Gamma \times v) - k^2 a_{\underline{V}}^{\kappa}(v\underline{n}, v'\underline{n})$$

$$\left[a_{\underline{V}}(\underline{\eta},\underline{\eta}) := \iint_{\Gamma\,\Gamma} G_k(x-y)\,\underline{\eta}(x)\,\underline{\eta}(y)\, dS(x,y) \right]$$

[Proof by integration by parts on Γ]

II.1. Boundary Integral Operators (BIOs)

[Coercivity]

<u>Thm</u>: There are rank-1 operators $F_V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $F_W : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ such that

$$\exists c_V > 0 : \left| a_V^0(\varphi, \varphi) + \langle \varphi, F_V \varphi \rangle_\Gamma \right| \geq c_V \|\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma)$$

$$\exists c_W > 0 : \left| a_W^0(v, v) + \langle F_W v, v \rangle_\Gamma \right| \geq c_W \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}(\Gamma)$$

II.1. Boundary Integral Operators (BIOs)

[Coercivity]

Thm: There are rank-1 operators $F_V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $F_W : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ such that

$$\exists c_V > 0 : \left| a_V^0(\varphi, \varphi) + \langle \varphi, F_V \varphi \rangle_\Gamma \right| \geq c_V \|\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma)$$

$$\exists c_W > 0 : \left| a_W^0(v, v) + \langle F_W v, v \rangle_\Gamma \right| \geq c_W \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}(\Gamma)$$

$d=3$:

$$G_k(x-y) - G_0(x,y) = \frac{e^{ik\|x-y\|} - 1}{4\pi \|x-y\|} \in C^0(\mathbb{R}^3 \times \mathbb{R}^3)$$

$\triangleright$ $\mathcal{G}_k - \mathcal{G}_0 : H^s(\mathbb{R}^3) \longrightarrow H^{s+4}(\mathbb{R}^3)$ continuous

II.1. Boundary Integral Operators (BIOs)

[Coercivity]

Thm: There are rank-1 operators $F_V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $F_W : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ such that

$$\exists c_V > 0: \left| a_V^0(\varphi,\varphi) + \langle \varphi, F_V \varphi \rangle_\Gamma \right| \geq c_V \|\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma)$$

$$\exists c_W > 0: \left| a_W^0(v,v) + \langle F_W v, v \rangle_\Gamma \right| \geq c_W \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}(\Gamma)$$

$d=3$: $$G_k(x-y) - G_0(x,y) = \frac{e^{ik\|x-y\|} - 1}{4\pi \|x-y\|} \in C^0(\mathbb{R}^3 \times \mathbb{R}^3)$$

▷ $\mathcal{G}_k - \mathcal{G}_0 : H^s(\mathbb{R}^3) \longrightarrow H^{s+4}(\mathbb{R}^3)$ continuous

▷ $T_k - T_0$ compact $(T = V, K, K', W)$ $\iff$ $\mathbb{A}_k - \mathbb{A}_0 : \mathcal{J}(\Gamma) \to \mathcal{J}(\Gamma)$ compact

II.1. Boundary Integral Operators (BIOs)

$\mathbb{T}$-Coercivity* of Electric Field Integral Operator

$$S_k = k\,\underline{V}_k + \frac{1}{k}\,\nabla_\Gamma \times V_k \circ \nabla_\Gamma \cdot$$

II.1. Boundary Integral Operators (BIOs)

[T-Coercivity* of Electric Field Integral Operator

$$\mathsf{S}_k = k \underline{\mathsf{V}}_k + \frac{1}{k} \nabla_\Gamma \times \mathsf{V}_k \circ \nabla_\Gamma \cdot$$

▷ $\mathsf{S}_k - \mathsf{S}_* : \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ compact

$$\Updownarrow$$

$$a_s^*(\underline{\eta}, \underline{\eta}) = k\, \underline{a}_V^0(\underline{\eta}, \underline{\eta}) - \frac{1}{k} a_V^0(\nabla_\Gamma \cdot \underline{\eta}, \nabla_\Gamma \cdot \underline{\eta})$$

$\underline{H}_t^{-1/2}(\Gamma)$-elliptic

$H^{-1/2}(\Gamma)$-elliptic

→ a_s^* indefinite

II.1. Boundary Integral Operators (BIOs)

[T-Coercivity* of Electric Field Integral Operator

$$S_k = k\,\underline{V}_k + \frac{1}{k}\,\nabla_\Gamma \times V_k \circ \nabla_\Gamma \cdot$$

$\triangleright$ $S_k - S_* : H^{-1/2}(\operatorname{div}_\Gamma,\Gamma) \longrightarrow H^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ compact

$\Updownarrow$

$$a_s^*(\underline{\eta},\underline{\eta}) = k\,a^{\circ}_{\underline{V}}(\underline{\eta},\underline{\eta}) - \frac{1}{k}\,a^{\circ}_V(\nabla_\Gamma\cdot\underline{\eta},\nabla_\Gamma\cdot\underline{\eta})$$

$\uparrow$ $\underline{H}_t^{-1/2}(\Gamma)$-elliptic

$\uparrow$ $H^{-1/2}(\Gamma)$-elliptic

$\longrightarrow$ a_s^* indefinite

---

* **Def**: Bilinear form $a: H\times H \to \mathbb{C}$ T-coercive, if there is

- a compact operator $K: H \longrightarrow H$
- an isomorphism $T: H \longrightarrow H$

$$\operatorname{Re}\{a(v,T\bar{v}) + \langle Kv,\bar{v}\rangle_{H'\times H}\} \geq c\,\|v\|_H^2 \quad \forall v \in H$$

II.1. Boundary Integral Operators (BIOs)

[T-Coercivity* of Electric Field Integral Operator

Def: Bilinear form $a : H \times H \to \mathbb{C}$ T-coercive, if there is

- a compact operator $K : H \to H'$
- an isomorphism $T : H \to H$

$$\operatorname{Re}\{ a(v, T\bar{v}) + \langle Kv, \bar{v} \rangle_{H' \times H} \} \geq c \|v\|_H^2 \quad \forall v \in H$$

▷ Fredholm alternative: a T-coercive & injective

$\Longleftrightarrow$ a induces isomorphism $A : H \to H'$

II.1. Boundary Integral Operators (BIOs)

[T-Coercivity* of Electric Field Integral Operator

$$a_S^*(\underline{\eta},\underline{\eta}') = k\,a_{\underline{V}}^{\circ}(\underline{\eta},\underline{\eta}') - \frac{1}{k}\,a_V^{\circ}(\nabla_\Gamma\cdot\underline{\eta},\nabla_\Gamma\cdot\underline{\eta}')$$

$\underline{H}_t^{-1/2}(\Gamma)$-elliptic $\qquad\qquad$ $H^{-1/2}(\Gamma)$-elliptic

$$\underline{\eta},\underline{\eta}' \in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) = \{\underline{\eta}\in\underline{H}_t^{-1/2}(\Gamma),\ \mathrm{div}_\Gamma\underline{\eta}\in H^{-1/2}(\Gamma)\} = H$$

II.1. Boundary Integral Operators (BIOs)

T-Coercivity* of Electric Field Integral Operator

$$a_s^*(\underline{\eta},\underline{\eta}') = k\,a_{\underline{V}}^{\circ}(\underline{\eta},\underline{\eta}') - \frac{1}{k}\,a_V^{\circ}(\nabla_\Gamma\cdot\underline{\eta},\nabla_\Gamma\cdot\underline{\eta}')$$

$\underline{H}_t^{-1/2}(\Gamma)$-elliptic $\qquad\qquad$ $H^{-1/2}(\Gamma)$-elliptic

$$\underline{\eta},\underline{\eta}' \in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) = \{\underline{\eta}\in\underline{H}_t^{-1/2}(\Gamma),\ \mathrm{div}_\Gamma\,\underline{\eta}\in H^{-1/2}(\Gamma)\} = H$$

Construction of T for electric field BIO:

Rely on projection $R : H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ (bounded)

- $\nabla_\Gamma\cdot R(\underline{\eta}) = \nabla_\Gamma\cdot\underline{\eta}$
- $R : H^{-1/2}(\mathrm{div},\Gamma) \to \underline{H}_t^{-1/2}(\Gamma)$ compact

II.1. Boundary Integral Operators (BIOs)

[T-Coercivity* of Electric Field Integral Operator

$$a_s^*(\underline{\eta}, \underline{\eta}') = k\, \underbrace{\mathring{a}_V}_{\underline{H}_t^{-1/2}(\Gamma)\text{-elliptic}}(\underline{\eta}, \underline{\eta}') - \frac{1}{k} \underbrace{\mathring{a}_V}_{H^{-1/2}(\Gamma)\text{-elliptic}}(\nabla_\Gamma \cdot \underline{\eta}, \nabla_\Gamma \cdot \underline{\eta}')$$

$$\underline{\eta}, \underline{\eta}' \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \{ \underline{\eta} \in \underline{H}_t^{-1/2}(\Gamma),\ \mathrm{div}_\Gamma \underline{\eta} \in H^{-1/2}(\Gamma) \} = H$$

Construction of T for electric field BIO:

Rely on projection $R : H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ (bounded)

- $\nabla_\Gamma \cdot R(\underline{\eta}) = \nabla_\Gamma \cdot \underline{\eta}$
- $R : H^{-1/2}(\mathrm{div}, \Gamma) \to \underline{H}_t^{-1/2}(\Gamma)$ compact

▷ $T = Id - 2R : H \to H$

II.2. Boundary Integral Equations (BIE)

II.2.1. Calderón Projectors

Representation formula [compact notation] : $U = \Psi^k([\gamma U])$

Jump relations : $\tilde{u} = [\Psi^k(\tilde{u})]\,, \quad \tilde{u} \in \mathcal{J}(\Gamma)$

Boundary integral operator : $A_k(\tilde{u}) = \{\Psi^k(\tilde{u})\}$

II.2. Boundary Integral Equations (BIE)

II.2.1. Calderón Projectors

Representation formula: [compact notation] $\qquad \mathcal{U} = \Psi^k([\gamma \mathcal{U}])$

Jump relations: $\qquad \tilde{u} = [\Psi^k(\tilde{u})]\,, \quad \tilde{u} \in \mathcal{J}(\Gamma)$

Boundary integral operator: $\qquad A_k(\tilde{u}) = \{\Psi^k(\tilde{u})\}$

▷ $$\gamma^{\pm}\Psi(\tilde{u}) = \pm \tfrac{1}{2}\tilde{u} + A_k(\tilde{u})\,, \quad \tilde{u} \in \mathcal{J}(\Gamma)$$

II.2. Boundary Integral Equations (BIE)

II.2.1. Calderón Projectors

Representation formula [compact notation]: $\quad u = \Psi^k([\gamma u])$

Jump relations: $\quad \tilde{u} = [\Psi^k(\tilde{u})]$, $\tilde{u} \in \mathcal{J}(\Gamma)$

Boundary integral operator: $\quad A_k(\tilde{u}) = \{\Psi^k(\tilde{u})\}$

$$\triangleright \quad \gamma^{\pm}\Psi(\tilde{u}) = \pm \tfrac{1}{2}\tilde{u} + A_k(\tilde{u}), \quad \tilde{u} \in \mathcal{J}(\Gamma)$$

<u>Thm</u>: For $\tilde{u} \in \mathcal{J}(\Gamma)$ $\exists$ radiating $u \in H(L, \Omega^{\pm})$: $\mathcal{L}_k u = 0$ in $\Omega^{\pm}$

$$\Longleftrightarrow \quad P^{\pm}\tilde{u} := \left(\tfrac{1}{2}\mathrm{Id} \pm A_k\right)\tilde{u} = \tilde{u}$$

exterior/interior Calderón projector

II.2.2. Direct BIE for Simple BVPs

Scalar case: $\mathcal{L}_k = -\Delta - k^2$ + Sommerfeld radiation cond.

$H(L,\Omega) = H^1(\Delta,\Omega)$, trace space $\mathcal{T}(\Gamma) = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$

Cauchy trace $\gamma = \begin{bmatrix} \gamma \\ \gamma_N \end{bmatrix}$ $\Rightarrow$ BIOs $\mathbb{A}_k = \begin{bmatrix} K_k & -V_k \\ -W_k & -K_k' \end{bmatrix}$

II.2.2. Direct BIE for Simple BVPs

Scalar case: $\mathcal{L}_k = -\Delta - k^2$ + Sommerfeld radiation cond.

$H(L,\Omega) = H^1(\Delta,\Omega)$, trace space $\mathcal{J}(\Gamma) = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$

Cauchy trace $\boldsymbol{\gamma} = \begin{bmatrix} \gamma \\ \gamma_N \end{bmatrix}$ $\Rightarrow$ BIOs $\mathbb{A}_k = \begin{bmatrix} K_k & -V_k \\ -W_k & -K'_k \end{bmatrix}$

<u>Thm</u>: For $\tilde{u} \in \mathcal{J}(\Gamma)$ $\exists$ radiating $U \in H(L,\Omega^\pm)$: $\mathcal{L}_k U = 0$ in $\Omega^\pm$

$$\Longleftrightarrow \quad P^\pm \tilde{u} := \left(\tfrac{1}{2} Id \pm \mathbb{A}_k\right)\tilde{u} = \tilde{u}$$

II.2.2. Direct BIE for Simple BVPs

Scalar case: $\mathcal{L}_k = -\Delta - k^2$ + Sommerfeld radiation cond.

$H(L,\Omega) = H^1(\Delta,\Omega)$, trace space $\mathcal{T}(\Gamma) = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$

Cauchy trace $\gamma = \begin{bmatrix} \gamma \\ \gamma_N \end{bmatrix} \Rightarrow$ BIOs $\mathbb{A}_k = \begin{bmatrix} K_k & -V_k \\ -W_k & -K_k' \end{bmatrix}$

$$\boxed{\begin{array}{l}\underline{\text{Thm}}: \text{ For } \tilde{u} \in \mathcal{T}(\Gamma)\ \exists \text{ radiating } u \in H(L,\Omega^{\pm}): \mathcal{L}_k u = 0 \text{ in } \Omega^{\pm} \\ \iff \quad P^{\pm}\tilde{u} := (\tfrac{1}{2}\mathrm{Id} \pm \mathbb{A}_k)\tilde{u} = \tilde{u}\end{array}}$$

$\triangleright$ If $(-\Delta - k^2)u = 0$ in Ω & u radiating

$$\Rightarrow \begin{bmatrix} \tfrac{1}{2} - K_k & V_k \\ W_k & \tfrac{1}{2} + K_k' \end{bmatrix} \begin{bmatrix} \gamma u \\ \gamma_N u \end{bmatrix} = \begin{bmatrix} \gamma u \\ \gamma_N u \end{bmatrix}$$

II.2.2. Direct BIE for Simple BVPs

Dirichlet problem : [scalar case]

given $g := \gamma u \in H^{1/2}(\Gamma)$

sought $\psi := \gamma_n u \in H^{-1/2}(\Gamma)$

$$\begin{bmatrix} 1/2 - K_k & V_k \\ W_k & 1/2 + K_k' \end{bmatrix} \begin{bmatrix} \gamma u \\ \gamma_n u \end{bmatrix} = \begin{bmatrix} \gamma u \\ \gamma_n u \end{bmatrix}$$

II.2.2. Direct BIE for Simple BVPs

Dirichlet problem: [scalar case]

given $g := \gamma u \in H^{1/2}(\Gamma)$

sought $\psi := \gamma_n u \in H^{-1/2}(\Gamma)$

$$\begin{bmatrix} \frac{1}{2} - K_\kappa & V_\kappa \\ W_\kappa & \frac{1}{2} + K'_\kappa \end{bmatrix} \begin{bmatrix} \gamma u \\ \gamma_n u \end{bmatrix} = \begin{bmatrix} \gamma u \\ \gamma_n u \end{bmatrix}$$

$$V_\kappa(\gamma_n u) = \left(\tfrac{1}{2} + K_\kappa\right)(g)$$

[BIE of the 1st kind in $H^{1/2}(\Gamma)$]

$\Updownarrow$

Seek $\psi \in H^{-1/2}(\Gamma)$:

$$a_V^\kappa(\psi, \psi') = \left\langle \left(\tfrac{1}{2} + K_\kappa\right)(g), \psi' \right\rangle_\Gamma \quad \forall \psi' \in H^{-1/2}(\Gamma)$$

II.2.2. Direct BIE for Simple BVPs

Dirichlet problem :
[scalar case]

given $g := \gamma u \in H^{1/2}(\Gamma)$

sought $\mu := \gamma_n u \in H^{-1/2}(\Gamma)$

$$\begin{bmatrix} \frac{1}{2} - K_k & V_k \\ W_k & \frac{1}{2} + K_k' \end{bmatrix} \begin{bmatrix} \gamma u \\ \gamma_n u \end{bmatrix} = \begin{bmatrix} \gamma u \\ \gamma_n u \end{bmatrix}$$

$$V_k(\gamma_n u) = (\tfrac{1}{2} + K_k)(g)$$

[BIE of the 1st kind in $H^{1/2}(\Gamma)$]

$\Updownarrow$

Seek $\mu \in H^{-1/2}(\Gamma)$:

$$a_V^{\kappa}(\mu, \mu') = \langle (\tfrac{1}{2} + K_k)(g), \mu' \rangle_\Gamma \quad \forall \mu' \in H^{-1/2}(\Gamma)$$

$$(\tfrac{1}{2} - K_k')(\gamma_n u) = W_k(g)$$

[BIE of the 2nd kind in $H^{-1/2}(\Gamma)$]

$\Downarrow$

Variational formulation ?

II.2.2. Direct BIE for Simple BVPs

On 2nd-kind BIE for scalar Dirichlet problem:

$$\mu \in H^{-1/2}(\Gamma): \left(\tfrac{1}{2} - K_k\right)(\mu) = W_k(g) \quad \text{in } H^{-1/2}(\Gamma)$$

- Variational formulation in $H^{-1/2}(\Gamma)$

$$\left(\left(\tfrac{1}{2} - K_k'\right)(\mu), \mu'\right)_{H^{-1/2}(\Gamma)} = \left(W_k(g), \mu'\right)_{H^{-1/2}(\Gamma)} \quad \forall \mu' \in H^{-1/2}(\Gamma)$$

$(\cdot,\cdot)_{H^{-1/2}(\Gamma)} \mathrel{\hat{=}}$ **non-local** inner product in $H^{-1/2}(\Gamma)$

<u>Thm</u>: (O. Steinbach) Coercive variational problem

BUT, difficult to use in context of Galerkin discretization

II.2.2. Direct BIE for Simple BVPs

On 2nd-kind BIE for scalar Dirichlet problem:

$$\mu \in H^{-1/2}(\Gamma): \quad \left(\tfrac{1}{2} - K'_k\right)(\mu) = W_k(g) \quad \text{in} \quad H^{-1/2}(\Gamma)$$

- Variational formulation in $L^2(\Gamma)$: $[g \in H^1(\Gamma)]$

$$\mu \in L^2(\Gamma): \quad \left\langle \left(\tfrac{1}{2} - K'_k\right)(\mu), \mu' \right\rangle = \left\langle W_k(g), \mu' \right\rangle \quad \forall \mu' \in L^2(\Gamma)$$

II.2.2. Direct BIE for Simple BVPs

On 2nd-kind BIE for scalar Dirichlet problem:

$$\mu \in H^{-1/2}(\Gamma): \left(\tfrac{1}{2} - K_k'\right)(\mu) = W_k(g) \quad \text{in} \quad H^{-1/2}(\Gamma)$$

- Variational formulation in $L^2(\Gamma)$: $[g \in H^1(\Gamma)]$

$$\mu \in L^2(\Gamma): \left\langle \left(\tfrac{1}{2} - K_k'\right)(\mu), \mu' \right\rangle = \left\langle W_k(g), \mu' \right\rangle \quad \forall \mu' \in L^2(\Gamma)$$

<u>Thm</u>: If Γ polyhedral, then

$$\left.\begin{aligned} K_k' &: L^2(\Gamma) \longrightarrow L^2(\Gamma) \\ W_k &: H^1(\Gamma) \longrightarrow L^2(\Gamma) \end{aligned}\right\} \text{continuous}$$

II.2.2. Direct BIE for Simple BVPs

On 2nd-kind BIE for scalar Dirichlet problem:

$$\mu \in H^{-1/2}(\Gamma): \quad \left(\tfrac{1}{2} - K_k'\right)(\mu) = W_k(g) \quad \text{in} \quad H^{-1/2}(\Gamma)$$

- Variational formulation in $L^2(\Gamma)$: $[g \in H^1(\Gamma)]$

$$\mu \in L^2(\Gamma): \left\langle \left(\tfrac{1}{2} - K_k'\right)(\mu), \mu' \right\rangle = \left\langle W_k(g), \mu' \right\rangle \quad \forall \mu' \in L^2(\Gamma)$$

<u>Thm</u>: If Γ polyhedral, then

$$\left.\begin{aligned} K_k' &: L^2(\Gamma) \longrightarrow L^2(\Gamma) \\ W_k &: H^1(\Gamma) \longrightarrow L^2(\Gamma) \end{aligned}\right\} \text{continuous}$$

<u>Thm</u>: If Γ C^2-smooth, then $K_k' : L^2(\Gamma) \to H^1(\Gamma)$ cont.

$$\left[\ (K'\varphi)(x) = \int_\Gamma \frac{(x-y)\cdot n(x)}{2^{d-1}\pi \|x-y\|^d}\, \varphi(y)\, dS(y)\ \right]$$

$\Downarrow$

$L^2(\Gamma)$-coercivity

II.2.2. Direct BIE for Simple BVPs

Neumann problem :
[scalar case]

given $\psi := \gamma_N U \in H^{-1/2}(\Gamma)$

sought $u := \gamma U \in H^{1/2}(\Gamma)$

$$\begin{bmatrix} 1/2 - K_k & V_k \\ W_k & 1/2 + K_k' \end{bmatrix} \begin{bmatrix} \gamma U \\ \gamma_N U \end{bmatrix} = \begin{bmatrix} \gamma U \\ \gamma_N U \end{bmatrix}$$

II.2.2. Direct BIE for Simple BVPs

Neumann problem : [scalar case]

given $\varphi := \gamma_N u \in H^{-1/2}(\Gamma)$

sought $u := \gamma u \in H^{1/2}(\Gamma)$

$$\begin{bmatrix} 1/2 - K_k & V_k \\ W_k & 1/2 + K_k' \end{bmatrix} \begin{bmatrix} \gamma u \\ \gamma_N u \end{bmatrix} = \begin{bmatrix} \gamma u \\ \gamma_N u \end{bmatrix}$$

$$W_k(\gamma u) = (1/2 - K_k')(\varphi)$$

[BIE of the 1st kind in $H^{-1/2}(\Gamma)$]

$\Updownarrow$

Seek $u \in H^{1/2}(\Gamma)$:

$$a_W^k(u, u') = \langle (1/2 - K_k')(\varphi), u' \rangle_\Gamma \quad \forall u' \in H^{1/2}(\Gamma)$$

II.2.2. Direct BIE for Simple BVPs

Neumann problem : given $\varphi := \gamma_N u \in H^{-1/2}(\Gamma)$
[scalar case] sought $u := \gamma u \in H^{1/2}(\Gamma)$

$$\begin{bmatrix} \frac{1}{2} - K_k & V_k \\ W_k & \frac{1}{2} + K_k' \end{bmatrix} \begin{bmatrix} \gamma u \\ \gamma_N u \end{bmatrix} = \begin{bmatrix} \gamma u \\ \gamma_N u \end{bmatrix}$$

$$W_\kappa(\gamma u) = (\tfrac{1}{2} - K_k')(\varphi)$$

[BIE of the 1st kind in $H^{-1/2}(\Gamma)$]

$\Updownarrow$

Seek $u \in H^{1/2}(\Gamma)$:

$$a_W^\kappa(u, u') = \langle(\tfrac{1}{2} - K_k')(\varphi), u'\rangle_\Gamma \quad \forall u' \in H^{1/2}(\Gamma)$$

$$(\tfrac{1}{2} - K_k)(\gamma u) = V_\kappa(\varphi)$$

[BIE of the 2nd kind in $H^{1/2}(\Gamma)$]

$\Downarrow$

Variational formulation in $L^2(\Gamma)$

[Coercive on C^2-boundaries]

II.2.2. Direct BIE for Simple BVPs

Maxwell case: $\mathcal{L}_k \underline{u} = \nabla \times (\nabla \times \underline{u}) - k^2 \underline{u}$ + Silver-Müller r.c.

$H(L, \Omega) = H^1(\mathrm{curl}^2, \Omega)$, trace space $\mathcal{T}(\Gamma) = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Cauchy trace $\gamma = \begin{bmatrix} \gamma_t \\ \gamma_N \end{bmatrix} \Rightarrow$ BIOs $A_k = \begin{bmatrix} -C_K & -S_K \\ -S_K & -C_K \end{bmatrix}$

II.2.2. Direct BIE for Simple BVPs

Maxwell case: $\mathsf{L}_k \underline{u} = \nabla \times (\nabla \times \underline{u}) - k^2 \underline{u}$ + Silver-Müller r.c.

$H(L,\Omega) = H(\mathbf{curl}^2,\Omega)$, trace space $\mathcal{T}(\Gamma) = H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \times H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

Cauchy trace $\quad \gamma = \begin{bmatrix} \gamma_t \\ \gamma_N \end{bmatrix} \Rightarrow$ BIOs $\quad \mathsf{A}_k = \begin{bmatrix} -C_k & -S_k \\ -S_k & -C_k \end{bmatrix}$

$\mathsf{L}_k \underline{u} = 0$ in Ω
$\underline{u}$ radiating

$$\triangleright \quad \begin{bmatrix} \tfrac{1}{2}\mathrm{Id} + C_k & S_k \\ S_k & \tfrac{1}{2}\mathrm{Id} + C_k \end{bmatrix} \begin{bmatrix} \gamma_t \underline{u} \\ \gamma_N \underline{u} \end{bmatrix} = \begin{bmatrix} \gamma_t \underline{u} \\ \gamma_N \underline{u} \end{bmatrix}$$

II.2.2. Direct BIE for Simple BVPs

Maxwell case: $\mathcal{L}_k \underline{\mathbf{u}} = \nabla\times(\nabla\times\underline{\mathbf{u}}) - k^2\underline{\mathbf{u}}$ + Silver-Müller r.c.

$H(\mathcal{L},\Omega) = H^1(\mathrm{curl}^2,\Omega)$, trace space $\mathcal{T}(\Gamma) = H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

Cauchy trace $\gamma = \begin{bmatrix}\gamma_t\\ \gamma_N\end{bmatrix}$ $\Rightarrow$ BIOs $A_k = \begin{bmatrix}-C_k & -S_k\\ -S_k & -C_k\end{bmatrix}$

$\mathcal{L}_k \underline{\mathbf{u}} = 0$ in Ω
$\underline{\mathbf{u}}$ radiating $\quad\triangleright\quad \begin{bmatrix}\frac12 Id + C_k & S_k\\ S_k & \frac12 Id + C_k\end{bmatrix}\begin{bmatrix}\gamma_t\underline{\mathbf{u}}\\ \gamma_N\underline{\mathbf{u}}\end{bmatrix} = \begin{bmatrix}\gamma_t\underline{\mathbf{u}}\\ \gamma_N\underline{\mathbf{u}}\end{bmatrix}$

- PEC scattering: given $g = \gamma_t\underline{\mathbf{u}}$, seek $\mu \in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

  1st-kind BIE: $S_k(\mu) = (\frac12 Id - C_k)(g)$ [EFIE]

  ↳ T coercive variational formulation

II.2.2. Direct BIE for Simple BVPs

Maxwell case: $\mathcal{L}_k \mathbf{u} = \nabla\times(\nabla\times\mathbf{u}) - k^2\mathbf{u}$ + Silver-Müller r.c.

$H(\mathcal{L},\Omega) = H^1(\mathbf{curl}^2,\Omega)$, trace space $\mathcal{T}(\Gamma) = H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

Cauchy trace $\gamma = \begin{bmatrix}\gamma_t\\ \gamma_N\end{bmatrix}$ $\Rightarrow$ BIOs $\mathbb{A}_k = \begin{bmatrix}-C_k & -S_k\\ -S_k & -C_k\end{bmatrix}$

$\mathcal{L}_k\mathbf{u}=0$ in Ω
$\mathbf{u}$ radiating $\triangleright$ $\begin{bmatrix}\frac12 Id + C_k & S_k\\ S_k & \frac12 Id + C_k\end{bmatrix}\begin{bmatrix}\gamma_t\mathbf{u}\\ \gamma_N\mathbf{u}\end{bmatrix} = \begin{bmatrix}\gamma_t\mathbf{u}\\ \gamma_N\mathbf{u}\end{bmatrix}$

- PEC scattering: given $g = \gamma_t\mathbf{u}$, seek $\mu \in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

1st-kind BIE: $S_k(\mu) = (\frac12 Id - C_k)(g)$ [EFIE]

2nd-kind BIE: $(\frac12 Id - C_k)(\mu) = S_k(g)$ [MFIE]

$\hookrightarrow$ Variational formulation in $L^2_t(\Gamma)$
(coercive on C^2-boundaries)

III.2.3. Indirect BIE for Simple BVPs

Idea: $\boxed{\text{Use layer potential trial expression for unknown (radiating) solution of BVP}}$

[scalar case, $\mathcal{L}_K = -\Delta - k^2$]

III.2.3. Indirect BIE for Simple BVPs

Idea: $\boxed{\text{Use layer potential trial expression for unknown (radiating) solution of BVP}}$

[scalar case, $\mathcal{L}_k = -\Delta - k^2$]

- Single layer ansatz: $u = \Psi_{SL}^k(\varphi)$, $\varphi \in H^{-1/2}(\Gamma)$

unknown

III.2.3. Indirect BIE for Simple BVPs

Idea : $\boxed{\text{Use layer potential trial expression for unknown (radiating) solution of BVP}}$

[scalar case, $\mathcal{L}_\kappa = -\Delta - k^2$]

• Single layer ansatz : $u = \Psi_{SL}^{k}(\varphi)$, $\varphi \in H^{-1/2}(\Gamma)$
$\uparrow$ unknown

Dirichlet problem : given $g = \gamma u \in H^{1/2}(\Gamma)$

$\triangleright$ 1-st kind BIE : $V_\kappa(\varphi) = g$ → coercive variational form.

III.2.3. Indirect BIE for Simple BVPs

Idea: $\boxed{\text{Use layer potential trial expression for unknown (radiating) solution of BVP}}$

[scalar case, $\mathcal{L}_K = -\Delta - k^2$]

- Single layer ansatz: $u = \Psi_{SL}^k(\varphi)$, $\varphi \in H^{-1/2}(\Gamma)$

$\uparrow$ unknown

Dirichlet problem: given $g = \gamma u \in H^{1/2}(\Gamma)$

▷ 1-st kind BIE: $V_K(\varphi) = g$ → coercive variational form.

Neumann problem: given $\psi = \gamma_n u \in H^{-1/2}(\Gamma)$

▷ 2nd-kind BIE: $(\tfrac{1}{2}\mathrm{Id} + K')(\varphi) = \psi$

↳ variational formulation in $L^2(\Gamma)$

III.2.3. Indirect BIE for Simple BVPs

[scalar case, $\mathcal{L}_\kappa = -\Delta - k^2$]

- Double layer ansatz: $u = \Psi_{DL}^{k}(\mu)$, $\underset{\text{unknown}}{\mu} \in H^{1/2}(\Gamma)$

III.2.3. Indirect BIE for Simple BVPs

[scalar case, $\mathcal{L}_K = -\Delta - k^2$]

- Double layer ansatz: $U = \Psi_{DL}^k(u)$, $u \in H^{1/2}(\Gamma)$

  u unknown

Dirichlet problem: given $g = \gamma U \in H^{1/2}(\Gamma)$

$\triangleright$ 2nd-kind BIE: $(K - \frac{1}{2}Id)(u) = g$

$\hookrightarrow$ Variational formulation in $L^2(\Gamma)$

III.2.3. Indirect BIE for Simple BVPs

[scalar case, $\mathcal{L}_K = -\Delta - k^2$]

- Double layer ansatz: $U = \Psi_{DL}^{k}(u)$, $u \in H^{1/2}(\Gamma)$

unknown

Dirichlet problem: given $g = \gamma U \in H^{1/2}(\Gamma)$

▷ 2nd-kind BIE: $(K - \tfrac{1}{2}Id)(u) = g$

↳ Variational formulation in $L^2(\Gamma)$

Neumann problem: given $\varphi = \gamma_n U \in H^{-1/2}(\Gamma)$

▷ 1st-kind BIE: $-W_K(u) = \varphi$

↳ Coercive variational formulation in $H^{1/2}(\Gamma)$

III. 2.4 Combined Field Integral Equations (CFIE)

[Scalar case, $\Omega = \Omega^+ \hat{=}$ unbounded exterior domain]

▷ Existence & uniqueness of radiating solutions of exterior Dirichlet and Neumann problems on Ω.

III.2.4 Combined Field Integral Equations (CFIE)

[Scalar case, $\Omega = \Omega^+ \triangleq$ unbounded exterior domain]

▷ Existence & uniqueness of radiating solutions of exterior Dirichlet and Neumann problems on Ω.

Direct BIEs from Calderón identities:

Dirichlet BVP
- Direct 1st-kind: $V_k(\varphi) = (\tfrac{1}{2}Id + K_k)(g)$
- Direct 2nd-kind: $(\tfrac{1}{2}Id - K_k')(\varphi) = W_k(g)$

Neumann BVP
- Direct 1st-kind: $W_k(u) = (\tfrac{1}{2}Id - K_k)(\varphi)$
- Direct 2nd-kind: $(\tfrac{1}{2}Id + K_k)(u) = V_k(\varphi)$

III.2.4 Combined Field Integral Equations (CFIE)

[Scalar case, $\Omega = \Omega^+ \hat{=}$ unbounded exterior domain]

▷ Existence & uniqueness of radiating solutions of exterior Dirichlet and Neumann problems on Ω.

Direct BIEs from Calderón identities:

Dirichlet BVP:
- Direct 1st-kind: $V_k(\varphi) = (\tfrac{1}{2}Id + K_k)(g)$
- Direct 2nd-kind: $(\tfrac{1}{2}Id - K_k')(\varphi) = W_k(g)$

Neumann BVP:
- Direct 1st-kind: $W_k(u) = (\tfrac{1}{2}Id - K_k)(\varphi)$
- Direct 2nd-kind: $(\tfrac{1}{2}Id + K_k)(u) = V_k(\varphi)$

Resonances: $\exists$ sequence of wave numbers $(k_i)_{i\in\mathbb{N}}$, $k_i \to \infty$ for $i \to \infty$ and resonant Dirichlet modes $u_i \in H_0^1(\Omega^-)\setminus\{0\}$

$$(-\Delta - k_i^2)u_i = 0 \text{ in } \Omega^-, \quad \gamma u_i = 0 \text{ on } \Gamma$$

III.2.4 Combined Field Integral Equations (CFIE)

$u_i \neq 0$: $(-\Delta - k_i^2) u_i = 0$ in Ω^-, $\gamma u_i = 0$ on Γ

III.2.4 Combined Field Integral Equations (CFIE)

$u_i \neq 0$: $(-\Delta - k_i^2) u_i = 0$ in Ω^-, $\gamma u_i = 0$ on Γ

$\triangleright$
$$\begin{bmatrix} \tfrac{1}{2}Id - K_{k_i} & V_{k_i} \\ W_{k_i} & \tfrac{1}{2}Id + K'_{k_i} \end{bmatrix} \begin{bmatrix} 0 \\ \gamma_N u_i \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma_N u_i \end{bmatrix}$$

III.2.4 Combined Field Integral Equations (CFIE)

$u_i \neq 0$: $(-\Delta - k_i^2) u_i = 0$ in Ω^-, $\gamma u_i = 0$ on Γ

▷ $$\begin{bmatrix} \tfrac{1}{2}\mathrm{Id} - K_{k_i} & V_{k_i} \\ W_{k_i} & \tfrac{1}{2}\mathrm{Id} + K'_{k_i} \end{bmatrix} \begin{bmatrix} 0 \\ \gamma_N u_i \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma_N u_i \end{bmatrix}$$

▷ $$\underbrace{\gamma_N u_i}_{\neq 0} \in \mathcal{N}(V_{k_i}) \cap \mathcal{N}(\tfrac{1}{2}\mathrm{Id} - K'_{k_i})$$

non-trivial kernels

III.2.4 Combined Field Integral Equations (CFIE)

$u_i \neq 0$: $(-\Delta - k_i^2) u_i = 0$ in Ω^- , $\gamma u_i = 0$ on Γ

▷ $$\begin{bmatrix} \frac{1}{2}Id - K_{k_i} & V_{k_i} \\ W_{k_i} & \frac{1}{2}Id + K'_{k_i} \end{bmatrix} \begin{bmatrix} 0 \\ \gamma_N u_i \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma_N u_i \end{bmatrix}$$

▷ $$\underbrace{\gamma_N u_i}_{\neq 0} \in \mathcal{N}(V_{k_i}) \cap \mathcal{N}(\tfrac{1}{2}Id - K'_{k_i})$$

non-trivial kernels

▷ Non-uniqueness of solutions of BIEs

III.2.4 Combined Field Integral Equations (CFIE)

[Scalar case, $\Omega = \Omega^+ \mathrel{\hat{=}}$ unbounded exterior domain, Helmholtz BVPs in Ω^+]

Idea: Complex combination of 1st and 2nd-kind BIEs

$$(\tfrac{1}{2}Id - K_k)\gamma u + V_k \gamma_n u = \gamma u \qquad \cdot i\eta$$

$$W_k(\gamma u) + (\tfrac{1}{2}Id + K_k')(\gamma_n u) = \gamma_n u$$

$(\eta \in \mathbb{R}\setminus\{0\})$

III.2.4 Combined Field Integral Equations (CFIE)

[Scalar case, $\Omega = \Omega^+ \hat{=}$ unbounded exterior domain, Helmholtz BVPs in Ω^+]

Idea: Complex combination of 1st and 2nd-kind BIEs

$$\left(\tfrac{1}{2}Id - K_k\right)\gamma u + V_k \gamma_n u = \gamma u$$

$$W_k(\gamma u) + \left(\tfrac{1}{2}Id + K_k'\right)(\gamma_n u) = \gamma_n u \quad \cdot i\eta \quad (\eta \in \mathbb{R}\setminus\{0\})$$

$$\triangleright \quad \left(\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\underset{\substack{\uparrow \\ \text{unknown}}}{\gamma_n u}) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - W_k\right)(\underset{\substack{\uparrow \\ \text{given}}}{\gamma u})$$

[Dirichlet problem: unknown … given]

III.2.4 Combined Field Integral Equations (CFIE)

[Scalar case, $\Omega = \Omega^+ \hat{=}$ unbounded exterior domain, Helmholtz BVPs in Ω^+]

Idea: Complex combination of 1st and 2nd-kind BIEs

$$\left(\tfrac{1}{2}Id - K_k\right)\gamma u + V_k \gamma_n u = \gamma u$$

$$W_k(\gamma u) + \left(\tfrac{1}{2}Id + K_k'\right)(\gamma_n u) = \gamma_n u \qquad \cdot\, i\eta \quad (\eta \in \mathbb{R}\setminus\{0\})$$

$$\triangleright \quad \left(\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\underbrace{\gamma_n u}_{\text{unknown}}) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - W_k\right)(\underbrace{\gamma u}_{\text{given}})$$

[Dirichlet problem: unknown, given]

Thm: $-\tfrac{1}{2}Id + K_k' + i\eta V_k : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ is injective for all $k > 0$.

III.2.4 Combined Field Integral Equations (CFIE)

Scalar case: CFIE operator for Dirichlet problem

$-\frac{1}{2}\mathrm{Id} + K_k' + i\eta V_k \;\hat{=}\;$ 2nd-kind BIE

$\rightarrow$ Variational formulation in $L^2(\Gamma)$ (coercive on C^2-boundaries)

III.2.4 Combined Field Integral Equations (CFIE)

Scalar case: CFIE operator for Dirichlet problem

$$-\tfrac{1}{2}Id + K_k' + i\eta V_k \;\hat{=}\; \text{2nd-kind BIE}$$

$\rightarrow$ Variational formulation in $L^2(\Gamma)$ (coercive on C^2-boundaries)

"Trace space crime" created CFIE:

(I) $\quad (\tfrac{1}{2}Id - K_k)\gamma u + V_k \gamma_n u = \gamma u \quad \leftarrow$ in $H^{1/2}(\Gamma)$

(II) $\quad W_k(\gamma u) + (\tfrac{1}{2}Id + K_k')(\gamma_n u) = \gamma_n u \quad \leftarrow$ in $H^{-1/2}(\Gamma)$

$\cdot\, i\eta$ $\quad (\eta \in \mathbb{R}\setminus\{0\})$

III.2.4 Combined Field Integral Equations (CFIE)

Scalar case: CFIE operator for Dirichlet problem

$-\frac{1}{2}\mathrm{Id} + K_k' + i\eta V_k \quad \hat{=} \quad$ 2nd-kind BIE

$\rightarrow$ Variational formulation in $L^2(\Gamma)$ (coercive on C^2-boundaries)

"Trace space crime" created CFIE:

(I) $\left(\frac{1}{2}\mathrm{Id} - K_k\right)\gamma u + V_k \gamma_n u = \gamma u$ $\quad\leftarrow$ in $H^{1/2}(\Gamma)$

(II) $W_k(\gamma u) + \left(\frac{1}{2}\mathrm{Id} + K_k'\right)(\gamma_n u) = \gamma_n u$ $\quad\leftarrow$ in $H^{-1/2}(\Gamma)$

$\cdot i\eta$ $(\eta \in \mathbb{R}\setminus\{0\})$

Regularized CFIE: $i\eta \cdot (\mathrm{I}) + M(\mathrm{II})$

- $M : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$ compact
- $\mathrm{Re}\{\langle \varphi, M\overline{\varphi}\rangle_\Gamma\} > 0 \quad \forall \varphi \in H^{-1/2}(\Gamma)\setminus\{0\}$

  [positive definite]

III.2.4 Combined Field Integral Equations (CFIE)

Regularized CFIE for exterior Dirichlet problem (direct)

$$\left(M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\varphi) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - MW_k\right)(g)$$

unknown Neumann trace   Dirichlet data

III.2.4 Combined Field Integral Equations (CFIE)

Regularized CFIE for exterior Dirichlet problem (direct)

$$\left(M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\varphi) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - M W_k\right)(g)$$

unknown Neumann trace (φ) Dirichlet data (g)

<u>Thm</u>: $\boxed{M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma) \text{ is injective}}$

$\hookrightarrow$ bijective, since coercive!

III.2.4 Combined Field Integral Equations (CFIE)

Regularized CFIE for exterior Dirichlet problem (direct)

$$\left(M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\psi) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - M W_k\right)(g)$$

unknown Neumann trace — Dirichlet data

Thm: $\boxed{M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \text{ is injective}}$

$\hookrightarrow$ bijective, since coercive!

A concrete regularizing operator:

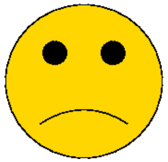
$$\mu \in H^{-1/2}(\Gamma) \Rightarrow M(\mu) \in H^1(\Gamma):$$

$$\langle \nabla_\Gamma M(\mu), \nabla_\Gamma v' \rangle_\Gamma + \langle M(\mu), v' \rangle_\Gamma = \langle \mu, v' \rangle_\Gamma \quad \forall v' \in H^1(\Gamma)$$

$\triangleright$ $M : H^{-1/2}(\Gamma) \to H^1(\Gamma)$ continuous

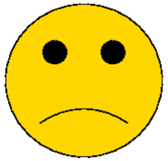
III.2.4 Combined Field Integral Equations (CFIE)

$$\left(M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\varphi) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - M W_k\right)(g)$$

 Products of non-local operators !

III.2.4 Combined Field Integral Equations (CFIE)

$$\left(M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\varphi) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - M W_k\right)(g)$$

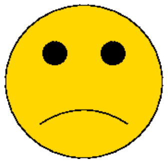


Products of non-local operators !

- Auxiliary variable $u := M\left(\left(-\tfrac{1}{2}Id + K_k'\right)\varphi + W_k g\right) \in H^1(\Gamma)$
- Mixed variational formulation :

III.2.4 Combined Field Integral Equations (CFIE)

$$\left(M\left(-\tfrac{1}{2}Id + K_k'\right) + i\eta V_k\right)(\varphi) = \left(i\eta\left(\tfrac{1}{2}Id + K_k\right) - M W_k\right)(g)$$



Products of non-local operators !

- Auxiliary variable $u := M\left(\left(-\tfrac{1}{2}Id + K_k'\right)\varphi + W_k g\right) \in H^1(\Gamma)$
- Mixed variational formulation:

$$\varphi \in H^{-1/2}(\Gamma),\quad u \in H^1(\Gamma)$$

$$i\eta\langle V_k\varphi, \varphi'\rangle_\Gamma + \langle u, \varphi'\rangle_\Gamma = i\eta\langle(\tfrac{1}{2}Id + K_k)g, \varphi'\rangle_\Gamma$$

$$\langle(\tfrac{1}{2}Id - K_k')\varphi, v\rangle + \langle\nabla_\Gamma u, \nabla_\Gamma v\rangle_\Gamma + \langle u, v\rangle = \langle W_k g, v\rangle$$

for all $\varphi' \in H^{-1/2}(\Gamma), v \in H^1(\Gamma)$

▷ $H^{-1/2}(\Gamma) \times H^1(\Gamma)$ - coercive