Adaptive Finite Element Methods
Lecture 3: Contraction Property and Optimal Convergence Rates

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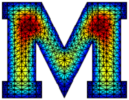

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School - Fundamentals and Practice of Finite Elements
Roscoff, France, April 16-20, 2018

Outline

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Adaptive Finite Element Method (AFEM)

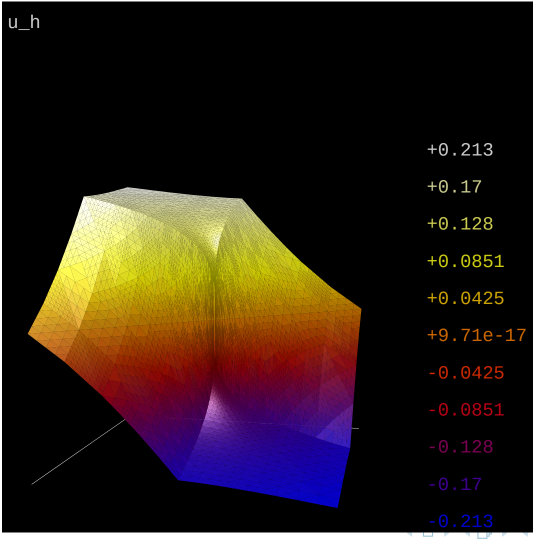
$$\text{AFEM}: \quad \text{SOLVE} \quad \to \quad \text{ESTIMATE} \quad \to \quad \text{MARK} \quad \to \quad \text{REFINE}$$

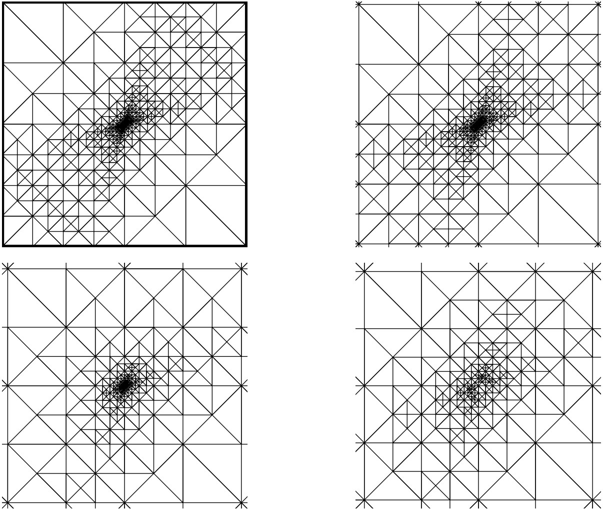
$$k \geq 0 \text{ loop counter} \quad \Rightarrow \quad (\mathcal{T}_k, \mathbb{V}(\mathcal{T}_k), U_k)$$

- $U_k = \text{SOLVE}(\mathcal{T}_k)$ computes the exact Galerkin solution $U_k \in \mathbb{V}(\mathcal{T}_k)$
  - dealing with L^2 data
  - exact linear algebra
- $\mathcal{E}_k = \text{ESTIMATE}(\mathcal{T}_k, U_k, f)$ computes local error indicators $e(z)$
  - localization of global H^{-1} norms to stars ω_z for $z \in \mathcal{N}_k = \mathcal{N}(\mathcal{T}_k)$
  - computation of residuals in weighted L^2 norms
- $\mathcal{M}_k = \text{MARK}(\mathcal{E}_k, \mathcal{T}_k)$ selects $\mathcal{M}_k \subset \mathcal{T}_k$ using Dörfler marking
  - $\mathcal{E}_k(\mathcal{M}_k) \geq \theta \mathcal{E}_k(\mathcal{T}_k)$ for $0 < \theta < 1$ (bulk chasing)
  - marked set $\mathcal{M}_k$ must be minimal for optimal rates
- $\mathcal{T}_{k+1} = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$ refines the marked elements $\mathcal{M}_k$ and outputs a conforming mesh $\mathcal{T}_{k+1}$ refinement of $\mathcal{T}_k$
  - uses $b \geq 1$ newest vertex bisection (Mitchell) for $d = 2$ to refine each $T \in \mathcal{M}_k$ so that T is bisected at least b times.

Example (Kellogg' 75): Checkerboard discontinuous coefficients

$u \approx r^{0.1} \Rightarrow u \in H^{1.1}(\Omega) \Rightarrow |u - U_k|_{H^1(\Omega)} \approx \#\mathcal{T}_k^{-0.05}$ ($\mathcal{T}_k$ quasi-uniform)





Discontinuous coefficients: Final graded grid (full grid with < 2000 nodes) (top left), and 3 zooms ($\times 10^3, 10^6, 10^9$); decay rate $N^{-1/2}$. Uniform grid would require $N \approx 10^{20}$ elements for a similar resolution.

AFEM: Main Results

- Convergence of AFEM: $U_k \to u$ as $k \to \infty$ without assuming that meshsize goes to zero, and with minimal assumptions regarding underlying problem and MARK.

- Contraction property of AFEM: there exist $0 < \alpha < 1$ and $\gamma > 0$ so that

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2 \le \alpha^2 \Big(\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2 \Big).$$

- Quasi-optimal convergence rates (for total error): if

$$\inf_{\#\mathcal{T} - \#\mathcal{T}_0 \le N} \; \inf_{V \in \mathbb{V}(\mathcal{T})} \Big(\|u - V\|_\Omega + \operatorname{osc}_{\mathcal{T}}(V, \mathcal{T}) \Big) \preccurlyeq N^{-s}$$
$$\Rightarrow \quad \|u - U_k\|_\Omega + \operatorname{osc}_{\mathcal{T}_k}(U_k, \mathcal{T}_k) \preccurlyeq (\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}.$$

- Sufficient conditions on (u, f, A) for total error decay N^{-s}.

Convergence of AFEM (Morin, Siebert, Veeser'08, Siebert'09)

Minimal assumption on MARK: for all $T \in \mathcal{T}_k$ such that

$$\mathcal{E}_k(U_k, T) = \max_{T' \in \mathcal{T}_k} \mathcal{E}(U_k, T') = \mathcal{E}_{k,\max} \quad \Rightarrow \quad T \in \mathcal{M}_k.$$

Lemma 1 (mesh-size function). If χ_k denotes the characteristic function of the union $\cup_{T \in \mathcal{T}_k \setminus \mathcal{T}_{k+1}} T$ of elements to be bisected and h_k is the mesh-size function of $\mathcal{T}_k$, then

$$\lim_{k \to \infty} \|h_k \chi_k\|_{L^\infty(\Omega)} = 0$$

This does not imply $h_k \to 0$ as $k \to \infty$ (no density argument).

Lemma 2 (convergence of largest estimator). $\mathcal{E}_{k,\max} \to 0$ as $k \to \infty$.

Theorem 2 (convergence). $U_k \to u$ and $\mathcal{E}_k(U_k) \to 0$ as $k \to \infty$.

This theory applies to problems satisfying a discrete inf-sup. It applies also to uniform refinement, so it provides no decay rate.

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Module ESTIMATE: Basic Properties

Reliability: Upper Bounds (Babuška-Miller, Stevenson)

- **Upper bound:** there exists a constant $C_1 > 0$, depending solely on the initial mesh $\mathcal{T}_0$ and the smallest eigenvalue $a_{\min}$ of A, such that

$$|\!|\!| u - U |\!|\!|_\Omega^2 \le C_1 \mathcal{E}_\mathcal{T}(U, \mathcal{T})^2$$

- **Localized upper bound:** if $U_* \in \mathbb{V}(\mathcal{T}_*)$ is the Galerkin solution for a conforming refinement $\mathcal{T}_*$ of $\mathcal{T}$, and $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ (refined set), then

$$|\!|\!| U - U_* |\!|\!|_\Omega^2 \le C_1 \mathcal{E}_\mathcal{T}(U, \mathcal{R})^2$$

Efficiency: Lower Bound (Babuška-Miller, Verfürth)

There exists a constant $C_2 > 0$, depending only on the shape regularity constant of $\mathcal{T}_0$ and the largest eigenvalue $a_{\max}$, such that

$$C_2 \mathcal{E}_\mathcal{T}(U, \mathcal{T})^2 \le |\!|\!| u - U |\!|\!|_\Omega^2 + \operatorname{osc}_\mathcal{T}(U, \mathcal{T})^2.$$

- **Reduction of Estimator:** For $\lambda = 1 - 2^{-b/d}$, $\mathcal{T}_* = \text{REFINE}(\mathcal{T}, \mathcal{M})$, and all $V \in \mathbb{V}(\mathcal{T})$ we have

$$\mathcal{E}^2_{\mathcal{T}_*}(V, \mathcal{T}_*) \leq \mathcal{E}^2_{\mathcal{T}}(V, \mathcal{T}) - \lambda \mathcal{E}^2_{\mathcal{T}}(V, \mathcal{M}).$$

- **Lipschitz Property:** The mapping $V \mapsto \mathcal{E}_{\mathcal{T}}(V, \mathcal{T})$ satisfies

$$|\mathcal{E}_{\mathcal{T}}(V, \mathcal{T}) - \mathcal{E}_{\mathcal{T}}(W, \mathcal{T})| \leq C_0 \| V - W \|_\Omega \qquad \forall V, W \in \mathbb{V}(\mathcal{T})$$

with a constant C_0 depending on $\mathcal{T}_0$, A, d and n.

This implies that for all $\delta > 0$

$$\mathcal{E}^2_{\mathcal{T}_*}(V_*, \mathcal{T}_*) \leq (1+\delta)\big(\mathcal{E}^2_{\mathcal{T}}(V, \mathcal{T}) - \lambda \mathcal{E}^2_{\mathcal{T}}(V, \mathcal{M})\big) + (1+\delta^{-1}) C_0^2 \| V_* - V \|^2_\Omega.$$

- **Dominance:** $\operatorname{osc}_{\mathcal{T}}(U, \mathcal{T}) \leq \mathcal{E}_{\mathcal{T}}(U, \mathcal{T})$

- **Pythagoras:** $\| u - U_* \|^2_\Omega = \| u - U \|^2_\Omega - \| U - U_* \|^2_\Omega$

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Module MARK: Dörfler Marking

- Given a mesh $\mathcal{T}$, indicators $\{\mathcal{E}_{\mathcal{T}}(U_{\mathcal{T}}, T)\}_{T\in\mathcal{T}}$, and a parameter $\theta \in (0,1]$, we select a subset $\mathcal{M}$ of $\mathcal{T}$ of marked elements such that

$$\mathcal{E}_{\mathcal{T}}(U,\mathcal{M}) \geq \theta \mathcal{E}_{\mathcal{T}}(U,\mathcal{T})$$

- The marked set $\mathcal{M}$ is minimal (this is crucial for optimal cardinality).

Module REFINE: Bisection

Binev, Dahmen, DeVore $(d=2)$, Stevenson $(d>2)$: If $\mathcal{T}_0$ has a suitable labeling, then there exists a constant $\Lambda_0 > 0$ only depending on $\mathcal{T}_0$ and d such that for all $k \geq 1$

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq \Lambda_0 \sum_{j=0}^{k-1} \#\mathcal{M}_j.$$

Module SOLVE: Multilevel Solvers

Chen, N, Xu'10: Optimal multigrid and BPX preconditioners for graded bisection grids, any polynomial degree $n \geq 1$, and any dimension $d \geq 2$.

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Contraction Property of AFEM

Vanishing Oscillation (Morin, N, Siebert'00)

We assume $\operatorname{osc}_{\mathcal{T}_k}(U_k) = 0$. If $\mathcal{T}_{k+1}$ satisfies an interior node property wrt $\mathcal{T}_k$, then we have the discrete lower bound

$$C_2 \mathcal{E}_k(U_k, \mathcal{M}_k)^2 \le |||U_{k+1} - U_k|||_\Omega^2$$

Therorem 2 (Contraction) For $\alpha := (1 - \theta^2 \frac{C_2}{C_1})^{1/2} < 1$ there holds

$$|||u - U_{k+1}|||_\Omega \le \alpha |||u - U_k|||_\Omega,$$

Proof: Recall Pythagoras

$$|||u - U_{k+1}|||_\Omega^2 = |||u - U_k|||_\Omega^2 - |||U_{k+1} - U_k|||_\Omega^2.$$

Combine the discrete lower bound with Dörfler marking and upper bound

$$|||U_{k+1} - U_k|||_\Omega^2 \ge C_2 \mathcal{E}_k(U_k, \mathcal{M}_k)^2 \ge C_2 \theta^2 \mathcal{E}_k(U_k)^2 \ge \frac{C_2}{C_1} \theta^2 |||u - U_k|||_\Omega^2$$

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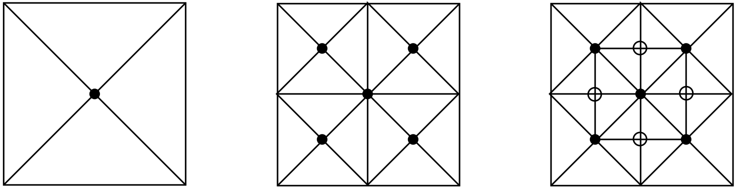
General Data: Contracting Quantities

- Energy error: $\|U_k - u\|_\Omega$ is monotone, but not strictly monotone (e.g. $U_{k+1} = U_k$).



$$\Omega = (0,1)^2, A = I, f = 1 \quad \Rightarrow \quad U_0 = U_1 = \frac{1}{12}\phi_0, \quad U_2 \neq U_1$$

- Residual estimator: $\mathcal{E}_k(U_k, \mathcal{T}_k)$ is not reduced by AFEM, and is not even monotone. But, if $U_{k+1} = U_k$, then $\mathcal{E}_k(U_k, \mathcal{T}_k)$ decreases strictly

$$\mathcal{E}^2_{k+1}(U_{k+1}, \mathcal{T}_{k+1}) = \mathcal{E}^2_{k+1}(U_k, \mathcal{T}_{k+1}) \leq \mathcal{E}^2_k(U_k, \mathcal{T}_k) - \lambda \mathcal{E}^2_k(U_k, \mathcal{M}_k)$$

- Heuristics: the quantity $\|U_k - u\|^2_\Omega + \gamma \mathcal{E}_k(U_k, \mathcal{T}_k)^2$ might contract!

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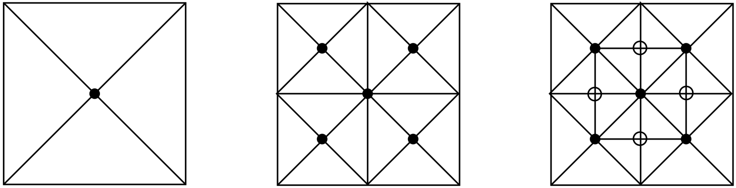

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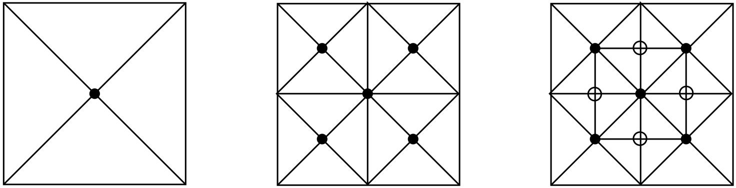
General Data: Contracting Quantities

- Energy error: $\|U_k - u\|_\Omega$ is monotone, but not strictly monotone (e.g. $U_{k+1} = U_k$).



$$\Omega = (0,1)^2, A = I, f = 1 \quad \Rightarrow \quad U_0 = U_1 = \frac{1}{12}\phi_0, \quad U_2 \neq U_1$$

- Residual estimator: $\mathcal{E}_k(U_k, \mathcal{T}_k)$ is not reduced by AFEM, and is not even monotone. But, if $U_{k+1} = U_k$, then $\mathcal{E}_k(U_k, \mathcal{T}_k)$ decreases strictly

$$\mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) = \mathcal{E}_{k+1}^2(U_k, \mathcal{T}_{k+1}) \leq \mathcal{E}_k^2(U_k, \mathcal{T}_k) - \lambda \mathcal{E}_k^2(U_k, \mathcal{M}_k)$$

- Heuristics: the quantity $\|U_k - u\|_\Omega^2 + \gamma \mathcal{E}_k(U_k, \mathcal{T}_k)^2$ might contract!

Contraction Property (Cascón, Kreuzer, Nochetto, Siebert' 08)

Theorem 3. There exist constants $\gamma > 0$ and $0 < \alpha < 1$, depending on the shape regularity constant of $\mathcal{T}_0$, the eigenvalues of A, and θ, such that

$$|\!|\!| u - U_{k+1} |\!|\!|_\Omega^2 + \gamma\, \mathcal{E}_{k+1}^2 \le \alpha^2 \left(|\!|\!| u - U_k |\!|\!|_\Omega^2 + \gamma\, \mathcal{E}_k^2 \right).$$

Main ingredients of the proof:

- Pythagoras: $|\!|\!| U_{k+1} - u |\!|\!|_\Omega^2 = |\!|\!| U_k - u |\!|\!|_\Omega^2 - |\!|\!| U_k - U_{k+1} |\!|\!|_\Omega^2$;
- a posteriori upper bound (not lower (or discrete lower) bound);
- reduction of the estimator;
- Dörfler marking (for estimator).

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Proof of Theorem 3

Error orthogonality $\|u - U_{k+1}\|_\Omega^2 = \|u - U_k\|_\Omega^2 - \|U_k - U_{k+1}\|_\Omega^2$ yields

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le \|u - U_k\|_\Omega^2 - \|U_k - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1})$$

Estimator reduction property implies

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le \|u - U_k\|_\Omega^2 - \|U_k - U_{k+1}\|_\Omega^2 + \gamma(1+\delta)\big(\mathcal{E}_k^2(U_k, \mathcal{T}_k) - \lambda \mathcal{E}_k^2(U_k, \mathcal{M}_k)\big) + \gamma(1+\delta^{-1})C_0^2 \|U_k - U_{k+1}\|_\Omega^2.$$

Choose $\gamma := \frac{1}{(1+\delta^{-1})\,C_0^2}$ to cancel $\|U_k - U_{k+1}\|_\Omega$:

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le \|u - U_k\|_\Omega^2 + \gamma(1+\delta)\mathcal{E}_k^2(U_k, \mathcal{T}_k) - \gamma(1+\delta)\lambda \mathcal{E}_k^2(U_k, \mathcal{M}_k).$$

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Proof of Theorem 3 (Continued)

Dörfler marking $\mathcal{E}_k(U_k, \mathcal{M}_k) \geq \theta \mathcal{E}_k(U_k, \mathcal{T}_k)$ yields

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \leq \|u - U_k\|_\Omega^2 - \frac{1}{2}\gamma(1+\delta)\lambda\theta^2 \mathcal{E}_k^2(U_k, \mathcal{T}_k)$$
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Applying the Upper Bound $\|u - U_k\|_\Omega^2 \leq C_1 \mathcal{E}_k^2(U_k, \mathcal{T}_k)$ gives

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \leq \Big(1 - \frac{1}{2}\gamma(1+\delta)\frac{\lambda\theta^2}{C_1}\Big)\|u - U_k\|_\Omega^2$$
$$+ (1+\delta)\Big(1 - \frac{\lambda\theta^2}{2}\Big)\gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k).$$

Choosing $\delta > 0$ sufficiently small so that

$$\alpha^2 := \max\Big\{1 - \frac{1}{2}\gamma(1+\delta)\frac{\lambda\theta^2}{C_1}, (1+\delta)\Big(1 - \frac{\lambda\theta^2}{2}\Big)\Big\} < 1,$$

we finally obtain the desired estimate

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Outline

AFEM: Design and Convergence

AFEM: Modules

AFEM: Contraction Property

Approximation Class

AFEM: Rate Optimality

Extensions and Limitations

The Total Error

- AFEM controls the new error quantity $\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k)$.
- Since estimator dominates oscillation

$$\operatorname{osc}_k(U_k, \mathcal{T}_k) \le \mathcal{E}_k(U_k, \mathcal{T}_k)$$

and there is a global lower bound,

$$C_2 \mathcal{E}_k^2(U_k, \mathcal{T}_k) \le \|u - U_k\|_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k)$$

$\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k)$ is equivalent to total error and error estimator:

$$\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k) \approx \|u - U_k\|_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k) \approx \mathcal{E}_k^2(U_k, \mathcal{T}_k)$$

- Total error: $E_{\mathcal{T}}(u, A, f; U) := \Big(\|u - U\|_\Omega^2 + \operatorname{osc}_{\mathcal{T}}^2(U, \mathcal{T}) \Big)^{1/2}$

The Total Error: Quasi-Best Approximation

- Operator with pw constant A (eg Laplace) and polynomial degree $n = 1$:

$$E_{\mathcal{T}}^2(u, A, f; U) = \|u - U\|_\Omega^2 + \|h(f - P_0 f)\|_\Omega^2$$

- Pre-asymptotics: $\epsilon = 2^{-K}, u(x) = \frac{1}{2}x(\epsilon - |x|)$ in $(-\epsilon, \epsilon)$ extended periodically ($\epsilon =$ scale of oscillation of $u \ll h = 2^{-k}$). Then

$$\|U - u\|_\Omega \approx 2^{-K} \ll 2^{-k} = \|hf\|_\Omega = \mathrm{osc}_{\mathcal{T}}(U, \mathcal{T}) = \mathcal{E}_{\mathcal{T}}(U, \mathcal{T})$$

and oscillation dominates the total error in the pre-asymptotic regime.

- **Quasi-Best Approximation:** There exists a constant $D > 0$ only depending on oscillation of A on $\mathcal{T}_0$ and on $\mathcal{T}_0$ such that

$$E_{\mathcal{T}}(u, A, f; U) \leq D \inf_{V \in \mathbb{V}(\mathcal{T})} E_{\mathcal{T}}(u, A, f; V).$$

The Total Error: Quasi-Best Approximation

- Operator with pw constant A (eg Laplace) and polynomial degree $n = 1$:

$$E^2_{\mathcal{T}}(u, A, f; U) = \|u - U\|^2_\Omega + \|h(f - P_0 f)\|^2_\Omega$$

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Approximation Class (for Total Error)

The set of all conforming triangulations with at most N elements more than in $\mathcal{T}_0$ is denoted

$$\mathbb{T}_N := \{\mathcal{T} \in \mathbb{T} \mid \#\mathcal{T} - \#\mathcal{T}_0 \leq N\}.$$

The quality of the best approximation to the total error in $\mathbb{T}_N$ is

$$\sigma_N(u; A, f) := \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{V\in\mathbb{V}(\mathcal{T})} E_{\mathcal{T}}(u, A, f; V)$$

For $0 < s \leq n/d$ the approximation class is finally given as

$$\mathbb{A}_s := \left\{(u, A, f) \mid \ |u, A, f|_s := \sup_{N\geq 0} N^s \sigma_N(u; A, f) < \infty\right\}.$$

Approximation of data is explicitly included in the definition of the class $\mathbb{A}_s$:

$$r(V) - P_{n-1}r(V) \quad \text{where} \quad r(V) = \operatorname{div}(A\nabla V) + f,$$

with $n \geq 1$. Nonlinear coupling between A and ∇U via oscillation!

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Sufficient Conditions for Membership in $\mathbb{A}_s$

- Let the dimension $d \geq 2$ and polynomial degree $n \geq 1$.
- Let $0 < p, q \leq \infty$, $0 < s \leq \frac{n}{d}$ such that $s > \frac{1}{p} - \frac{1}{2}$, $s > \frac{1}{q}$.

  If the triple (u, A, f) satisfies

$$u \in B^{1+sd}_{p,p}(\Omega), \quad A \in B^{sd}_{q,q}(\Omega), \quad f \in B^{sd}_{p,p}(\Omega),$$

  then $(u, A, f) \in \mathbb{A}_s$. Moreover, $\operatorname{osc}_{\mathcal{T}}(f)$ exhibits a faster decay $s + \frac{1}{d}$.

- Note that

$$\operatorname{sob}(B^{1+sd}_{p,p}) = 1 + sd - \frac{d}{p} > 1 - \frac{d}{2} = \operatorname{sob}(H^1) \quad \Rightarrow \quad B^{1+sd}_{p,p}(\Omega) \subset H^1(\Omega)$$

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$$\operatorname{sob}(B^{sd}_{q,q}) = sd - \frac{d}{q} > 0 - \frac{0}{\infty} = \operatorname{sob}(L^\infty) \quad \Rightarrow \quad B^{sd}_{q,q}(\Omega) \subset L^\infty(\Omega)$$

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Quasi-Optimal Cardinality: Vanishing Oscillation (Stevenson' 07)

Stevenson's insight: *any marking strategy that reduces the energy error relative to the current value must contain a substantial portion of $\mathcal{E}_{\mathcal{T}}(U,\mathcal{T})$, and so it can be related to Dörfler Marking.*

Lemma 3 (Dörfler Marking). Let $\theta < \theta_* = \sqrt{\frac{C_2}{C_1}}$, and $\mu = 1 - \frac{\theta^2}{\theta_*^2}$. Let $\mathcal{T}_*$ be a conforming refinement of $\mathcal{T}$, and $U_* \in \mathbb{V}(\mathcal{T}_*)$ satisfy

$$\|u - U_*\|_\Omega^2 \le \mu \|u - U\|_\Omega^2.$$

Then the refinement set $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ satisfies Dörfler marking with θ

$$\mathcal{E}_{\mathcal{T}}(U,\mathcal{R}) \ge \theta \mathcal{E}_{\mathcal{T}}(U,\mathcal{T}).$$

Proof: Use lower bound followed by Pythagoras equality

$$\begin{aligned}(1-\mu) C_2 \mathcal{E}_{\mathcal{T}}^2(U,\mathcal{T}) &\le (1-\mu)\|u - U\|_\Omega^2 \\ &\le \|u - U\|_\Omega^2 - \|u - U_*\|_\Omega^2 = \|U - U_*\|_\Omega^2.\end{aligned}$$

Finally, resort to the discrete lower bound

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$$\mathcal{E}_{\mathcal{T}}(U,\mathcal{R}) \ge \theta \mathcal{E}_{\mathcal{T}}(U,\mathcal{T}).$$

Proof: Use lower bound followed by Pythagoras equality

$$\begin{aligned}(1-\mu) C_2 \mathcal{E}_{\mathcal{T}}^2(U,\mathcal{T}) &\le (1-\mu) |\!|\!| u - U |\!|\!|_\Omega^2 \\ &\le |\!|\!| u - U |\!|\!|_\Omega^2 - |\!|\!| u - U_* |\!|\!|_\Omega^2 = |\!|\!| U - U_* |\!|\!|_\Omega^2.\end{aligned}$$

Finally, resort to the discrete lower bound

$$(1-\mu) C_2 \mathcal{E}_{\mathcal{T}}^2(U,\mathcal{T}) \le |\!|\!| U - U_* |\!|\!|_\Omega^2 \le C_1 \mathcal{E}_{\mathcal{T}}^2(U,\mathcal{R}).$$

Quasi-Optimal Cardinality: Vanishing Oscillation (Continued)

$$\mathcal{A}_s: \ |v|_s := \sup_{N>0} \left(N^s \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{V\in\mathbb{V}(\mathcal{T})} \|v - V\|_\Omega \right) < \infty \ \Rightarrow \ \|v - U_N\|_\Omega \le |u|_s N^{-s}.$$

Lemma 4 (Cardinality of $\mathcal{M}_k$) If Dörfler marking chooses minimal set, and $u \in \mathcal{A}_s$, then the k-th marked set $\mathcal{M}_k$ generated by AFEM satisfy

$$\#\mathcal{M}_k \preccurlyeq |u|_s^{\frac{1}{s}} \|u - U_k\|_\Omega^{-\frac{1}{s}}.$$

Proof: Let $\varepsilon^2 = \mu\|u - U_k\|_\Omega^2$. Since $u \in \mathcal{A}_s$ there exist $\mathcal{T}_\varepsilon \in \mathbb{T}$ and $U_\varepsilon \in \mathbb{V}(\mathcal{T}_\varepsilon)$ such that

$$\|u - U_\varepsilon\|_\Omega^2 \le \varepsilon^2, \qquad \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \preccurlyeq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

We introduce the overlay $\mathcal{T}_* = \mathcal{T}_\varepsilon \oplus \mathcal{T}_k$, and exploit that $\mathcal{T}_* \ge \mathcal{T}_\varepsilon$ to get

$$\|u - U_*\|_\Omega^2 \le \|u - U_\varepsilon\|_\Omega^2 \le \varepsilon^2 = \mu\|u - U\|_\Omega^2.$$

This implies $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ satisfies Dörfler marking with $\theta < \theta_*$. Since $\mathcal{M}_k$ is minimal, we conclude

$$\#\mathcal{M}_k \le \#\mathcal{R} \le \#\mathcal{T}_* - \#\mathcal{T}_k \le \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \preccurlyeq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

Quasi-Optimal Cardinality: Vanishing Oscillation (Continued)

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Quasi-Optimal Cardinality: Vanishing Oscillation (Continued)

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Quasi-Optimal Cardinality: Vanishing Oscillation (Continued)

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This implies $\mathcal{R} = \mathcal{R}_{\mathcal{T} \to \mathcal{T}_*}$ satisfies Dörfler marking with $\theta < \theta_*$. Since $\mathcal{M}_k$ is minimal, we conclude

$$\#\mathcal{M}_k \leq \#\mathcal{R} \leq \#\mathcal{T}_* - \#\mathcal{T}_k \leq \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \preccurlyeq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

Quasi-Optimal Cardinality: General Data (Cascón, Kreuzer, Nochetto, Siebert' 08)

Lemma 5 (Dörfler Marking) Let $\theta < \theta_* = \sqrt{\frac{C_2}{1+C_1(1+C_3)}}$, with C_3 explicitly depending on A and $\mathcal{T}_0$, and $\mu = \frac{1}{2}(1 - \frac{\theta^2}{\theta_*^2})$. Let $\mathcal{T}_* \leq \mathcal{T}$ and $U_* \in \mathbb{V}(\mathcal{T}_*)$ satisfy

$$|||u - U_*|||_\Omega^2 + \operatorname{osc}_{\mathcal{T}_*}^2(U_*, \mathcal{T}_*) \leq \mu\Big(|||u - U|||_\Omega^2 + \operatorname{osc}_{\mathcal{T}}^2(U, \mathcal{T})\Big).$$

Then the refinement set $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ satisfies Dörfler marking with θ

$$\mathcal{E}_{\mathcal{T}}(U, \mathcal{R}) \geq \theta \mathcal{E}_{\mathcal{T}}(U, \mathcal{T}).$$

Lemma 6 (Cardinality of $\mathcal{M}_k$). If Dörfler marking chooses a minimal set $\mathcal{M}_k$, and $(u, A, f) \in \mathbb{A}_s$, then the k-th mesh $\mathcal{T}_k$ and marked set $\mathcal{M}_k$ generated by AFEM satisfy

$$\#\mathcal{M}_k \preccurlyeq |(u, A, f)|_s^{\frac{1}{s}} \Big(|||U_k - u|||_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k)\Big)^{-\frac{1}{2s}}.$$

Theorem 4 (Quasi-Optimal Cardinality of AFEM)

If $(u, A, f) \in \mathbb{A}_s$ for $s > 0$, then AFEM produces a sequence $\{\mathcal{T}_k, U_k\}_{k=0}^{\infty}$ of conforming bisection meshes and discrete solutions such that

$$\Big(\|U_k - u\|_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k)\Big)^{1/2} \preccurlyeq |u, A, f|_s \big(\#\mathcal{T}_k - \#\mathcal{T}_0\big)^{-1/s}.$$

- Counting DOF (Binev, Dahmen, DeVore '04, Stevenson '06):

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \preccurlyeq \sum_{j=0}^{k-1} \#\mathcal{M}_j \preccurlyeq \sum_{j=0}^{k-1} \Big(\|U_j - u\|_\Omega^2 + \operatorname{osc}_j^2(U_j, \mathcal{T}_j)\Big)^{-\frac{1}{2s}}.$$

- Contraction Property of AFEM:

$$\|U_k - u\|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k) \le \alpha^{2(k-j)} \Big(\|U_j - u\|_\Omega^2 + \mathcal{E}_j^2(U_j, \mathcal{T}_j)\Big),$$

whence

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \preccurlyeq \Big(\|U_k - u\|_\Omega^2 + \gamma \underbrace{\mathcal{E}_k^2(U_k, \mathcal{T}_k)}_{\ge \operatorname{osc}_k^2(U_k, \mathcal{T}_k)}\Big)^{-\frac{1}{2s}} \underbrace{\sum_{j=0}^{k} \alpha^{\frac{j}{s}}}_{<(1-\alpha^{\frac{1}{s}})^{-1}}.$$

Outline

References

- R.H. Nochetto, K.G. Siebert and A. Veeser, *Theory of adaptive finite element methods: an introduction*, in *Multiscale, Nonlinear and Adaptive Approximation*, R. DeVore and A. Kunoth eds, Springer (2009), 409-542.
- R.H. Nochetto and A. Veeser, *Primer of adaptive finite element methods*, in *Multiscale and Adaptivity: Modeling, Numerics and Applications*, CIME Lectures, eds R. Naldi and G. Russo, Springer, (2012), 125–225.
- A. Bonito, J.M. Cascón, K. Mekchay, P. Morin and R.H. Nochetto, *High-order AFEM for the Laplace-Beltrami operator on parametric surfaces: convergence rates*, Found. Comp. Math. 16 (2016), 1473-1539.

Extensions of Theory

- **Non-Residual Estimators** (Cascón, Nochetto; Kreuzer, Siebert' 10).
- **Non-conforming meshes** (Bonito, Nochetto' 10).
- **Adaptive dG (interior penalty)** (Bonito, Nochetto' 10).
- **Raviart-Thomas mixed methods** (Chen, Holst, Xu' 09).
- **Edge elements for Maxwell** (Zhong, Chen, Shu, Wittum, Xu' 10).
- **Laplace-Beltrami on parametric surfaces** (Bonito, Cascón, Mekchay, Morin, Nochetto' 13, 16).
- **Local H^1-norm** (Demlow' 10) and **L^2-norm** (Demlow, Stevenson' 10).
- **H^{-1}-data** (Stevenson' 07; Cohen, DeVore, Nochetto' 12; Kreuzer, Veeser' 18).
- **Discontinuous coefficients** (Bonito, DeVore, Nochetto' 13).
- **Hierarchical splines** (Buffa, Giannelli' 16; Buffa, Garau' 16; Morin, Nochetto, Pauletti' 18).

Limitations of Theory

- **Pythagoras or variants:** does not apply to saddle point problems.
- **Quasi-orthogonality property:** applies to some saddle point problems (Taylor-Hood elements for Stokes (Feischl' 18)).
- **Other norms** such as $L^\infty, L^p, W^1_\infty$.