Adaptive Finite Element Methods
Lecture 4: Extensions I

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Outline

AFEM for PDEs with Discontinuous Coefficients (w. A. Bonito and R. DeVore)

Nonresidual Estimators (w. J.M. Cascón)

Nonconforming Meshes (w. A. Bonito)

Outline

Continuous Dependence on Data

Model problem: consider elliptic PDE of the form

$$-\operatorname{div}(A\nabla u) = f$$

- $A = (a_{ij}(x))_{i,j=1}^d$ uniformly positive definite and bounded

$$\lambda_{\min}(A)|y|^2 \le y^t A(x) y \le \lambda_{\max}(A)|y|^2 \quad \forall\, x \in \Omega,\ y \in \mathcal{R}^d;$$

- The discontinuities of A are not match by the sequence of meshes $\mathcal{T}$;
- The forcing $f \in W_p^{-1}(\Omega)$ for some $p > 2$.

Goal: Design and study an AFEM able to handle such an A.

Difficulty: PDE perturbation results hinge on approximation of A in L^∞

$$\|u - \widehat{u}\|_{H_0^1(\Omega)} \le \lambda_{\min}^{-1}(\widehat{A})\Big(\|f - \widehat{f}\|_{H^{-1}(\Omega)} + \|A - \widehat{A}\|_{L_\infty(\Omega)}\|f\|_{H^{-1}(\Omega)}\Big)$$

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Perturbation Argument

Theorem (perturbation). Let $p \geq 2, q = 2p/(p-2) \in [2,\infty]$ and $\nabla u \in L^p(\Omega)$. Then

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Question: can we guarantee that $\nabla u \in L^p(\Omega)$ with $p > 2$ but $A \in L^\infty(\Omega)$?

Proposition (Meyers). Let $\widetilde{K} > 0$ be so that the solution $\widetilde{u}$ of the Laplacian satisfies

$$\|\nabla \widetilde{u}\|_{L^p(\Omega)} \leq \widetilde{K}\|f\|_{W_p^{-1}(\Omega)}.$$

Then the solution u of $-\operatorname{div}(A\nabla u) = f$ satisfies

$$\|\nabla u\|_{L^p(\Omega)} \leq K\|f\|_{W_p^{-1}(\Omega)}$$

if $2 \leq p < p^*$ and $K = \frac{1}{\lambda_{\max}(A)} \frac{\widetilde{K}^{\eta(p)}}{1-\widetilde{K}^{\eta(p)}\left(1-\frac{\lambda_{\min}(A)}{\lambda_{\max}(A)}\right)}$ with $\eta(p) = \frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p^*}}$.

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DISC: AFEM for Discontinuous Diffusion Matrices

Given $0 < \omega \leq \omega_0$ sufficiently small (but explicit) and any $\beta < 1$, let

DISC$(\mathcal{T}_0, \epsilon_1)$
  $k = 1$
  LOOP
    $[\mathcal{T}_k(f), f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$
    $[\mathcal{T}_k(A), A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$
    $[\mathcal{T}_k, U_k] =$ **PDE**$(\mathcal{T}_k(A), A_k, f_k, \varepsilon_k/2)$
    $\epsilon_{k+1} = \beta\epsilon_k$
    $k \leftarrow k+1$
  END LOOP
END **DISC**

Modules of DISC

- $[\mathcal{T}_k(f), f_k] = \textbf{RHS}(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$

  gives a mesh $\mathcal{T}_k(f) \geq \mathcal{T}_{k-1}$ and a pw polynonial approximation f_k of f on $\mathcal{T}_k(f)$ such that $\|f - f_k\|_{H^{-1}(\Omega)} \leq \omega\epsilon_k$.

- $[\mathcal{T}_k(A), A_k] = \textbf{COEFF}(\mathcal{T}_k(f), A, \omega\varepsilon_k)$

  gives a mesh $\mathcal{T}_k(A) \geq \mathcal{T}_k(f)$ and a piecewise polynomial approximation A_k of A on $\mathcal{T}_k(A)$ such that $\|A - A_k\|_{L^q(\Omega)} \leq \omega\epsilon_k$ and its eigenvalues satisfy uniformly in k

$$C^{-1}\lambda_{\min}(A) \leq \lambda(A_k) \leq C\lambda_{\max}(A).$$

- $[\mathcal{T}_k, U_k] = \textbf{PDE}(\mathcal{T}_k(A), A_k, f_k, \varepsilon_k/2)$

  gives a mesh $\mathcal{T}_k \geq \mathcal{T}_k(A)$ and a Galerkin solution $U_k \in \mathbb{V}(\mathcal{T}_k)$ of a PDE with oscillation free data (A_k, f_k) and error tolerance $\varepsilon_k/2$.

Approximation Classes

- **Approximating u:** $u \in \mathcal{A}^s(H_0^1(\Omega))$ with $s \le m_u/d$ if given any ε there exists a non-conforming refinement $\mathcal{T}_n$ of $\mathcal{T}_0$ with n more elements and a continuous piecewise polynomial V over $\mathcal{T}_n$ of degree $\le m_u$ such that

$$\|u - V\|_{H_0^1(\Omega)} \le \varepsilon, \qquad n \le |u|_{\mathcal{A}^s}^{1/s} \varepsilon^{-1/s}.$$

- **Approximating f:** $f \in \mathcal{B}^s(H^{-1}(\Omega))$ with $s \le m_f/d$ if given any ε there exists a non-conforming refinement $\mathcal{T}_n$ of $\mathcal{T}_0$ with n more elements and a piecewise polynomial V over $\mathcal{T}_n$ of degree $\le m_f$ such that

$$\|f - V\|_{H^{-1}(\Omega)} \le \varepsilon, \qquad n \le |f|_{\mathcal{B}^s}^{1/s} \varepsilon^{-1/s}.$$

- **Approximating A:** $A \in \mathcal{M}^s(L_q(\Omega))$ with $s \le m_A/d$ if given any ε there exists a non-conforming refinement $\mathcal{T}_n$ of $\mathcal{T}_0$ with n more elements and a piecewise polynomial matrix V over $\mathcal{T}_n$ of degree $\le m_A$ such that

$$\|A - V\|_{L_q(\Omega)} \le \varepsilon, \qquad n \le |A|_{\mathcal{M}^s}^{1/s} \varepsilon^{-1/s}.$$

Optimality of DISC

Theorem (optimality). Assume that the right side f is in $\mathcal{B}^{s_f}(H^{-1}(\Omega))$ and that the diffusion matrix A is positive definite, in $L_\infty(\Omega)$ and in $\mathcal{M}^{s_A}(L_q(\Omega))$ for $q := \frac{2p}{p-2}$. Let $\mathcal{T}_0$ be the initial subdivision and $U_k \in \mathbb{V}(\mathcal{T}_k)$ be the Galerkin solution obtained at the k-th iteration of the algorithm. Then, whenever $u \in \mathcal{A}^{s_u}(H_0^1(\Omega))$, we have for $k \geq 1$

$$\|u - U_k\|_{H_0^1(\Omega)} \leq \epsilon_k,$$

and

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \Big(|A|^{1/s_*}_{\mathcal{M}^{s_*}(L_q(\Omega))} + |f|^{1/s_*}_{\mathcal{B}^{s_*}(H^{-1}(\Omega))} + |u|^{1/s_*}_{\mathcal{A}^{s_*}(H_0^1(\Omega))} \Big) \epsilon_k^{-1/s_*},$$

with $s_* = \min(s_u, s_A, s_f)$.

Counterexample: s_u cannot be achieved if $s_A, s_f < s_u$.

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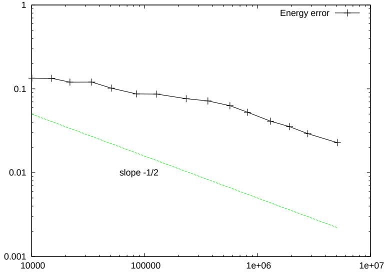
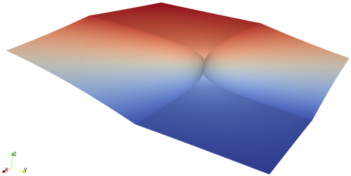
Proving Theorem (optimality)

- Approximation of f and A: this uses GREEDY
- Enforcing positive definiteness: the approximation of A must be uniformly positive definite and bounded.
- The module **PDE** approximates the solution u_k of the PDE with data (A_k, f_k), which is not u. However, u_k inherits the degree of approximation s_u of u at scales larger than ε_k.
- The module **PDE** perform a fixed number of steps to reduce the error from ε_k to $\varepsilon_k/2$.

Checkerboard Example: $u \approx r^{1.25}$





Checkerboard: The parameters are chosen so that the solution $u \in H^{1+s}(\Omega)$, $s < 0.25$. (Left) Energy error versus number of degrees of freedom. The optimal rate of convergence ≈ -0.5 is recovered. (Right) The Galerkin solution together with the underlying partition after 6 iterations of the algorithm **DISC**. The discontinuity of A is never captured by the partitions and the singularities of both A and ∇u drive the refinements.

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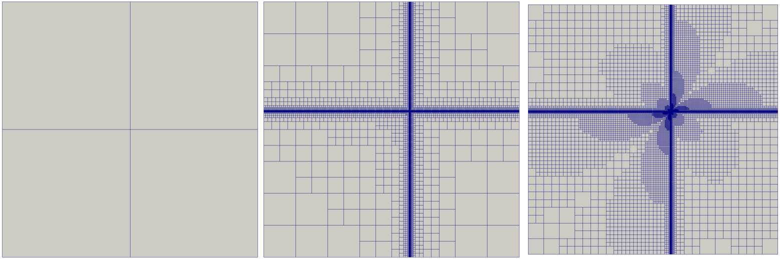

Checkerboard: Sequence of partitions (from left to right) generated by **DISC** with $\omega = 0.8$. The initial partition (first) is made of four quadrilaterals, The algorithm refines at early stages only to capture the discontinuity in the diffusion coefficient (second). Later the singularity of u comes into play and, together with that of A, drives the refinement (third). The corresponding subdivision consists of 5 million degrees of freedom. The smallest cell has a diameter of 2^{-8} which illustrates the strongly graded mesh constructed by **DISC**.

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Continuous Model Problem

Consider the following linear elliptic symmetric PDE in weak form:

$$u \in V : \quad \mathcal{B}(u, v) := \int_\Omega \boldsymbol{A} \nabla u \cdot \nabla v \, dx = \langle f, v \rangle_\Omega \quad \forall v \in V$$

where

- $\Omega \subset \mathcal{R}^d$ $(d \in \mathbb{N})$ is a polyhedral domain that is triangulated by $\mathcal{T}_0$;
- $\boldsymbol{A} \in L^\infty(\Omega; \mathcal{R}^{d \times d})$ is pw. Lipschitz over $\mathcal{T}_0$ and uniformly spd;
- $f \in L^2(\Omega)$ and $\langle \cdot, \cdot \rangle_\Omega$ is the L^2 scalar product.

The bilinear form $\mathcal{B}$ defines the corresponding energy-norm by

$$|\!|\!| v |\!|\!|_\Omega := \mathcal{B}(v, v)^{1/2},$$

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AFEM

We consider the most standard adaptive iteration

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}$$

- Module SOLVE: $U_{\mathcal{T}} = \text{SOLVE}(\mathcal{T})$ computes the Ritz projection $U_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ to u:
  - assuming exact integration;
  - assuming exact solution of the discrete system;
- Module ESTIMATE: $\{\mathcal{E}_{\mathcal{T}}(T)\}_{T\in\mathcal{T}} = \text{ESTIMATE}(U_{\mathcal{T}}, \mathcal{T})$ computes (local) error indicators yielding the estimator:
  - non-residual estimator;
- Module MARK: $\mathcal{M} = \text{MARK}(\{\mathcal{E}_{\mathcal{T}}(T), \mathcal{T}\}_{T\in\mathcal{T}})$ selects a subset $\mathcal{M} \subset \mathcal{T}$ of elements subject to refinement:
  - Dörfler marking (single marking according to local error indicator);
- Module REFINE: $\mathcal{T}_* = \text{REFINE}(\mathcal{T}, \mathcal{M})$ refines all elements in $\mathcal{M}$ and outputs a conforming refinement $\mathcal{T}_* \geq \mathcal{T}$:
  - All elements of $\mathcal{M}$ are bisected $b \geq 1$ (no interior node property).

Extension to Nonresidual Estimators

- Hierarchical estimators (Bornemann, Kornhuber, Veeser)
- Star estimators (Morin, Nochetto, Siebert; Carstensen, Funken)
- Star estimators (Parés, Diez, Huerta)
- $H(\mathrm{div})$ estimators (Braess, Hoppe, Schöberl; Ern, Smears, Vohralik)
- Gradient recovery (Zienkiewicz, Zhu)

Star Estimator (Morin, Nochetto, Siebert' 02; Carstensen, Funken '99)

- **Ideal star estimator:** Let ω_z be a star for $z \in \mathcal{N}$ and define

$$W(\omega_z) := \{v \in H^1_{loc}(\omega_z) : \int_{\omega_z} v\phi_z = 0, \int_{\omega_z} |\nabla v|^2 \phi_z < \infty\}$$

  Let $\zeta_z \in W(\omega_z)$ solve the variational problem with weight ϕ_z

$$\int_{\omega_z} A\nabla\zeta_z \nabla\varphi\, \phi_z = \int_{\omega_z} f\varphi\, \phi_z - \int_{\omega_z} A\nabla U \nabla(\varphi\, \phi_z).$$

- **Local error indicator:**

$$\eta_z := \Big(\int_{\omega_z} A\nabla\zeta_z \nabla\zeta_z\, \phi_z \Big)^{\frac{1}{2}}.$$

- **Global error estimator:**

$$\eta = \eta(U, \mathcal{T}) := \Big(\sum_{z\in\mathcal{N}} \eta_z^2 \Big)^{\frac{1}{2}}.$$

- **Similar estimators:**
  - **Fractional Diffusion:** L. Chen, Nochetto, Otárola, Salgado '15.
  - **Hierarchical B-splines:** Morin, Nochetto, Pauletti '18.

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Constant Free Upper Bound

Let $e := u - U$, $\varphi = e$, $\|e\|_\Omega^2 = \int_\Omega A\nabla e \nabla e$, recall $\sum_{z\in\mathcal{N}} \phi_z = 1$, and compute

$$\|e\|_\Omega^2 = \langle \mathcal{R}(U), \varphi\rangle = \int_\Omega f\varphi - \int_\Omega A\nabla U \nabla \varphi$$

$$= \sum_{z\in\mathcal{N}} \int_\Omega f\varphi\phi_z - \int_\Omega A\nabla U \nabla(\varphi\phi_z)$$

$$= \sum_{z\in\mathcal{N}} \int_{\omega_z} f(\varphi - c_\varphi)\phi_z - \int_{\omega_z} A\nabla U \nabla\big((\varphi - c_\varphi)\phi_z\big) = \sum_{z\in\mathcal{N}} \int_{\omega_z} A\nabla\zeta_z \nabla\varphi\,\phi_z$$

for constant $c_\varphi = \frac{\int_{\omega_z}\varphi\phi_z}{\int_{\omega_z}\phi_z}$ because of Galerkin orthogonality

$$\int_{\omega_z} f\phi_z - \int_{\omega_z} A\nabla U \nabla \phi_z = 0.$$

Therefore applying Cauchy-Schwarz

$$\|e\|_\Omega^2 \le \Big(\underbrace{\sum_{z\in\mathcal{N}} \int_{\omega_z} A\nabla\zeta_z\nabla\zeta_z\phi_z}_{=\eta_z^2}\Big)^{\frac{1}{2}} \Big(\underbrace{\sum_{z\in\mathcal{N}} \int_{\omega_z} A\nabla\varphi\nabla\varphi\phi_z}_{=\int_\Omega A\nabla\varphi\nabla\varphi = \|e\|_\Omega^2}\Big)^{\frac{1}{2}} \quad\Rightarrow\quad \|e\|_\Omega^2 \le \eta(U,\mathcal{T}).$$

Practical Star Estimator

- **Computability:** the ideal indicator η_z is not computable.
- **Polynomial degree:** Let U be piecewise linear Galerkin solution.
- **Local discrete space:** Let $\mathcal{P}_0^2(\omega_z)$ be the space of piecewise quadratic polynomials on the star ω_z that vanish on $\partial\omega_z$.
- **Local problem:** Let $\xi_z \in \mathcal{P}_0^2(\omega_z)$ solve

$$\int_{\omega_z} A\nabla\xi_z\nabla\varphi\phi_z = \int_{\omega_z} f\varphi\phi_z - \int_{\omega_z} A\nabla U\nabla\varphi\phi_z \quad \forall\ \varphi \in \mathcal{P}_0^2(\omega_z)$$

- **Local indicator:**

$$\eta_z^2 = \eta_z^2(U,\mathcal{T}) = \int_{\omega_z} A\nabla\xi_z\nabla\eta_z\phi_z$$

- **Global error estimator:**

$$\eta_{\mathcal{T}}(U,\mathcal{T}) = \left(\sum_{z\in\mathcal{N}} \eta_z^2\right)^{\frac{1}{2}}.$$

Properties of Estimators

- **Global Upper Bound:** there exists C_1 depending on $\mathcal{T}_0$ so that

$$|\!|\!| U_\mathcal{T} - u |\!|\!|_\Omega^2 \le C_1 \Big(\eta_\mathcal{T}^2(U_\mathcal{T}, \mathcal{T}) + \operatorname{osc}_\mathcal{T}^2(U_\mathcal{T}, \mathcal{T}) \Big) =: \mathcal{E}_\mathcal{T}^2(U_\mathcal{T}, \mathcal{T})$$

- **Localized Upper Bound:** if $\mathcal{T} \le \mathcal{T}_*$ and $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ is the set of refined elements of $\mathcal{T}$ to go to $\mathcal{T}_*$, then

$$|\!|\!| U_{\mathcal{T}_*} - U_\mathcal{T} |\!|\!|_\Omega^2 \le C_1 \Big(\eta_\mathcal{T}^2(U_\mathcal{T}, \mathcal{R}) + \operatorname{osc}_\mathcal{T}^2(U_\mathcal{T}, \mathcal{R}) \Big) =: \mathcal{E}_\mathcal{T}^2(U_\mathcal{T}, \mathcal{R})$$

- **Discrete Local Lower Bound:** if $\mathcal{T} \le \mathcal{T}_*$ and $\mathcal{G}^\ell_{\mathcal{T}\to\mathcal{T}_*}$ is the set of simplices of $\mathcal{R}_{\mathcal{T}\to\mathcal{T}_*} \subset \mathcal{T}$ which are bisected at least ℓ times in $\mathcal{T}_*$, then

$$C_2 \eta_\mathcal{T}^2(U_\mathcal{T}, \mathcal{G}^\ell_{\mathcal{T}\to\mathcal{T}_*}) \le |\!|\!| U_\mathcal{T} - U_{\mathcal{T}_*} |\!|\!|_\Omega^2 + \operatorname{osc}_\mathcal{T}^2(U_\mathcal{T}, \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*})$$

- **Interior Node Property:** this is guaranteed by the prescribed ℓ levels of refinement ($\ell = 3$ for $d = 2$; $\ell = 6$ for $d = 3$) Then $U_\mathcal{T}$ and $U_{\mathcal{T}_*}$ cannot be consecutive Galerkin solutions but rather obtained after ℓ iterations of AFEM.

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- **Discrete Local Lower Bound:** if $\mathcal{T} \le \mathcal{T}_*$ and $\mathcal{G}^\ell_{\mathcal{T}\to\mathcal{T}_*}$ is the set of simplices of $\mathcal{R}_{\mathcal{T}\to\mathcal{T}_*} \subset \mathcal{T}$ which are bisected at least ℓ times in $\mathcal{T}_*$, then

$$C_2\eta_\mathcal{T}^2(U_\mathcal{T},\mathcal{G}^\ell_{\mathcal{T}\to\mathcal{T}_*}) \le \|U_\mathcal{T} - U_{\mathcal{T}_*}\|_\Omega^2 + \mathrm{osc}_\mathcal{T}^2(U_\mathcal{T},\mathcal{R}_{\mathcal{T}\to\mathcal{T}_*})$$

- **Interior Node Property:** this is guaranteed by the prescribed ℓ levels of refinement ($\ell = 3$ for $d = 2$; $\ell = 6$ for $d = 3$) Then $U_\mathcal{T}$ and $U_{\mathcal{T}_*}$ cannot be consecutive Galerkin solutions but rather obtained after ℓ iterations of AFEM.

Nonresidual Estimators (Continued)

Module MARK: uses Dörfler marking with the total error indicators $\{\mathcal{E}_{\mathcal{T}}(U_{\mathcal{T}}, T)\}_{T\in\mathcal{T}}$ to select $\mathcal{M}$

$$\mathcal{M} = \mathrm{MARK}\ (\{\mathcal{E}_{\mathcal{T}}(U_{\mathcal{T}}, T)\}_{T\in\mathcal{T}}, \mathcal{T})$$

Module REFINE: bisects elements of $\mathcal{M}$ at least $b \geq 1$ times and updates the refinement flag $\rho_{\mathcal{T}}(T)$ of all elements $T \in \mathcal{T}$ to guarantee ℓ levels of refinement of marked elements

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Theorem (Contraction Property of AFEM). There exists $0 < \alpha < 1$ and $\gamma > 0$, depending $\mathcal{T}_0, \theta, b, C_1, C_2$, and data A, such that

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The proof is a combination of those by Mekchay-Nochetto '05 and Cascón-Kreuzer-Nochetto-Siebert '08 (Lecture 2).

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Comparison with Residual Estimators

- AFEM uses one Dörfler marking according to $\mathcal{E}_k(U_k, \mathcal{T}_k)$ as before.
- AFEM enforces an interior node property after ℓ refinement steps ($\ell = 2$ for $d = 2$ and $\ell = 6$ for $d = 3$).
- The estimator $\eta_{\mathcal{T}}(V, \mathcal{T})$ does not dominate the error nor the oscillation. However

$$|||u - U_k|||_{\Omega}^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k) \approx \eta_k^2(U_k, \mathcal{T}_k) + \operatorname{osc}_k^2(U_k, \mathcal{T}_k) \approx \mathcal{E}_k^2(U_k, \mathcal{T}_k)$$

- The estimator $\eta_{\mathcal{T}}(V, \mathcal{T})$ does not have a reduction property.
- Proof uses the discrete local lower bound.
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Quasi-Optimal Decay Rates

- REFINE creates a conforming refinement $\mathcal{T}_*$ of $\mathcal{T}$ by bisection and guarantees that every marked element satisfies the interior node property after ℓ refinement steps;
- MARK chooses a set $\mathcal{M}$ with minimal cardinality;
- The Dörfler parameter θ satisfies $0 < \theta < \theta_*$ for a suitable $\theta_* < 1$.
- Triple (u, f, A) is in the approximation class $\mathbb{A}_s$ $(0 < s \leq 1/2)$: given any $\varepsilon > 0$ there exists $\mathcal{T}_\varepsilon$ such that

$$|\!|\!| u - U |\!|\!|_\Omega + \operatorname{osc}_\mathcal{T}(U, \mathcal{T}) \leq \varepsilon, \qquad \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \lesssim |u, f, A|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

Theorem (optimal cardinality of AFEM) Let $\{\mathcal{T}_k, V(\mathcal{T}_k), U_k\}_{k=0}^\infty$ be the sequence of conforming meshes, conforming finite element spaces, and Galerkin solutions generated by AFEM. If $(u, f, A) \in \mathbb{A}_s$, then

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Outline

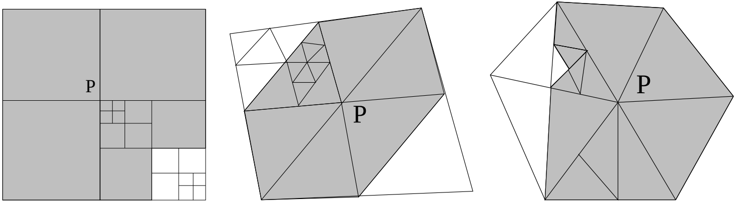
AFEM for PDEs with Discontinuous Coefficients (w. A. Bonito and R. DeVore)

Nonresidual Estimators (w. J.M. Cascón)

Nonconforming Meshes (w. A. Bonito)

Extension to Nonconforming Meshes

- Hanging nodes: quad-refinement, red refinement, bisection showing domain of influence of conforming node P.



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- Quad refinements in 2d and 3d are used in `deal.II` (Bangerth, Hartmann, Kanschat). It does not require a suitable initial labeling.
- Complexity of REFINE (Bonito and Nochetto' 10): there exists a constant $\Lambda_0 > 0$ only depending on $\mathcal{T}_0$ and d such that for all $k \geq 1$

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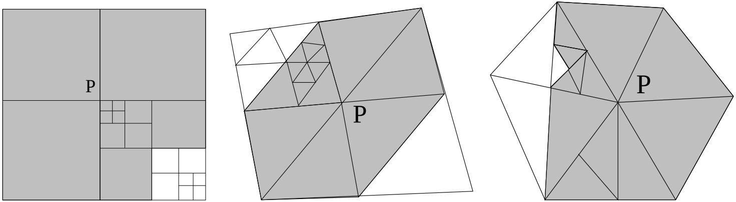

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Optimal Cardinality with Nonconforming Meshes

- The matrix A is symmetric, uniformly positive definite and Lipschitz in each element of $\mathcal{T}_0$;
- The forcing $f = -\operatorname{div}(A\nabla u)$ is in $L^2(\Omega)$;
- $\mathcal{T}_* =$ REFINE$(\mathcal{T})$ creates a non-conforming mesh $\mathcal{T}_*$ from $\mathcal{T}$ with a fixed level of nonconformity (e.g. one hanging node per edge);
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