Bondary Element Methods: Derivation, Analysis, and Implementation

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Seminar for Applied Mathematics, ETH Zürich

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Fundamentals and practice of finite elements
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IV. Calderón Preconditioning

Preconditioning

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Best-known: $\mathbf{A}$ s.p.d. and conjugate gradient (CG) method

$$\left\|\mathbf{e}^{(k)}\right\| \leq \frac{2\gamma^k}{1+\gamma^{2k}} \left\|\mathbf{e}^{(0)}\right\| \;, \quad \gamma := \frac{\sqrt{\kappa(\mathbf{CA})} - 1}{\sqrt{\kappa(\mathbf{CA})} + 1} \;.$$

$\kappa(\mathbf{CA}) \mathrel{\hat=}$ spectral condition number

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Krylov methods:
- CG
- CGS, CGN
- MINRES, GMRES, ORTHODIR
- BiCG, BICGStab

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- $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ $\triangleright$ $\mathrm{cond}(A_V) = O(h^{-1})$
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$\hookrightarrow$ 1st-order (Ψ-do differential) operators

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- EFIE : $S_\kappa = k\underline{V} + \underbrace{\frac{1}{k}\nabla_T \times V_\kappa(\nabla_T \cdot)}_{\text{order } 1}$

  $H_t^{-1/2}(\Gamma) \to H_t^{1/2}(\Gamma)$

▷ $\mathrm{cond}(A_S) = O(h^{-2})$

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$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

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Spectral condition number $\kappa(D_h^{-1} B_h D_h^{-*} A_h) \leq \|A_h\| \, \|A_h^{-1}\| \; \|B_h\| \, \|B_h^{-1}\| \, \dfrac{\|d\|^2}{c_D^2}$

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$$\kappa(\mathbf{A}^H \mathbf{C}^H \mathbf{C} \mathbf{A}) \leq \left(\|A_h\| \, \|A_h^{-1}\| \; \|B_h\| \, \|B_h^{-1}\| \frac{\|d\|^2}{c_D^2} \right)^2$$

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▷ $\Gamma \,\hat{=}$ boundary of a domain $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

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trace space of $\boldsymbol{H}(\mathbf{curl}; \Omega)$

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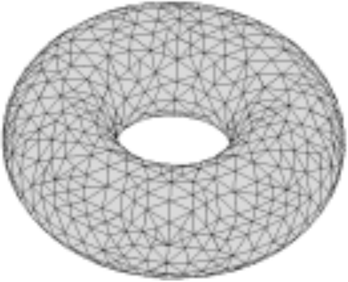

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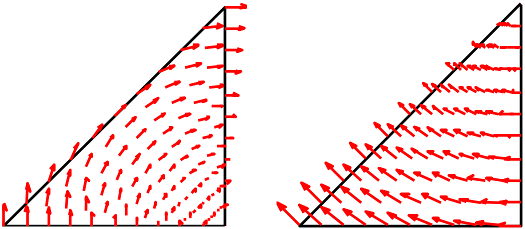

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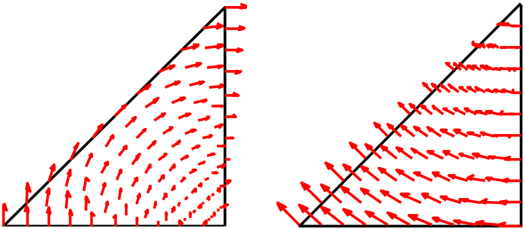
If $k \neq$ interior resonant frequency of Ω, then $a_{|V_h}$ satisfies h-uniform inf-sup condition on sufficiently fine and shape regular meshes.



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$$\exists c > 0: \quad \sup_{\boldsymbol{v} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} \frac{\int_\Gamma \boldsymbol{u} \cdot (\bar{\boldsymbol{v}} \times \boldsymbol{n}) \,\mathrm{d}S}{\|\boldsymbol{v}\|_{-\frac{1}{2}, \mathrm{div}_\Gamma}} \geq c \|\boldsymbol{u}\|_{-\frac{1}{2}, \mathrm{div}_\Gamma} \;\; \forall \boldsymbol{u} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

"Self Duality" of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

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This is flawed !

S. Christiansen and J.-C. Nédélec, *A preconditioner for the electric field integral equation based on Calderón formulas*, SIAM J. Numer. Anal., 40 (2002), pp. 1100–1135.

$\exists$ subspace $N_h \subset V_h$, $C, c > 0$: $\quad \dim N_h \geq c \dim V_h$ such that

$$\forall \boldsymbol{u}_h \in N_h: \quad \sup_{\boldsymbol{v}_h \in V_h} \frac{d(\boldsymbol{u}_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \leq C h^{1/2} \,\|\boldsymbol{u}_h\|_{-\frac{1}{2},\mathrm{div}_\Gamma} \ .$$

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- Simplest Galerkin space: $V_h \mathrel{\hat{=}}$ piecewise constants

$\boxed{V_h \not\subset W \quad \succ \quad W_h = V_h \text{ not an option:} \qquad \text{What is } W_h\ \textbf{?}}$

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- Simplest

Required $\dim V_h = \dim W_h$, $W_h \subset C^0(\Gamma)$

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(Discrete) Exterior Calculus: $\wedge$-Pairing

Summary: First kind boundary integral operators:

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scalar single layer: $\mathsf{A}: \quad H^{-\frac{1}{2}}(\Gamma) \quad \mapsto \quad H^{\frac{1}{2}}(\Gamma)$

dual w.r.t. $\langle \cdot, \cdot \rangle_\Gamma$

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$H^{\frac{1}{2}}(\Gamma)$	=	trace space of $H^1(\Omega)$	➢	space of 0-forms on Γ
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$$\int_\Gamma \underbrace{u}_{\text{0-form}} \, \underbrace{v}_{\text{2-form}} \, \mathrm{d}S$$

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0-form 2-form 1-forms

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"vector proxies": $\underbrace{\int_\Gamma u}_{\text{0-form}}\, \underbrace{v}_{\text{2-form}} \,\mathrm{d}S$, $\int_\Gamma \underbrace{\mathbf{u} \cdot \mathbf{v}}_{\text{1-forms}} \times \mathbf{n} \,\mathrm{d}S$

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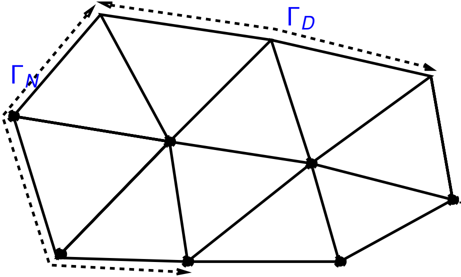

Recall: discrete Hodge operators based on dual meshes

$$\langle \star\omega_h, \eta_h \rangle = \int_\Omega \omega_h \wedge \eta_h$$

for all discrete $n-l$-forms ω_h, discrete l-forms η_h

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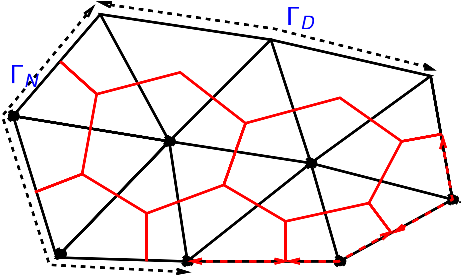

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Dual Meshes

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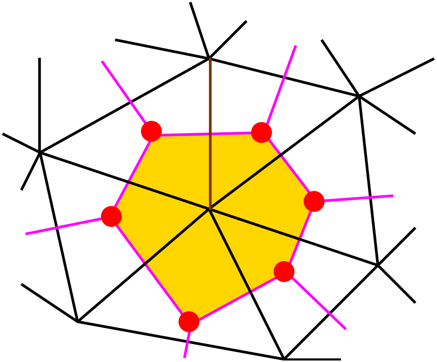
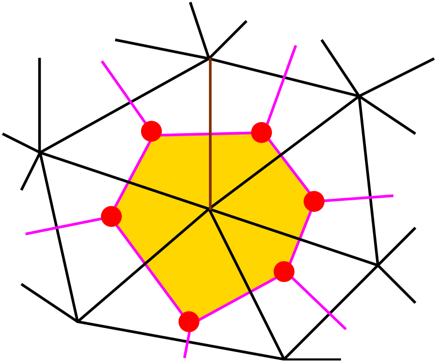


mesh Γ_h	$\leftrightarrow$	dual mesh $\widetilde{\Gamma_h}$
nodes	$\leftrightarrow$	cells
edges	$\leftrightarrow$	edges
cells	$\leftrightarrow$	nodes

Dual Meshes



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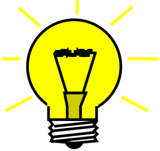
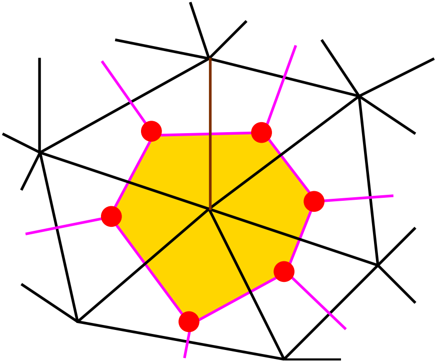


Idea:

$$V_h \leftrightarrow l\text{-facets of } \Gamma_h \quad \Leftrightarrow \quad W_h \leftrightarrow 2-l\text{-facets of } \widetilde{\Gamma}_h$$

Dual Meshes



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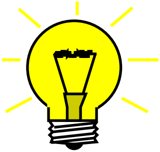

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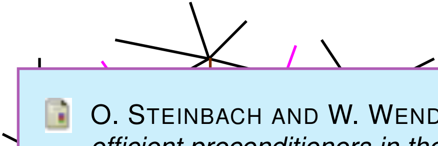
($l = 2 \;\hat{=}$ scalar single layer, $l = 1 \;\hat{=}$ EFIE)

Dual Meshes



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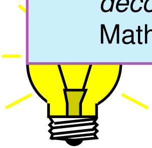

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$$V_h$$
$$\updownarrow$$
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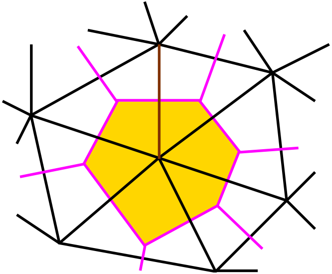

$\leftrightarrow$

Duality via Dual Meshes: Scalar Case

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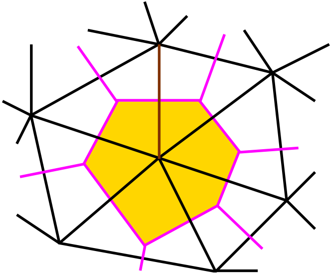
$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{p.w. constant on cells of } \Gamma_h & \longleftrightarrow & \text{“p.w. linear” } \textbf{\&}\ C^0 \text{ on } \widetilde{\Gamma}_h \end{array}$$



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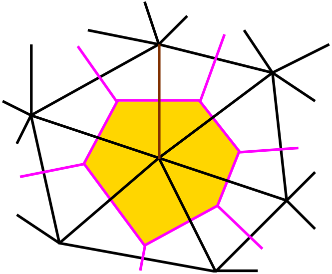

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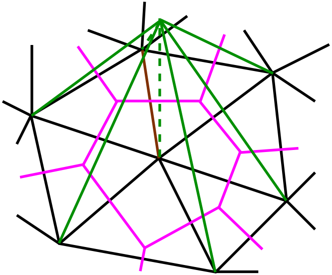
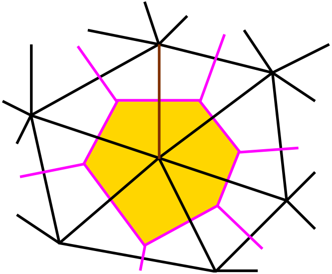


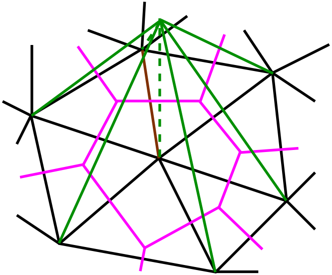
Duality via Dual Meshes: Scalar Case

For $V = H^{-\frac{1}{2}}(\Gamma)$:

$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{p.w. constant on cells of } \Gamma_h & \longleftrightarrow & \text{“p.w. linear” \& } C^0 \text{ on } \widetilde{\Gamma}_h \end{array}$$



$\leftrightarrow$



h-uniform stability of discrete duality paiting $(u, v) \mapsto \int uv \,\mathrm{d}S$ on $V_h \times W_h$

Duality via Dual Meshes: Vectorial Case

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For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: $\quad$ ($\leftrightarrow$ electric field integral equation)

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$$\begin{array}{c} V_h \\ \updownarrow \\ RT_0 \text{ elements on } \Gamma_h \end{array}$$

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

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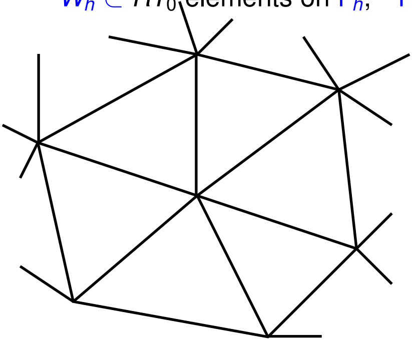
Idea: $W_h \subset RT_0$ elements on $\widehat{\Gamma}_h$, $\widehat{\Gamma}_h$ = barycentric refinement of Γ_h



Duality via Dual Meshes: Vectorial Case

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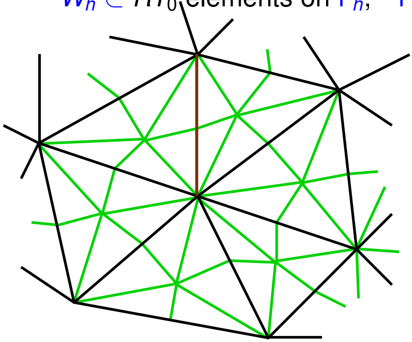

Duality via Dual Meshes: Vectorial Case

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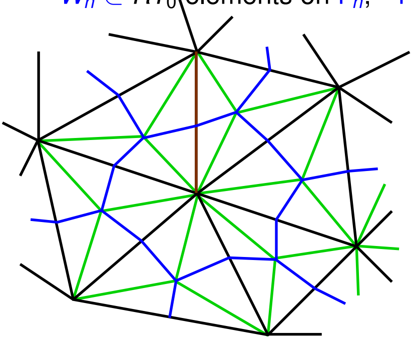

Duality via Dual Meshes: Vectorial Case

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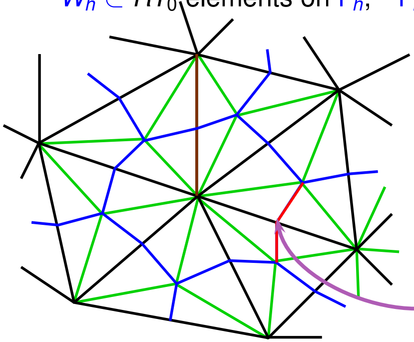

supporting edge of $\widetilde{\Gamma}_h$

Duality via Dual Meshes: Vectorial Case

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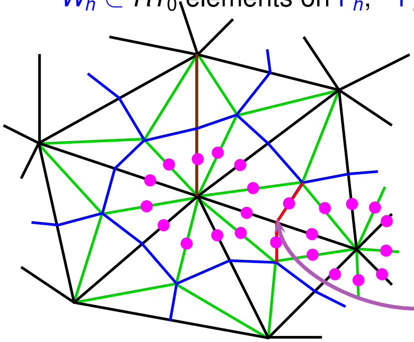

$\triangleleft$ Basis function of W_h

$\bullet$: edges of barycentric refinement with nonzero contribution to basis function at —

supporting edge of $\widetilde{\Gamma}_h$

Duality via Dual Meshes: Vectorial Case

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A. BUFFA AND S. CHRISTIANSEN, *A dual finite element complex on the barycentric refinement*, Math. Comp., 76 (2007), pp. 1743–1769.

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Duality via Dual Meshes: Vectorial Case

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F. ANDRIULLI, K. COOLS, H. BAGCI, F. OLYSLAGER, A. BUFFA, S. CHRISTIANSEN, AND E. MICHIELSSEN, *A multiplicative Calderon preconditioner for the electric field integral equation*, IEEE Trans. Antennas and Propagation, 56 (2008), pp. 2398–2412.

F. P. ANDRIULLI, K. COOLS, I. BOGAERT, AND E. MICHIELSSEN, *On a well-conditioned electric field integral operator for multiply connected geometries*, IEEE Transactions on Antennas and Propagation, 61 (2013), pp. 2077–2087.

Dual Mesh OPC: Algebraic View

- $A = V_\kappa$ (single layer BIO), $V = H^{-1/2}(\Gamma)$
  $B = W_\kappa$ (hypersingular BIO); $W = H^{1/2}(\Gamma)$

Dual Mesh OPC: Algebraic View

- $A = V_\kappa$ (single layer BIO), $\mathcal{V} = H^{-1/2}(\Gamma)$

  $B = W_\kappa$ (hypersingular BIO); $\mathcal{W} = H^{1/2}(\Gamma)$

$\Rightarrow \quad \mathcal{V}_h = \mathcal{W}^2(\mathcal{T}_h)\,, \quad \mathcal{W}_h = \mathcal{W}^0(\tilde{\mathcal{T}}_h)\,.$

Dual Mesh OPC: Algebraic View

- $A = V_\kappa$ (single layer BIO), $V = H^{-1/2}(\Gamma)$
  $B = W_\kappa$ (hypersingular BIO), $W = H^{1/2}(\Gamma)$

$\Rightarrow$ $V_h = \mathcal{W}^2(\mathcal{T}_h)$, $W_h = \mathcal{W}^0(\tilde{\mathcal{T}}_h)$.

$W_h \subset \mathcal{W}^0(\hat{\mathcal{T}}_h) \Rightarrow \exists$ matrix $\underline{C}$

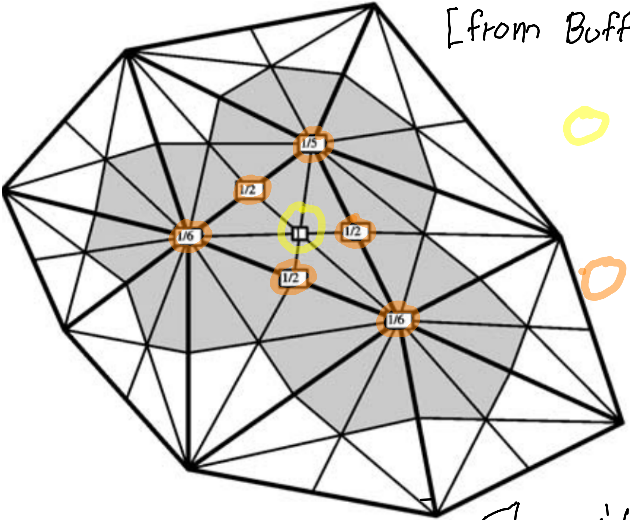
$$\hat{\vec{x}} = C \tilde{\vec{x}}$$

coefficients in $\mathcal{W}^0(\hat{\mathcal{T}}_h)$ [std. basis]

coefficients in $\mathcal{W}^0(\tilde{\mathcal{T}}_h)$

C defines $\mathcal{W}^0(\tilde{\mathcal{T}}_h)$

Dual Mesh OPC: Algebraic View



[from Buffa & Christiansen]

○ $\hat{=}$ location of dual basis function (↔ dual vertex)

○ $\hat{=}$ contributing basis functions on barycentric refinement

◁ weights ↔ column of C

Discrete Duality of $W^1(\mathcal{T}_h) - W^1(\widetilde{\mathcal{T}}_h)$

[from Buffa & Christiansen]