Bondary Element Methods: Derivation, Analysis, and Implementation

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Chapter III : Boundary Element Methods

Preface

BEM = FEM for BIE

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Remember: BIE in variational form

$\boxed{\text{1st-kind BIE (set in natural trace spaces)}}$

2nd-kind BIEs (set in $L^2(\Gamma)/L^2_t(\Gamma)$)

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Natural option : Galerkin discretization

Need finite-dimensional subspaces of trace spaces.

$$\left[\, H^{1/2}(\Gamma) \;-\; H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \;-\; H^{-1/2}(\Gamma) \,\right]$$

Focus : piecewise polynomial subspaces

III.1 Surface Meshes ($d=3$)

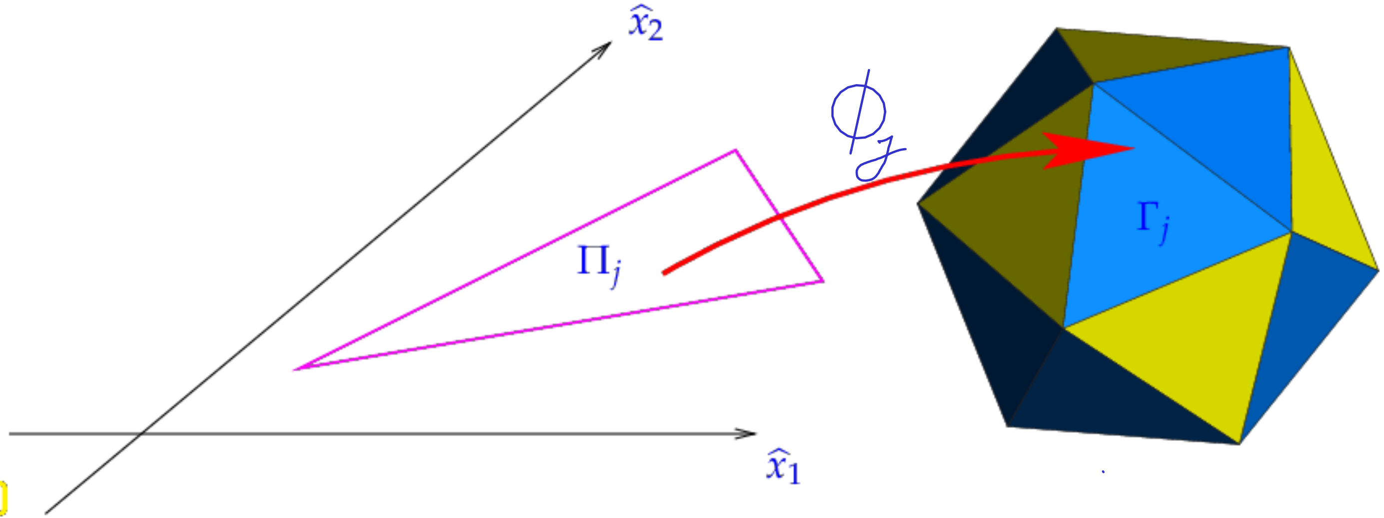

Fig. 41

$\Gamma \hat{=}$ surface of a curvilinear polyhedron

III.1 Surface Meshes $d=3$)

$\Pi_j \subset \mathbb{R}^2 \triangleq$ parameter domain

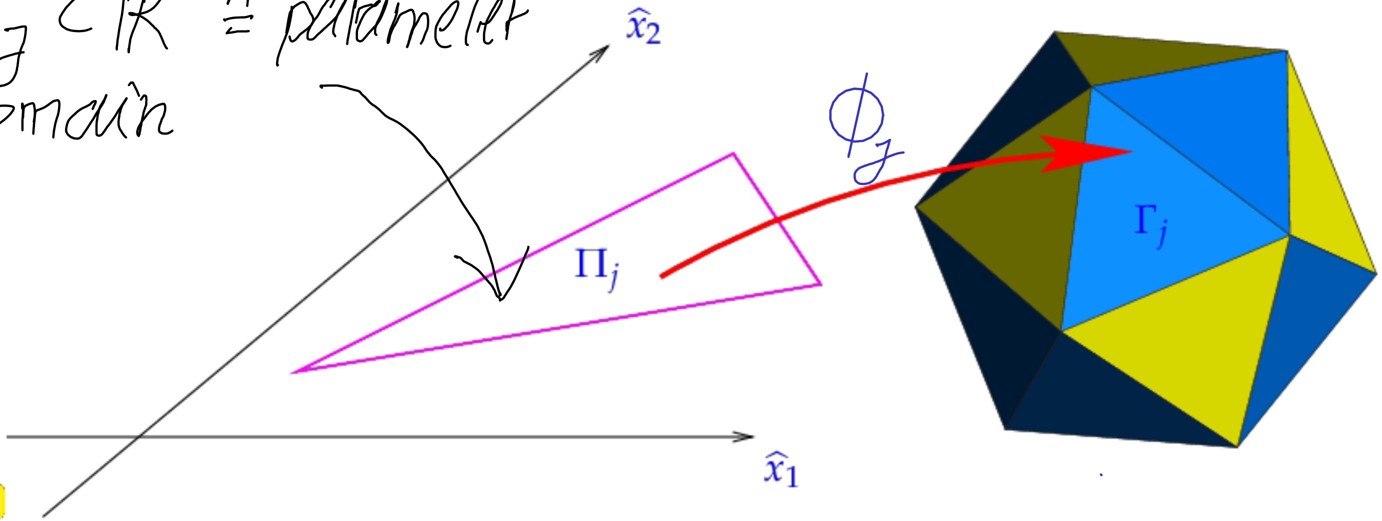

Fig. 41

$\Gamma \triangleq$ surface of a curvilinear polyhedron

▷ Each face has a smooth parameterization $\phi_j : \Pi_j \to \Gamma_j$

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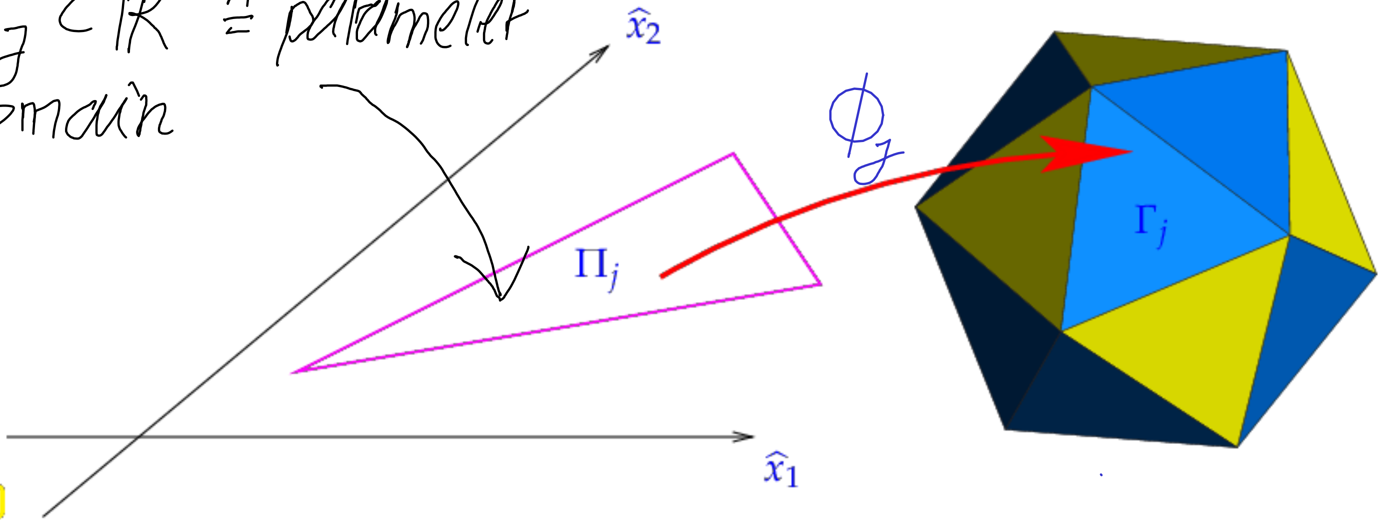

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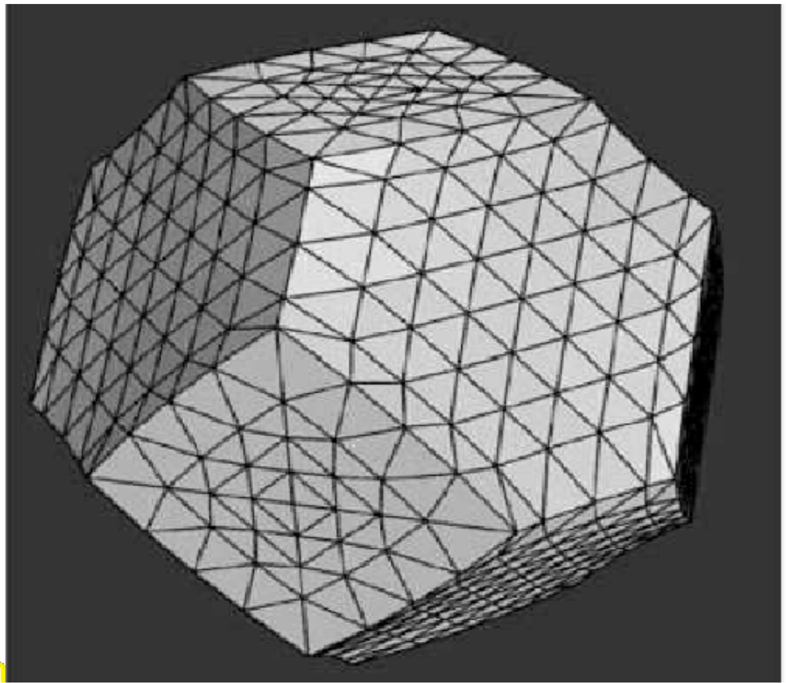
Fig. 44

Compatible triangulations of parameter domains

$\Downarrow$

conforming surface triangulations

III.2. (Lowest-order) BE from FE

Traces of FE spaces
$\Downarrow$
BE for trace spaces

[Assume: surface mesh $\mathcal{T}_h$ is trace of a volume mesh]

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III.2.1 Discrete Differential Forms in 3D

↳ Concrete: Simplicial Whitney forms

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Traces of FE spaces

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BE for trace spaces

[Assume: surface mesh $\mathcal{T}_h$ is trace of a volume mesh]

III.2.1 Discrete Differential Forms in 3D

↳ Concrete: Simplicial Whitney forms

Recall exterior calculus perspective:

function spaces $\longleftrightarrow$ spaces of differential forms

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla"] \arrow[d, leftrightarrow] & H(\mathbf{curl},\Omega) \arrow[r, "\mathbf{curl}"] \arrow[d, leftrightarrow] & H(\mathrm{div},\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, leftrightarrow] & L^2(\Omega) \arrow[d, leftrightarrow] \\
0\text{-forms} & 1\text{-forms} & 2\text{-forms} & 3\text{-forms}
\end{tikzcd}$$

III.2.1 Discrete Differential Forms in 3D

[Lowest order]

Local trial/test spaces on simplex T:

$\ell = 0$: $\quad W^0(T) = \{ x \mapsto a \cdot x + \beta,\ a \in \mathbb{R}^3, \beta \in \mathbb{R} \}$

$\ell = 1$: $\quad W^1(T) = \{ x \mapsto a \times x + b,\ a, b \in \mathbb{R}^3 \}$

$\ell = 2$: $\quad W^2(T) = \{ x \mapsto \alpha x + b,\ \alpha \in \mathbb{R}, b \in \mathbb{R}^3 \}$

$\ell = 3$ $\quad W^3(T) = \{ x \mapsto \alpha,\ \alpha \in \mathbb{R} \}$

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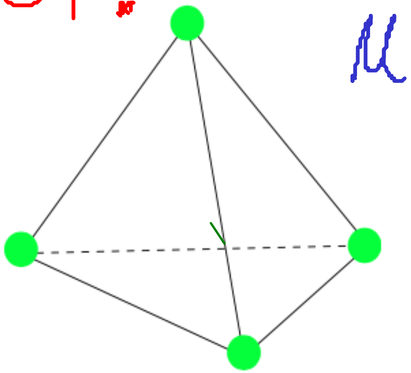
▷ (Incomplete) $\mathcal{P}_1$ spaces

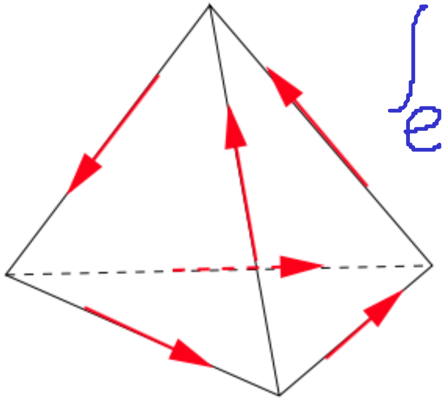
III.2.1 Discrete Differential Forms in 3D

[Lowest order]

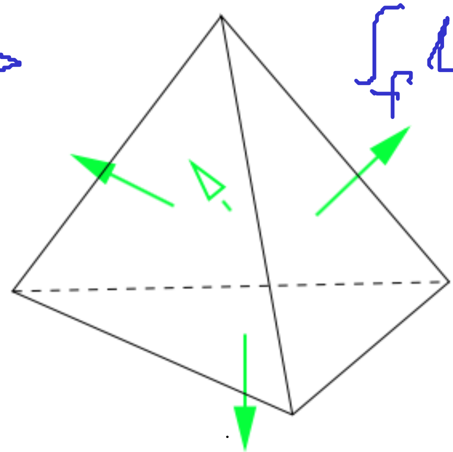
Dof:



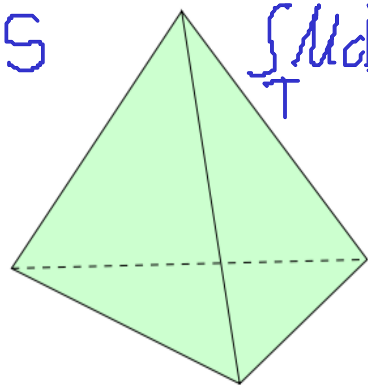
$u(x_i)$



$\int_e u\cdot d\vec{s}$



$\int_f u\cdot n\,dS$



$\int_T u\,dx$

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▷ (Incomplete) $\mathcal{P}_1$ spaces

III.2.1 Discrete Differential Forms in 3D

[Lowest order]

$\mathcal{T}_h \hat{=}$ simplicial mesh of polyhedron Ω

$W^\ell(\mathcal{T}_h) \hat{=} \mathrm{Span}\{ f_{|T} \in W^\ell(T), \text{d.o.f. unique} \}$

d

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dof ensure $\begin{cases} C^0\text{-continuity} & (\ell=0) \\ \text{tangential cont.} & (\ell=1) \\ \text{normal cont.} & (\ell=2) \end{cases} \Rightarrow \begin{array}{l} W^0(\mathcal{T}_h) \subset H^1(\Omega) \\ W^1(\mathcal{T}_h) \subset H(\mathrm{curl}, \Omega) \\ W^2(\mathcal{T}_h) \subset H(\mathrm{div}, \Omega) \end{array}$

5

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d.o.f. $\Rightarrow$ local finite element projectors

shape function

$\ell = 0: \quad \Pi_h^0 V = \sum_i V(x_i)\, b_i$

$\ell = 1: \quad \Pi_h^1 V = \sum_{ij} \int_{[x_i, x_j]} V \cdot d\vec{s}\ b_{ij}, \quad V \in C^0(\bar{\Omega})$

$\ell = 2: \quad \Pi_h^2 V = \sum_{i,j,k} \int_{[x_i, x_j, x_k]} V \cdot n\, dS \cdot b_{ijk}$

III.2.2. Discrete DeRham Complex

$$
\begin{tikzcd}
W^0(\mathcal{T}_h) \arrow[r, "\nabla"] & W^1(\mathcal{T}_h) \arrow[r, "\nabla\times"] & W^2(\mathcal{T}_h) \arrow[r, "\nabla\cdot"] & W^3(\mathcal{T}_h) \\
C^\infty(\overline{\Omega}) \arrow[u, "\Pi_h^0"'] \arrow[r, "\nabla"] & C^\infty(\overline{\Omega}) \arrow[u, "\Pi_h^1"'] \arrow[r, "\nabla\times"] & C^\infty(\overline{\Omega}) \arrow[u, "\Pi_h^2"'] \arrow[r, "\nabla\cdot"] & C^\infty(\overline{\Omega}) \arrow[u, "\Pi_h^3"']
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\end{tikzcd}
$$

$\Rightarrow$ Commuting diagram property

$$
\Longleftrightarrow
\begin{cases}
\Pi^1 \circ \nabla = \nabla \circ \Pi^0 \\
\Pi^2 \circ \nabla\times = \nabla\times \circ \Pi^1 \\
\Pi^3 \circ \nabla\cdot = \nabla\cdot \circ \Pi^2
\end{cases}
$$

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$$\Longleftrightarrow \begin{cases} \Pi^1 \circ \nabla = \nabla \circ \Pi^0 \\ \Pi^2 \circ \nabla\times = \nabla\times \circ \Pi^1 \\ \Pi^3 \circ \nabla\cdot = \nabla\cdot \circ \Pi^2 \end{cases}$$

Projectors Π_h^ℓ unbounded on $\begin{cases} H^1(\Omega), & \ell = 0 \\ H(\mathrm{curl},\Omega), & \ell = 1 \\ H(\mathrm{div},\Omega), & \ell = 2 \end{cases}$

III.2.3. Surface Whitney Forms

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla"] \arrow[d, "\gamma"'] & H(\mathbf{curl}, \Omega) \arrow[r, "\nabla\times"] \arrow[d, "\gamma_t"'] & H(\mathrm{div}, \Omega) \arrow[r, "\nabla\cdot"] \arrow[d, "\gamma_n"'] & L^2(\Omega) \\
H^{1/2}(\Gamma) \arrow[r, "\nabla_\Gamma\times"] & H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \arrow[r, "\nabla_\Gamma\cdot"] & H^{-1/2}(\Gamma) &
\end{tikzcd}$$

$\leftarrow$ continuous & surjective traces

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$\leftarrow$ continuous & surjective traces

$$\Downarrow$$

$$
\begin{tikzcd}
\mathcal{W}^0(\mathcal{T}_h) \arrow[r, "\nabla"] \arrow[d, "\gamma"'] & \mathcal{W}^1(\mathcal{T}_h) \arrow[r, "\nabla\times"] \arrow[d, "\gamma_t"'] & \mathcal{W}^2(\mathcal{T}_h) \arrow[r, "\nabla\cdot"] \arrow[d, "\gamma_n"'] & \mathcal{W}^3(\mathcal{T}_h) \\
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\end{tikzcd}
$$

$\leftarrow$ continuous & surjective traces

$\mathcal{T}_h \;\hat{=}\;$ simplicial mesh of Ω : $\quad \Gamma_h := \mathcal{T}_h|_\Gamma$

III.2.3. Surface Whitney Forms

$$
\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla"] \arrow[d, phantom, "\cup"] & H(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] \arrow[d, phantom, "\cup"] & H(\mathrm{div},\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, phantom, "\cup"] & L^2(\Omega) \arrow[d, phantom, "\cup"] \\
W^0(\mathcal{T}_h) \arrow[r, "\nabla"] \arrow[d, "\gamma"'] & W^1(\mathcal{T}_h) \arrow[r, "\nabla\times"] \arrow[d, "\gamma_t"'] & W^2(\mathcal{T}_h) \arrow[r, "\nabla\cdot"] \arrow[d, "\gamma_n"'] & W^3(\mathcal{T}_h) \\
W^0(\Gamma_h) \arrow[r, "\nabla_\Gamma\times"] \arrow[d, phantom, "\cap"] & W^1(\Gamma_h) \arrow[r, "\nabla_\Gamma\cdot"] \arrow[d, phantom, "\cap"] & W^2(\Gamma_h) \arrow[d, phantom, "\cap"] & \\
H^{1/2}(\Gamma_h) & H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) & H^{-1/2}(\Gamma) &
\end{tikzcd}
$$

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\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla"] & H(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] & H(\mathrm{div},\Omega) \arrow[r, "\nabla\cdot"] & L^2(\Omega) \\
\cup & \cup & \cup & \cup \\
W^0(\mathcal{T}_h) \arrow[r, "\nabla"] \arrow[d, "\gamma"'] & W^1(\mathcal{T}_h) \arrow[r, "\nabla\times"] \arrow[d, "\gamma_t"'] & W^2(\mathcal{T}_h) \arrow[r, "\nabla\cdot"] \arrow[d, "\gamma_n"'] & W^3(\mathcal{T}_h) \\
W^0(\Gamma_h) \arrow[r, "\nabla_\Gamma\times"] & W^1(\Gamma_h) \arrow[r, "\nabla_\Gamma\cdot"] & W^2(\Gamma_h) & \\
\cap & \cap & \cap & \\
H^{1/2}(\Gamma_h) & H^{-1/2}(\mathrm{div}_\Gamma,\Gamma) & H^{-1/2}(\Gamma) &
\end{tikzcd}
$$

$W^0(\Gamma_h) \triangleq \Gamma_h$-p.w. linear C^0-BE space

$W^2(\Gamma_h) \triangleq \Gamma_h$-p.w. constant BE space

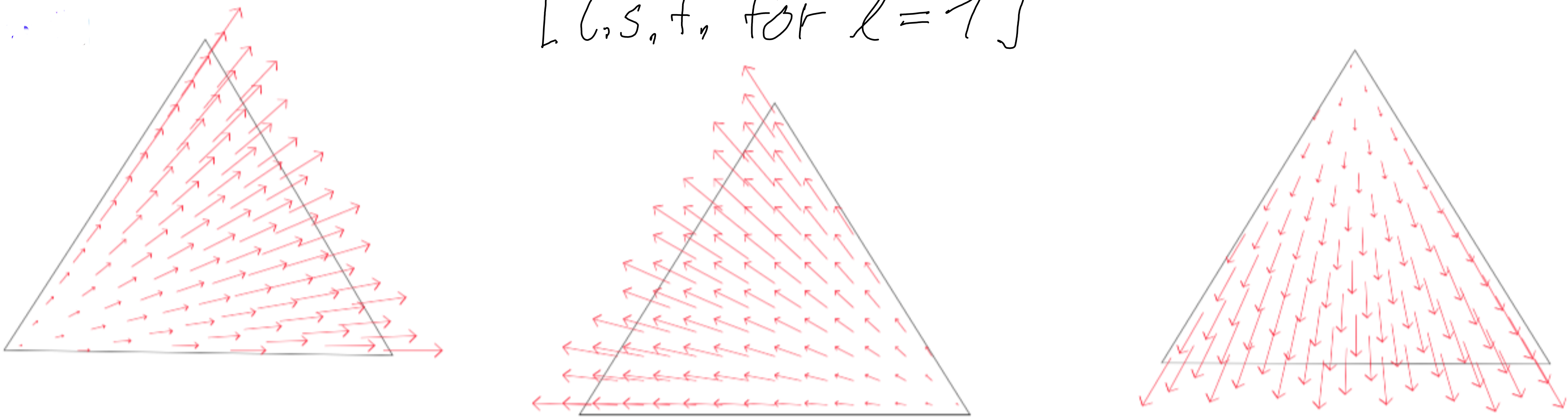
$W^1(\Gamma_h) \triangleq$ Surface RT_0-elements, RWG space

III.2.3. Surface Whitney Forms

Local shape functions

= traces of volume l.s.f. associated with T

[l.s.f. for $\ell = 1$]



D.o.f.

= Volume d.o.f. w/ domain of integration $\subset T$

▷ Local projectors $\Pi^\ell : C^0(T) \to W^\ell(\mathcal{T}_h)$

III.2.3. Surface Whitney Forms

Local shape functions

= traces of volume l.s.f. associated with Γ

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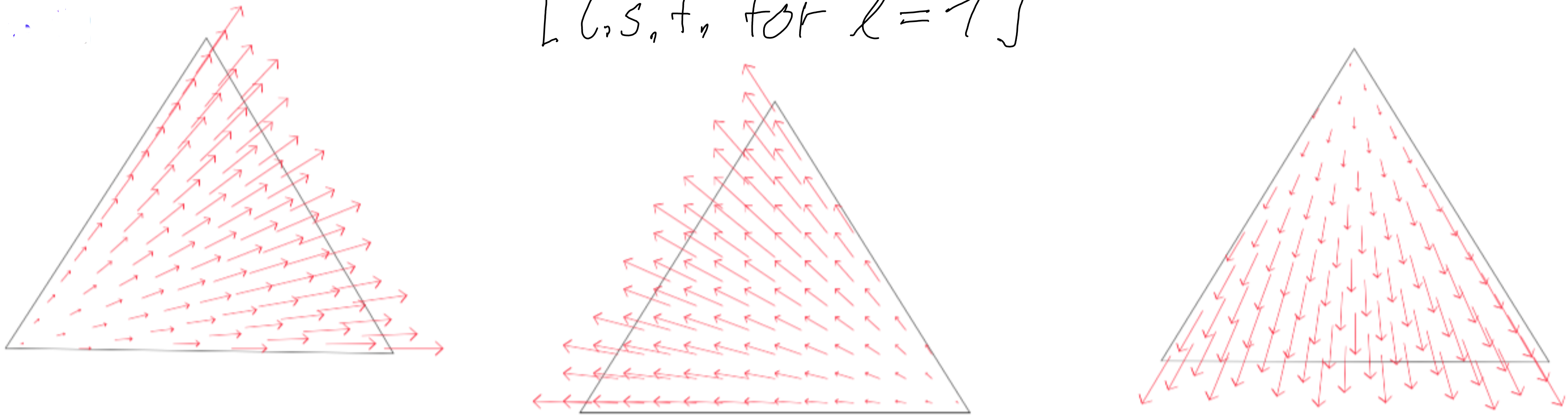

III.2.4. $W^\ell(\mathcal{T}_h)$: Parametric Construction

$\hat{\mathcal{J}}_j \hat{=}$ triangulation of $\hat{\Pi}_j$

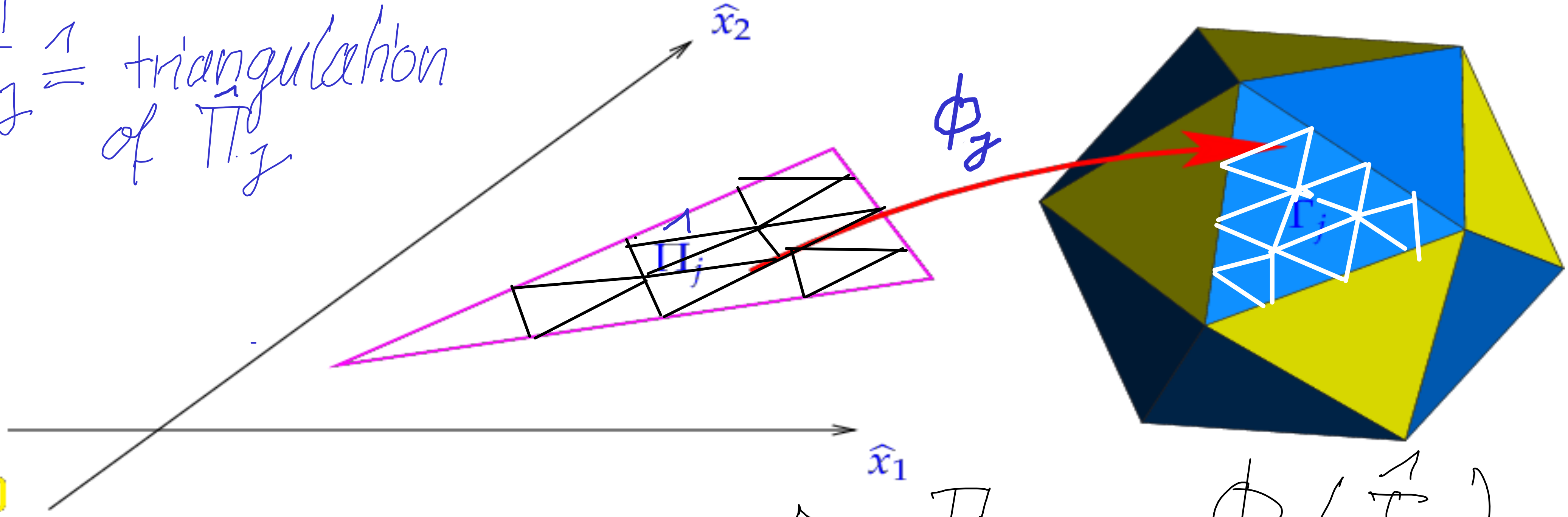

Fig. 41

$\triangleright$ $\mathcal{T}_h|_{\Gamma_j} = \Phi_j(\hat{\mathcal{J}}_j)$

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$\hat{\mathcal{T}}_j \hat{=}$ triangulation of $\hat{\Pi}_j$

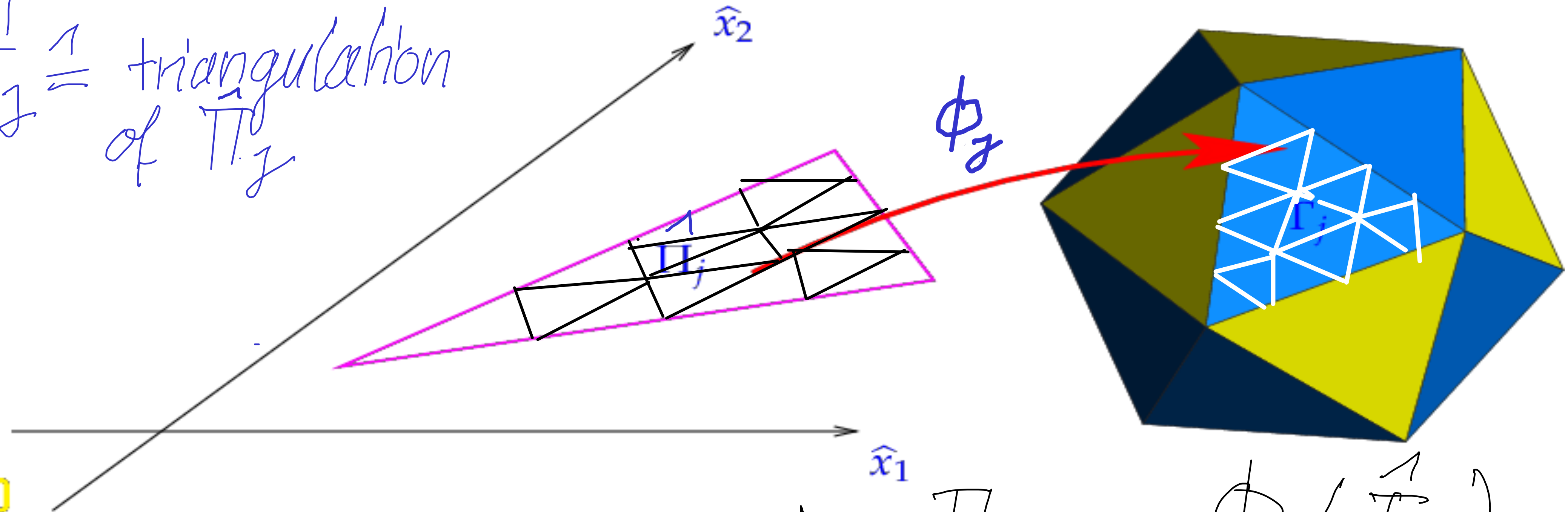

Fig. 41

$\triangleright\ \mathcal{T}_h|_{\Gamma_j} = \phi_j(\hat{\mathcal{T}}_j)$

Transformations:

$\ell = 0$: $\quad u(x) = \hat{u}(\hat{x})$

$\ell = 1$: $\quad \underline{u}(x) = g^{-1/2}(\hat{x})\, D\phi(\hat{x})\, \underline{\hat{u}}(\hat{x})$

$\ell = 2$: $\quad \varphi(x) = g^{-1}(\hat{x})\, \hat{\varphi}(\hat{x})$

$[\ G(\hat{x}) = D\phi(\hat{x})^T D\phi(\hat{x}),\quad g(\hat{x}) = \det G(\hat{x}),\ x = \phi(\hat{x})\ ]$

III.2.5. $W^{\ell}(\mathcal{T}_h)$: Approximation Properties

- "Inherent" estimates from volume
- Mapping techniques
- Interpolation in Sobolev scale

III.2.5. $W^\ell(\mathcal{T}_n)$: Approximation Properties

- "Inherit" estimates from volume
- Mapping techniques
- Interpolation in Sobolev scale

▷ $\ell = 0$; [$C > 0$ depends on shape regularity]

$$\inf_{v_n \in W^0(\mathcal{T}_n)} \| u - u_n \|_{H^{1/2}(\Gamma)} \leq C h^{s-1/2} |u|_{H^s(\Gamma)}$$

$$[\, 1/2 \leq s \leq 3/2 \,]$$

III.2.5. $W^\ell(\mathcal{T}_h)$: Approximation Properties

- "Inherit" estimates from volume
- Mapping techniques
- Interpolation in Sobolev scale

▷ $\ell = 0$: [$C > 0$ depends on shape regularity]

$$\inf_{v_h \in W^0(\mathcal{T}_h)} \|u - u_n\|_{H^{1/2}(T)} \le C h^{s-1/2} |u|_{H^s(T)}$$

$$\left[\tfrac{1}{2} \le s \le \tfrac{3}{2}\right]$$

▷ $\ell = 1$:

$$\inf_{\underline{v}_h \in W^1(\mathcal{T}_h)} \|\underline{u} - \underline{v}_h\|_{H^{-1/2}(\mathrm{div}_\Gamma, T)} \le C h^{s+1/2} \left(\|\underline{u}\|_{H^s(T)} + \|\nabla_\Gamma \cdot \underline{u}\|_{H^s(T)} \right)$$

$$\left[-\tfrac{1}{2} \le s \le \tfrac{1}{2}\right]$$

III.2.5. $W^{\ell}(\mathcal{T}_n)$: Approximation Properties

- "Inherit" estimates from volume
- Mapping techniques
- Interpolation in Sobolev scale

▷ $\ell = 0$: [$C>0$ depends on shape regularity]

$$\inf_{v_n \in W^0(\Gamma_n)} \|u - u_n\|_{H^{1/2}(\Gamma)} \leq C h^{s-1/2} |u|_{H^s(\Gamma)}$$

$[\tfrac{1}{2} \leq s \leq \tfrac{3}{2}]$

▷ $\ell = 1$:

$$\inf_{\mathbf{v}_n \in W^1(\Gamma_n)} \|\mathbf{u} - \mathbf{v}_n\|_{H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \leq C h^{s+1/2} \left(\|\mathbf{u}\|_{H^s(\Gamma)} + \|\nabla_\Gamma \cdot \mathbf{u}\|_{H^s(\Gamma)} \right)$$

$[-\tfrac{1}{2} \leq s \leq \tfrac{1}{2}]$

▷ $\ell = 2$:

$$\inf_{\varphi_h \in W^2(\Gamma_h)} \|\psi - \varphi_h\|_{H^{-1/2}} \leq C h^{s+1/2} \|\psi\|_{H^s(\Gamma)}$$

$[-\tfrac{1}{2} \leq s \leq 1]$

III.3. Discrete inf-sup Conditions

III.3.1. Galerkin Discretization of Coercive V.P.

$(W_h)_{h\in\mathbb{H}} \stackrel{\Delta}{=}$ asymptotically dense family of finite-dimensional subspaces $W_h \subset H$

$A : H \times H \to \mathbb{C}$: bounded sesqui-linear form

$\hookrightarrow$ injective : $A(u,v) = 0 \;\; \forall v \in H \implies u = 0$

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$\mathcal{K} : H\times H \longrightarrow \mathbb{C}$: compact sesqui-linear form

Garding inequ. : $|\mathcal{A}(u,\bar{u}) + \mathcal{K}(u,\bar{u})| \geq c\|u\|_H^2 \;\forall u$

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Garding inequ. : $|\mathsf{A}(u,\bar{u}) + \mathsf{K}(u,\bar{u})| \geq c \|u\|_H^2 \;\; \forall u$

$\triangleright \;\; \exists \; \begin{matrix} h_0 > 0 \\ c > 0 \end{matrix} : \; \sup_{v_h \in W_h} \dfrac{\mathsf{A}(u_h, v_h)}{\|v_h\|_H} \geq c \|u_h\|_H^2 \quad \forall u_h \in W_h$

III.3.1. Galerkin Discretization of Coercive V.P.

Proof:

$\mathcal{A} \Longleftrightarrow A : H \longrightarrow H'$ bijective

$\mathcal{K} \Longleftrightarrow K : H \longrightarrow H'$ compact

▷ Continuous "inf-sup candidate": $v(u) := u + A^{-1}Ku$

$P_h : H \longrightarrow W_h \stackrel{\wedge}{=}$ orthogonal projections

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▷ Continuous "inf-sup candidate": $v(u) := u + A^{-1}Ku$

$P_h : H \longrightarrow W_h \ \hat{=}$ orthogonal projections

▷ Discrete "inf-sup candidate"

$$v_h(u_h) = u_h + P_h A^{-1} K u_h$$

III.3.1. Galerkin Discretization of Coercive V.P.

Proof:

$\mathcal{A} \Leftrightarrow A : H \to H'$ bijective

$\mathcal{K} \Leftrightarrow K : H \to H'$ compact

▷ Continuous "inf-sup candidate": $v(u) := u + A^{-1}Ku$

$P_h : H \to W_h \;\hat{=}\;$ orthogonal projections

▷ Discrete "inf-sup candidate"

$$v_h(u_h) = u_h + P_h A^{-1} K u_h$$

$[A^{-1}K \text{ compact}]$

$$Id - P_h \xrightarrow{h \to 0} 0 \text{ pointwise} \Longrightarrow \|(Id - P_h)A^{-1}K\| \xrightarrow{h \to 0} 0$$

$$|\mathcal{A}(u_h, v_h(u_h))| \geq |\mathcal{A}(u_h, v(u_h))| - |\mathcal{A}(u_h, (Id - P_h)A^{-1}Ku_h)|$$

III.3.2. Galerkin Discretization of T-Coercive V.P.

(14)

$(W_h)_{h \in \mathbb{H}} \triangleq$ asymptotically dense family of finite-dimensional subspaces $W_h \subset H$

$A : H \times H \to \mathbb{C}$: bounded sesqui-linear form

$\hookrightarrow$ injective : $A(u,v) = 0 \quad \forall v \in H \implies u = 0$

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$K : H \times H \to \mathbb{C}$: compact sesqui-linear form

$T : H \to H$: bijective linear operator

T-coercivity : $|A(u, T\bar{u}) + K(u,\bar{u})| \geq c \|u\|_H^2 \;\forall u$

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T-coercivity : $|\mathsf{A}(u,\mathsf{T}\overline{u}) + \mathsf{K}(u,\overline{u})| \geq c\|u\|_H^2 \;\forall u$

$\triangleright$ ~~$\exists \begin{matrix} h_0>0 \\ c>0 \end{matrix} : \sup\limits_{v_h\in W_h} \dfrac{\mathsf{A}(u_h,v_h)}{\|v_h\|_H} \geq c\|u_h\|_H^2 \quad \forall u_h\in W_h$~~

III.3.2. Galerkin Discretization of T-Coercive V.P.

Stalled proof:

▷ Continuous "inf-sup candidate": $v(u) := Tu + A^{-1}Ku$

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▷ Discrete "inf-sup candidate"

$$\cancel{v_h(u_h) = Tu_h + P_h A^{-1}Ku_h}$$

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$$v_h(u_h) = \widetilde{P_h} T u_h + P_h A^{-1} K u_h$$

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[$A^{-1}K$ compact]

$Id - P_h \xrightarrow{h\to 0} 0$ pointwise $\Longrightarrow \|(Id-P_h)A^{-1}K\| \xrightarrow{h\to 0} 0$

$$|\mathsf{A}(u_h, v_h(u_h))| \geq |\mathsf{A}(u_h, v(u_h))| - |\mathsf{A}(u_h, (Id-\tilde{P}_h)(T + A^{-1}K)u_h)|$$

Required: $\|(Id-\tilde{P}_h)T\| \xrightarrow{h\to 0} 0$ on W_h

III.4 Edge BEM for EFIE

III.4.1 Discrete Variational Problem

EFIE : Seek $\underline{\mu} \in H := H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

$$\langle S_k(\underline{\mu}), \underline{v} \rangle_{t,\Gamma} = \underbrace{k \langle \underline{V}_k \underline{\mu}, \underline{v} \rangle_\Gamma - \frac{1}{k} \langle V_k (\nabla_\Gamma \cdot \underline{\mu}), \nabla_\Gamma \cdot \underline{v} \rangle_\Gamma}_{=: \mathcal{A}(\underline{\mu}, \underline{v})} = \ldots \quad \forall \underline{v} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

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$H_h = W^1(\mathcal{T}_h) \stackrel{\Delta}{=}$ Surface Whitney 1-forms
↳ asymptotically dense for $h \to 0$

▷ Seek $\underline{\mu}_h \in W^1(\mathcal{T}_h)$

$$\mathcal{A}(\underline{\mu}_h, \underline{v}_h) = -\langle \gamma_t E_{inc}, \underline{v}_h\rangle_{t,\Gamma}, \quad \forall \underline{v}_h \in W^1(\mathcal{T}_h)$$

III.4.2: EFIE: T-coercivity recalled

for $\underline{\mu}, \underline{v} \in H := H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

$$\mathcal{A}(\underline{\mu}, \underline{v}) = k \langle \underline{V}_k \underline{\mu}, \underline{v} \rangle_\Gamma - \frac{1}{k} \langle V_k \nabla_\Gamma \cdot \underline{\mu}, \nabla_\Gamma \cdot \underline{v} \rangle$$

$$T := \mathrm{Id} - 2R : H \longrightarrow H$$

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$$\shortparallel$$

- $T := \mathrm{Id} - 2R : H \longrightarrow H$

with projection $R : H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ (bounded)

$\nabla_\Gamma \cdot R(\underline{\eta}) = \nabla_\Gamma \cdot \underline{\eta}$ ($\nabla_\Gamma \cdot$ preserving)

$R : H^{-1/2}(\mathrm{div}, \Gamma) \longrightarrow \underline{L}^2_t(\Gamma)$ compact

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for $\underline{\mu}, \underline{\nu} \in H := H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

$$\mathcal{A}(\underline{\mu}, \underline{\nu}) = k \langle V_k \underline{\mu}, \underline{\nu} \rangle_\Gamma - \frac{1}{k} \langle V_k \nabla_\Gamma \cdot \underline{\mu}, \nabla_\Gamma \cdot \underline{\nu} \rangle$$

- $(\underline{\mu}, \underline{\nu}) \longrightarrow \langle \underline{V}_k \underline{\mu}, \underline{\nu} \rangle_\Gamma$ is $H_t^{-1/2}(\Gamma)$-elliptic
- $(\varphi, \psi) \longrightarrow \langle V_k \varphi, \psi \rangle_\Gamma$ is $H^{-1/2}(\Gamma)$-elliptic
- $$T := \mathrm{Id} - 2R : H \longrightarrow H$$

with projection $R : H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ (bounded)

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$R : H^{-1/2}(\mathrm{div}, \Gamma) \longrightarrow \underline{L}_t^2(\Gamma)$ compact

III.4.3. EFIE: Discrete inf-sup Candidate

Goal: Find $\tilde{P}_h : H \to W_x'(\mathcal{J}_h)$: $\|(Id - \tilde{P}_h)T\| \xrightarrow{h \to 0} 0$

[Shape-regular family of meshes assumed]

III.4.3. EFIE: Discrete inf-sup Candidate

Goal: Find $\tilde{P}_h : H \longrightarrow W_x^1(\mathcal{J}_h)$: $\|(Id - \tilde{P}_h)T\| \xrightarrow{h \to 0} 0$

[Shape-regular family of meshes assumed]

Try $\tilde{P}_h = \Pi_h^1$ [local projectors]

$$(Id - \Pi_h^1)(2R - Id) = 2(Id - \Pi_h^1)R$$

III.4.3. EFIE: Discrete inf-sup Candidate

Goal: Find $\tilde{P}_h : H \to W_x^1(\mathcal{T}_h)$: $\|(Id - \tilde{P}_h)T\| \xrightarrow{h \to 0} 0$

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Try $\tilde{P}_h = \Pi_h^1$ [local projectors]

$$(Id - \Pi_h^1)(2R - Id) = 2(Id - \Pi_h^1)R .$$

Lemma: If $\underline{\varphi} \in H_x^s(\Gamma)$, $\nabla_\Gamma \cdot \underline{\varphi} \in W^2(\mathcal{T}_h)$,

$$\|(Id - \Pi_h^1)\underline{\varphi}\|_{L^2(\Gamma)} \le C\, h^s \, \|\underline{\varphi}\|_{H_x^s}$$

with $C > 0$ independent of $\underline{\varphi}$ and h.

▷ $\|(Id - \Pi_h^1)R\,\underline{\mu}\|_H \le \|(Id - \Pi_h^1)R\,\underline{\mu}\|_{L^2(\Gamma)} \lesssim h^s \, \|R\,\underline{\mu}\|_{H_x^{1/2}(\Gamma)}$

III. 5. Assembly of Galerkin BE Matrices

III. 5. 1. Singular Integration

Surface / curve triangulation $\mathcal{T}_h = \{\pi_j\}_{j=1}^{M}$

▷ Entries of BE Galerkin matrix $A \in \mathbb{C}^{N,N}$ by summing terms of the form

$$\underbrace{\int_{\pi_\ell} \int_{\pi_j}}_{} K(x,y)\, p(x) q(x)\, dS(x,y) \quad , \quad 1 \leq \ell, j \leq M$$

↳ singular for $x = y$

↳ for all pairs!

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K singular for $x = y$

[e.g.: $k(x,y) = \dfrac{e^{ik\|x-y\|}}{4\pi\|x-y\|}$, $k(x,y) = \dfrac{(x-y)\cdot n(y)}{4\pi\|x-y\|^d}$]

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Singularity subtraction $\dfrac{e^{ik\|z\|}-1}{\|z\|} = \|z\|\,\underbrace{R_1(\|z\|^2) + R_2(\|z\|^2)}_{\text{analytic}}$

III.5.1. Prelude: Collocation for Single layer BIO

(20)

BIE: $$V_\Gamma \varphi(x) = \int_\Gamma \frac{1}{4\pi \|x-y\|} \varphi(y)\, dS(y) = \underbrace{g(x)}_{\in C^0(\Gamma)}, \quad x \in \Gamma$$

$[\ \varphi \in H^{-1/2}(\Gamma)\ ]$

Trial space: $\varphi \approx \varphi_h \in W^2(\Gamma_h)$

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Trial space: $\varphi \approx \varphi_h \in \mathcal{W}^2(\Gamma_h)$

Collocation conditions: [$d=3$]

$$(V\varphi_h)(c_e) = \int_\Gamma \frac{\varphi(y)}{4\pi\|c_e-y\|}\,dS(y) = g(c_e)$$

$\forall c_e \in \{$ centers of panels of $\Gamma_h\}$

III.5.1. Prelude: Collocation for Single layer BIO

20

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$\forall c_\ell \in \{$ centers of panels of $\Gamma_h \}$

▷ Compute $\displaystyle\int_{\pi_j} \frac{1}{4\pi\|c_\ell - y\|}\, dS(y)$

↳ singular for $c_\ell \in \pi_j$!

III.5.1. Prelude: Collocation for Single layer BIO

Compute

$$\int_{\pi_y} \frac{1}{4\pi\|y\|}\, dS(y) \quad ?$$

[O = center of π_y]

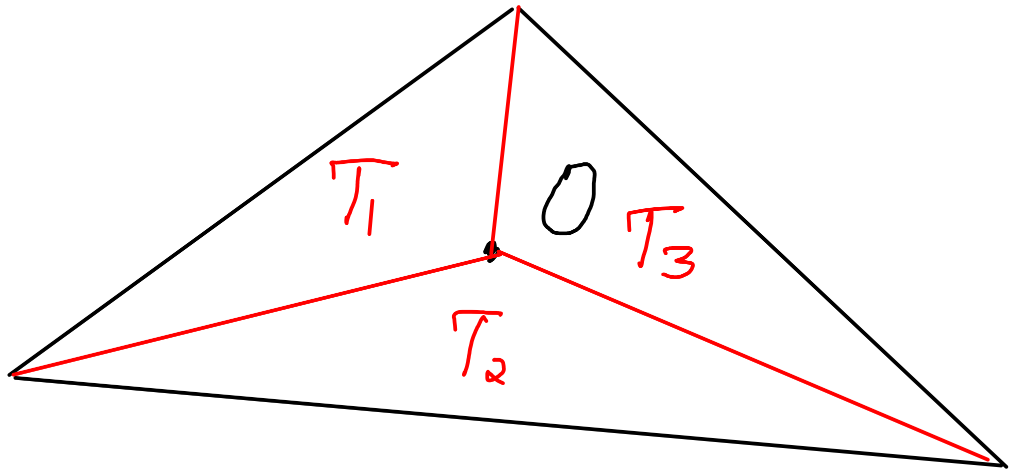
III.5.1. Prelude: Collocation for Single layer BIO

Compute $\displaystyle\int_{\pi_y} \frac{1}{4\pi\|y\|}\, dS(y)$?

[O = center of π_y]

▷ Split triangle π_y

$$\int_{\pi_y} \dots dS(y) = \sum_{\ell=1}^{3} \int_{T_\ell} \dots dS(y)$$



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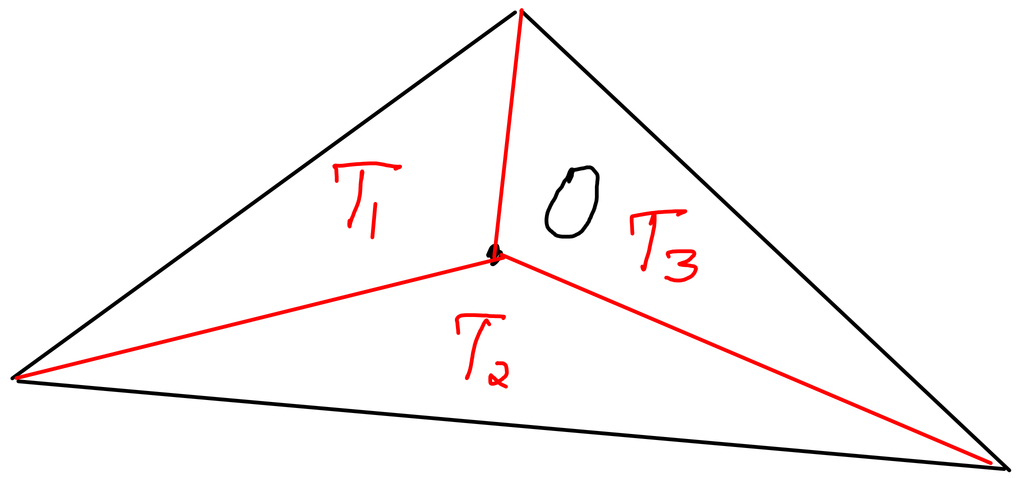
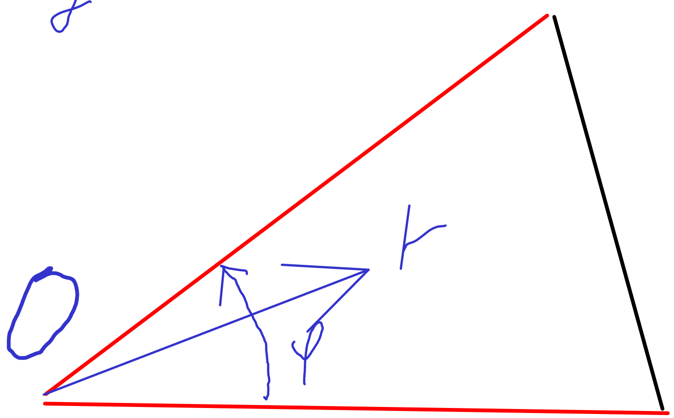


$$\int_{\pi_y} \dots dS(y) = \sum_{\ell=1}^{3} \int_{T_\ell} \dots dS(y)$$

Transformation to polar coordinates cancels singularity!

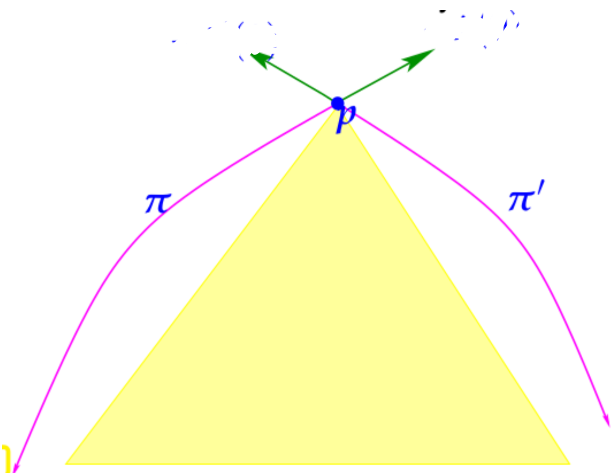
$$\int_T \frac{1}{\|y\|}\, dy = \int_0^{\varphi_0} \int_0^{r(\varphi)} dr\, d\varphi$$



III. 5.2. Toy problem: Double layer BIO in 2D

2D: $\Gamma \hat{=}$ closed curve, $\Gamma_h = \{\pi_j\}$, $\pi_j \hat{=}$ open curves

$\pi, \pi' \in \Gamma_h$ adjacent panels, $\pi \cap \pi' = \{p\}$,



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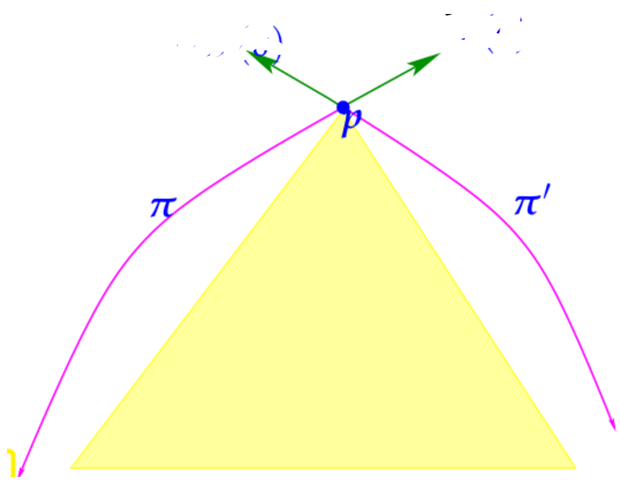
Arclength parametrization

$\phi : [0, |\pi|] \to \pi$, $\|\dot{\phi}\| = 1$

$\phi' : [0, |\pi'|] \to \pi'$, $\|\dot{\phi}'\| = 1$, $\phi(0) = \phi'(0) = p$

analytic



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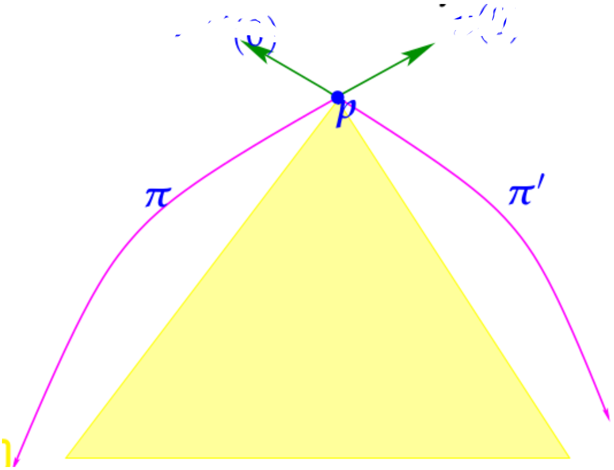
analytic

After pullback to parameter intervals

▷ Compute $$\int_0^{|\pi|}\int_0^{|\pi'|} \frac{(\phi(s)-\phi'(t))\cdot n(\phi'(t))}{\|\phi(s)-\phi(t)\|^2} F(s)G(t)\,ds\,dt$$

analytic



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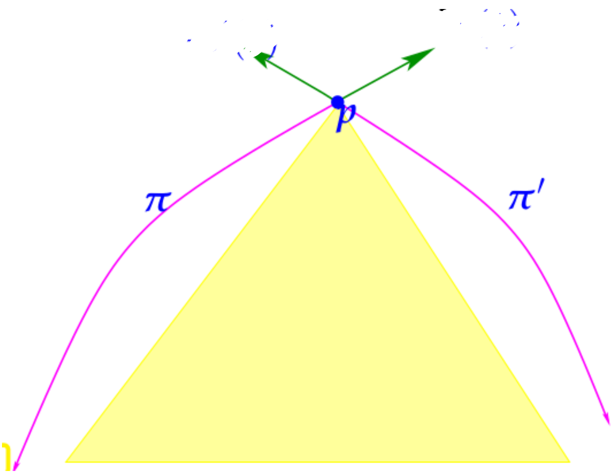
analytic



◁ Minimal angle assumption

$$\dot{\phi}(0) \cdot \dot{\phi}'(0) \le c_\angle < 1$$

III.5.2. Toy problem : Double layer BIO in 2D

$$\int_0^{|\pi'|}\int_0^{|\pi'|} \frac{(\phi(s)-\phi'(t))\cdot n(\phi'(t))}{\|\phi(s)-\phi(t)\|^2} F(s)G(t)\,ds\,dt$$

Polar coordinates $s = r\cos\varphi$, $t = r\sin\varphi$

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$$\int_0^{|\pi|}\int_0^{|\pi'|} \frac{(\Phi(s)-\Phi'(t))\cdot n(\Phi'(t))}{\|\Phi(s)-\Phi(t)\|^2}\, F(s)G(t)\,ds\,dt$$

Polar coordinates $s = r\cos\varphi$, $t = r\sin\varphi$

▷ [Taylor] $\Phi(s)-\Phi(t) = r\,\underline{b}(r,\varphi)$

with $\underline{b}(0,\varphi) \neq 0$, $(r,\varphi) \to \underline{b}(r,\varphi)$ analytic

▷ $$\frac{(\Phi(s)-\Phi'(t))\cdot n(\Phi'(t))}{\|\Phi(s)-\Phi(t)\|^2} = \frac{1}{r}\,\frac{\underline{b}(r,\varphi)\cdot \underline{n}(\Phi'(r\sin\varphi))}{\|\underline{b}(r,\varphi)\|^2}$$

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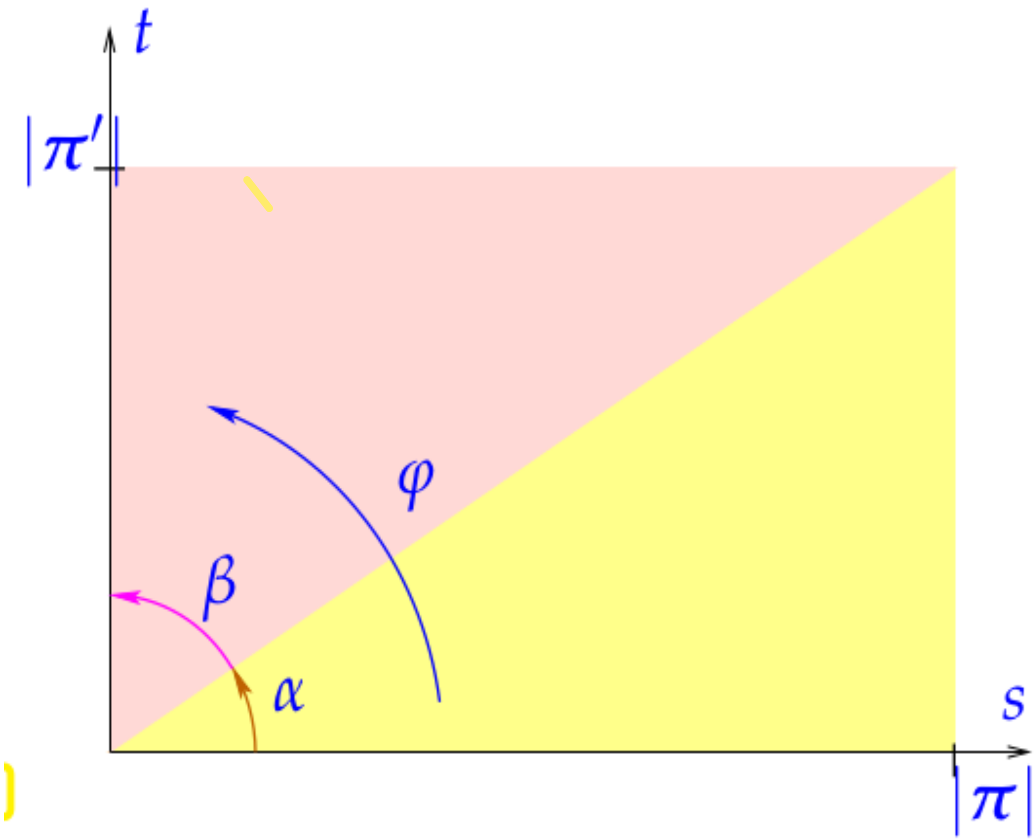
$$\triangleright\quad \frac{(\Phi(s)-\Phi'(t))\cdot n(\Phi'(t))}{\|\Phi(s)-\Phi(t)\|^2} = \frac{1}{r}\underbrace{\frac{\underline{b}(r,\varphi)\cdot \underline{n}(\Phi'(r\sin\varphi))}{\|\underline{b}(r,\varphi)\|^2}}_{\text{analytic in } (r,\varphi)}$$

cancelled by metric factor in polar coordinates!

III. 5.2. Toy problem: Double layer BIO in 2D



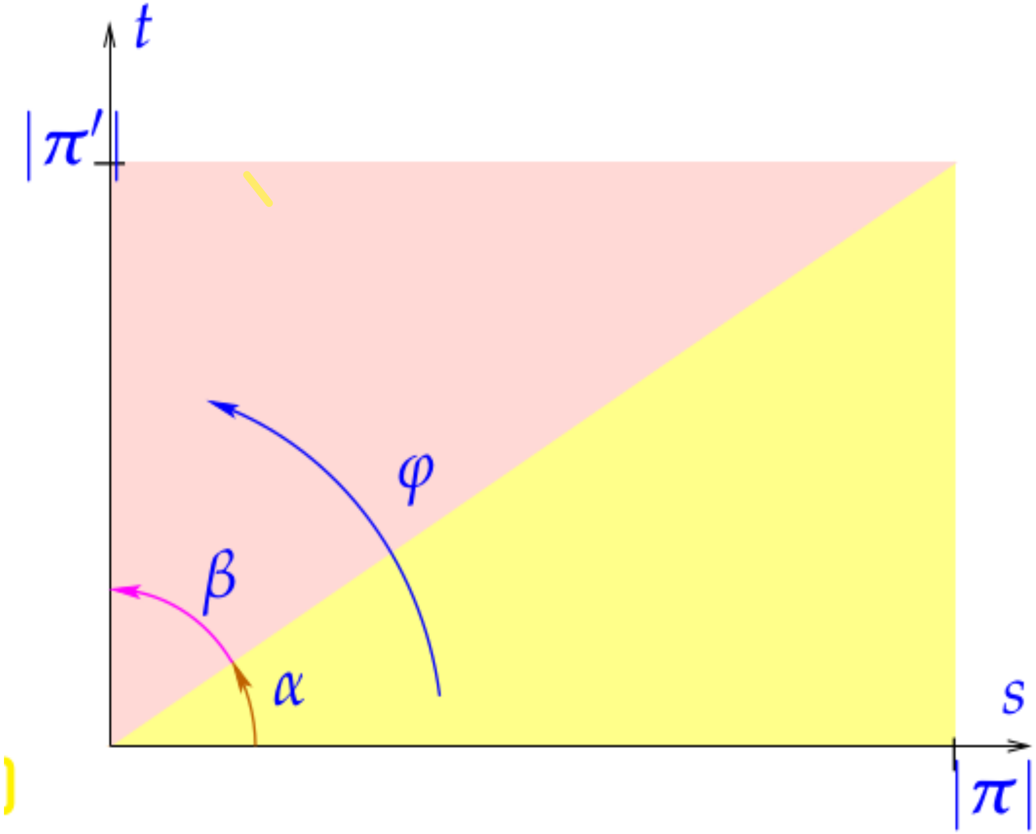
$\tan\alpha = |\pi'|/|\pi|$, $\tan\beta = |\pi|/|\pi'|$

$$\int_0^{|\pi|}\int_0^{|\pi'|} \dots \, dt\,ds$$

$$= \int_0^{\alpha}\int_0^{|\pi|/\cos\varphi} \dots \, r\,dr\,d\varphi$$

$$+ \int_0^{\beta}\int_0^{|\pi'|/\sin\varphi} \dots \, r\,dr\,d\varphi$$

III.5.2. Toy problem: Double layer BIO in 2D



$\tan\alpha = |\pi'|/|\pi|$, $\tan\beta = |\pi|/|\pi'|$

$$\int_0^{|\pi|}\int_0^{|\pi'|} \dots \, dt\, ds$$

$$= \int_0^{\alpha}\int_0^{|\pi|/\cos\varphi} \dots \, r\, dr\, d\varphi$$

$$+ \int_0^{\beta}\int_0^{|\pi'|/\sin\varphi} \dots \, r\, dr\, d\varphi$$

with integrands analytic in (r, φ)

$\triangleright$ Exponential convergence of Gauss quadrature

III.5.3. Sauter-Schwab Quadrature

Focus: Single layer BIO, self-interaction of panel

$$I := \int_\pi \int_\pi \frac{1}{\|x-y\|} \, dS(x,y) = \int_{\hat K} \int_{\hat K} \frac{1}{\|\Phi(s)-\Phi(t)\|} F(t) G(s) \, dt \, ds$$

analytic

$\hat K \hat{=}$ reference triangle, vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\hat K \times \hat K = \{ [s_1, s_2, t_1, t_2] \in \mathbb{R}_+^4 , \; t_1 + t_2 < 1, \; s_1 + s_2 < 1 \}$$

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$\hat K \mathrel{\hat=}$ reference triangle, vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$$\hat K \times \hat K = \{[s_1, s_2, t_1, t_2] \in \mathbb{R}_+^4,\ t_1 + t_2 < 1,\ s_1 + s_2 < 1\}$$

Confine singularity to one coordinate

$$\tilde s = s,\quad \tilde z = s - t \iff s = \tilde s,\ t = \tilde s - \tilde z$$

III.5.3. Sauter-Schwab Quadrature

Focus: Single layer BIO, self-interaction of panel

$$I := \int_{\pi}\int_{\pi} \frac{1}{\|x-y\|}\, dS(x,y) = \int_{\hat{K}}\int_{\hat{K}} \frac{1}{\|\Phi(s)-\Phi(t)\|}\, F(t)\, G(s)\, dt\, ds$$

analytic

$\hat{K} \,\hat{=}\,$ reference triangle, vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$$\hat{K}\times\hat{K} = \{[s_1,s_2,t_1,t_2]\in\mathbb{R}_+^4,\ t_1+t_2<1,\ s_1+s_2<1\}$$

Confine singularity to one coordinate

$$\tilde{s} = s,\quad \tilde{z} = s-t \iff s = \tilde{s},\quad t = \tilde{s}-\tilde{z}$$

$\triangleright$
$$I = \iint_{D} \frac{1}{\|\Phi(\tilde{s})-\Phi(\tilde{s}-\tilde{z})\|}\, F(\tilde{s}-\tilde{z})\, G(\tilde{s})\, d\tilde{z}\, d\tilde{s}$$

III.5.3. Sauter-Schwab Quadrature

$$I = \iint_D \frac{1}{\|\Phi(\hat{z}) - \Phi(\hat{s}-\hat{z})\|} F(\hat{s}-\tilde{z})\, G(\hat{s})\, d\tilde{z}\, d\tilde{s}$$

with

$$D = \left\{ [\widehat{s}_1, \widehat{s}_2, \widehat{z}_1, \widehat{z}_2]^\top \in \mathbb{R}^4 : \begin{array}{l} \widehat{s}_1, \widehat{s}_2 > 0,\ \widehat{s}_1 - \widehat{z}_1 > 0,\ \widehat{s}_2 - \widehat{z}_2 > 0, \\ \widehat{s}_1 + \widehat{s}_2 - (\widehat{z}_1 + \widehat{z}_2) < 1 \end{array} \right\}$$

III.5.3. Sauter-Schwab Quadrature

$$I = \iint_D \frac{1}{\|\Phi(\tilde{s}) - \Phi(\tilde{s}-\tilde{z})\|} F(\hat{s}-\hat{z})\, G(\tilde{s})\, d\hat{z}\, d\tilde{s}$$

with
$$D = \left\{ [\hat{s}_1, \hat{s}_2, \hat{z}_1, \hat{z}_2]^\top \in \mathbb{R}^4 : \begin{array}{l} \hat{s}_1, \hat{s}_2 > 0,\ \hat{s}_1 - \hat{z}_1 > 0,\ \hat{s}_2 - \hat{z}_2 > 0, \\ \hat{s}_1 + \hat{s}_2 - (\hat{z}_1 + \hat{z}_2) < 1 \end{array} \right\}$$

Polar coordinates in singular direction

$$\tilde{z}_1 = r\cos\varphi, \quad \tilde{z}_2 = r\sin\varphi$$

▷ $$\frac{\|\Phi(\tilde{s}) - \Phi(\tilde{s}-\tilde{z})\|}{\|\tilde{z}\|} := \underbrace{B(r,\varphi,\tilde{s})}_{\neq 0}$$ analytic in $(r,\varphi,\tilde{s})$ on D [by Taylor expansion]

III.5.3. Sauter-Schwab Quadrature

$$I = \iint_D \frac{1}{\|\Phi(\tilde{s}) - \Phi(\tilde{s}-\tilde{z})\|} F(\tilde{s}-\tilde{z})\, G(\tilde{s})\, d\tilde{z}\, d\tilde{s}$$

with $D = \left\{ [\hat{s}_1, \hat{s}_2, \hat{z}_1, \hat{z}_2]^\top \in \mathbb{R}^4 : \begin{array}{l} \hat{s}_1, \hat{s}_2 > 0,\ \hat{s}_1 - \hat{z}_1 > 0,\ \hat{s}_2 - \hat{z}_2 > 0, \\ \hat{s}_1 + \hat{s}_2 - (\hat{z}_1 + \hat{z}_2) < 1 \end{array} \right\}$

Polar coordinates in singular direction

$\tilde{z}_1 = r\cos\varphi\,,\quad \tilde{z}_2 = r\sin\varphi$

▷ $\dfrac{\|\Phi(\tilde{s}) - \Phi(\tilde{s}-\tilde{z})\|}{\|\tilde{z}\|} := \underbrace{B(r,\varphi,\tilde{s})}_{\neq 0}$ analytic in $(r,\varphi,\tilde{s})$ on D [by Taylor expansion]

$$I = \iint_D \frac{1}{\|\tilde{z}\|} \frac{F(\tilde{s}-\tilde{z})\, G(\tilde{s})}{B(\tilde{z},\tilde{s})}\, d\tilde{z}\, d\tilde{s} = \iint_D \frac{F(\tilde{s}-\tilde{z})\, G(\tilde{s})}{B(r,\varphi,\tilde{s})}\, d\tilde{s}\, dr\, d\varphi$$

analytic on D

III.5.3. Sauter-Schwab Quadrature

$$I = \iint_D \frac{F(\tilde{s}-\tilde{z})\,G(\tilde{s})}{B(r,\varphi,\tilde{s})}\,d\tilde{s}\,dr\,d\varphi$$

[polar coordinates!]

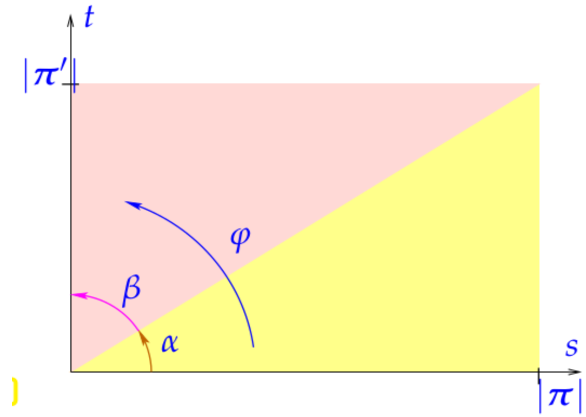
III.5.3. Sauter-Schwab Quadrature

$$I = \iint_D \frac{F(\tilde{s}-\tilde{z})\,G(\tilde{s})}{B(r,\varphi,\hat{s})}\,d\tilde{s}\,dr\,d\varphi$$

[polar coordinates !]

Split $D \subset \mathbb{R}^4$ into simplices $\Longleftrightarrow$

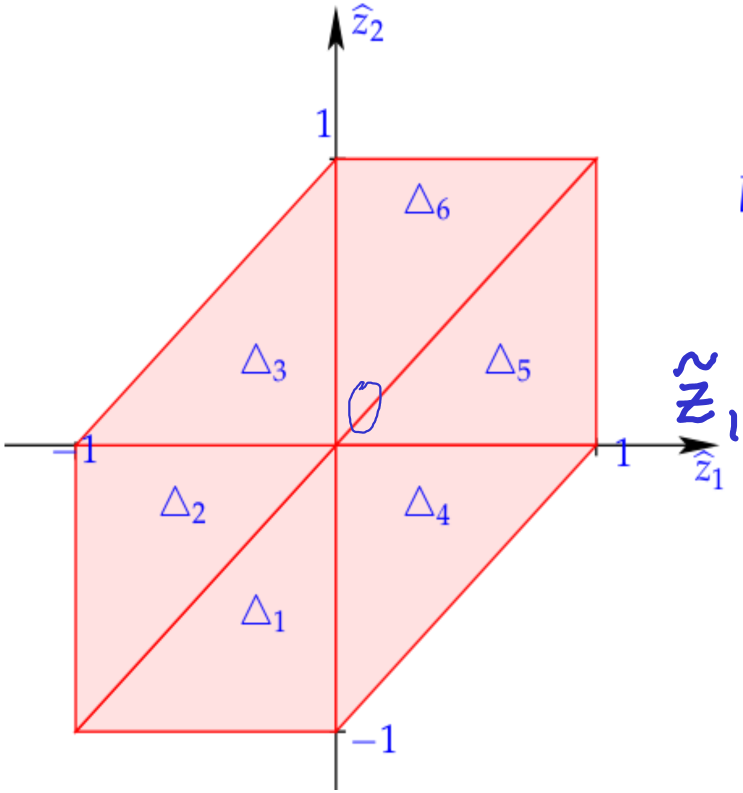
[$D = \hat{K} \times \hat{K} \hat{=}$ polytope]



$\tan\alpha = |\pi'|/|\pi|$, $\tan\beta = |\pi|/|\pi'|$

$$\begin{aligned}
D = &\{-1 < \hat{z}_1 < 0, -1 < \hat{z}_2 < \hat{z}_1, -\hat{z}_2 < \hat{s}_1 < 1, -\hat{z}_2 < \hat{s}_2 < \hat{s}_1\} \quad \bigcup \\
&\{-1 < \hat{z}_1 < 0, \hat{z}_1 < \hat{z}_2 < 0, \hat{z}_1 < \hat{s}_1 < 1, -\hat{z}_2 < \hat{s}_2 < \hat{s}_1 + \hat{z}_1 - \hat{z}_2\} \quad \bigcup \\
&\{-1 < \hat{z}_1 < 0, 0 < \hat{z}_2 < 1 + \hat{z}_1, \hat{z}_2 - \hat{z}_1 < \hat{s}_1 < 1, 0 < \hat{s}_2 < \hat{s}_1 + \hat{z}_1 - \hat{z}_2\} \quad \bigcup \\
&\{0 < \hat{z}_1 < 1, -1 + \hat{z}_1 < \hat{z}_2 < 0, -\hat{z}_2 < \hat{s}_1 < 1 - \hat{z}_1, -\hat{z}_2 < \hat{s}_2 < \hat{s}_1\} \quad \bigcup \\
&\{0 < \hat{z}_1 < 1, 0 < \hat{z}_2 < \hat{z}_1, 0 < \hat{s}_1 < 1 - \hat{z}_1, 0 < \hat{s}_2 < \hat{s}_1\} \quad \bigcup \\
&\{0 < \hat{z}_1 < 1, \hat{z}_1 < \hat{z}_2 < 1, \hat{z}_2 - \hat{z}_1 < \hat{s}_2 < 1 - \hat{z}_1, 0, \hat{s}_2 < \hat{z}_1 - \hat{z}_2 + \hat{s}_1\} \\
=: &D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \,.
\end{aligned}$$

III.5.3. Sauter-Schwab Quadrature



$$D = \{\hat{z} \in \Delta_1, \overbrace{-\hat{z}_2 < \hat{s}_1 < 1, -\hat{z}_2 < \hat{s}_2 < \hat{s}_1}^{\text{triangle in } \hat{s}\text{-plane}}\} \;\cup$$
$$\{\hat{z} \in \Delta_2, \hat{z}_1 < \hat{s}_1 < 1, -\hat{z}_2 < \hat{s}_2 < \hat{s}_1 + \hat{z}_1 - \hat{z}_2\} \;\cup$$
$$\{\hat{z} \in \Delta_3, \hat{z}_2 - \hat{z}_1 < \hat{s}_1 < 1, 0 < \hat{s}_2 < \hat{s}_1 + \hat{z}_1 - \hat{z}_2\} \;\cup$$
$$\{\hat{z} \in \Delta_4, -\hat{z}_2 < \hat{s}_1 < 1 - \hat{z}_1, -\hat{z}_2 < \hat{s}_2 < \hat{s}_1\} \;\cup$$
$$\{\hat{z} \in \Delta_5, 0 < \hat{s}_1 < 1 - \hat{z}_1, 0 < \hat{s}_2 < \hat{s}_1\} \;\cup$$
$$\{\hat{z} \in \Delta_6, \hat{z}_2 - \hat{z}_1 < \hat{s}_2 < 1 - \hat{z}_1, 0, \hat{s}_2 < \hat{z}_1 - \hat{z}_2 + \hat{s}_1\},$$

polar coordinates here

$$\int\int_D \dots \mathrm{d}\hat{s}\,\mathrm{d}r\,\mathrm{d}\varphi = \int_{\Delta_1} \int_{-\hat{z}_2}^{1} \int_{-\hat{z}_2}^{\hat{s}_1} \dots \mathrm{d}\hat{s}_2 \mathrm{d}\hat{s}_1 \mathrm{d}r\,\mathrm{d}\varphi + \int_{\Delta_2} \int_{-\hat{z}_1}^{1} \int_{-\hat{z}_2}^{\hat{s}_1+\hat{z}_1-\hat{z}_2} \dots \mathrm{d}\hat{s}_2 \mathrm{d}\hat{s}_1 \mathrm{d}r\,\mathrm{d}\varphi +$$
$$\int_{\Delta_3} \int_{\hat{z}_2-\hat{z}_1}^{1} \int_{0}^{\hat{s}_1+\hat{z}_1-\hat{z}_2} \dots \mathrm{d}\hat{s}_2 \mathrm{d}\hat{s}_1 \mathrm{d}r\,\mathrm{d}\varphi + \int_{\Delta_4} \int_{-\hat{z}_2}^{1-\hat{z}_1} \int_{-\hat{z}_2}^{\hat{s}_1} \dots \mathrm{d}\hat{s}_2 \mathrm{d}\hat{s}_1 \mathrm{d}r\,\mathrm{d}\varphi$$
$$\int_{\Delta_5} \int_{0}^{1-\hat{z}_1} \int_{0}^{\hat{s}_1} \dots \mathrm{d}\hat{s}_2 \mathrm{d}\hat{s}_1 \mathrm{d}r\,\mathrm{d}\varphi + \int_{\Delta_6} \int_{\hat{z}_2-\hat{z}_1}^{1-\hat{z}_1} \int_{0}^{\hat{z}_1-\hat{z}_2+\hat{s}_1} \dots \mathrm{d}\hat{s}_2 \mathrm{d}\hat{s}_1 \mathrm{d}r\,\mathrm{d}\varphi\,.$$

Exponential convergence of tensor product Gauss quadrature!