Adaptive Finite Element Methods
Lecture 7: hp-AFEM - Convergence, Optimality and p-Robustness

Ricardo H. Nochetto

Department of Mathematics and
Institute for Physical Science and Technology
University of Maryland

Joint work with

Claudio Canuto, Politecnico di Torino, Italy
Rob Stevenson, Korteweg-de Vries Institute for Mathematics, The Netherlands
Marco Verani, Politecnico di Milano, Italy

School - Fundamentals and Practice of Finite Elements
Roscoff, France, April 16-20, 2018

Outline

Setting for hp-Adaptivity

Basic hp-Adaptive Algorithm

hp-Adaptive Approximation

Practical Realizations of the hp-Adaptive Algorithm

Saturation Properties

References

Challenges for hp-Adaptive Approximation

- **Example 1: Non-degenerate function.** (Babuška-Gui, DeVore-Scherer) Let $u(x) = x^\alpha$ with $\alpha < 1$ on $I_0 = [0,1]$. The exponential rate of convergence hinges on assumption
$$C_2 \leq \frac{E(I,p+1)}{E(I,p)} \leq C_1$$
for all intervals $I \subset [0,1]$ and all p, with C_1, C_2 independent of I and p. This is a non-degeneracy condition for all frequences.

- **Example 2: Lacunary function.** Let u be a highly oscillatory polynomial of degree $2^L - 1$ such that on each dyadic subinterval K_ℓ of $[0,1]$ of generation $0 \leq \ell < L$, u is L^2-orthogonal to linear polynomials with vanishing mean. Such polynomial exists (counting argument) and leads to a sequence of piecewise constant functions u_ℓ for all $0 \leq \ell \leq L$, which in K_ℓ equals the mean-value of u. This function u is extremely sparse, but the sparsity gap is responsible for early h-refinements. This suggest the need of coarsening.

PDE and Error Functional

- **PDE:** Consider the model elliptic problem in Ω

$$-\Delta u = f \quad \Omega, \qquad u = 0 \quad \partial\Omega.$$

- **Assumptions on Data:**
  - **Domain:** Ω is a bounded polygonal domain in $\mathbb{R}^2$
  - **Forcing:** $f \in L^2(\Omega)$
- ***hp*-partition:** $\mathcal{D} = \{D\}$, with $D = (K_D, p_D)$ an hp-element with meshsize h_D and polynomial degree p_D.
- **Local error functional:** Given $(v, f) \in H^1(K_D) \times L^2(K_D)$, and a penalty parameter $\kappa > 0$, consider the error

$$\mathrm{e}_D(v, f) := |v - P^1_{p_D} v|^2_{H^1(K_D)} + \frac{1}{\kappa} \|p_D^{-1} h_D (f - P^0_{p_D - 1} f)\|^2_{L^2(K_D)}$$

  where $P^1_{p_D} v$ and $P^0_{p_D-1} f$ are the H^1 and L^2 projections onto the local spaces V_{K_D} and F_{D_K} of polynomials of degree p_D and $p_D - 1$.
- **Global error functional:** given an hp-partition $\mathcal{D}$, let

$$\mathrm{E}_{\mathcal{D}}(v, f) := \sum_{D \in \mathcal{D}} e_D(v, f).$$

Monotonicity of Error Functionals

- **Local monotonicity:** The quantity $e_D = e_D(v, f)$ is non-increasing under both 'h-refinements' and 'p-enrichments', in the sense that
  - h-**refinement**

    $$e_{D'} + e_{D''} \leq e_D \quad \text{if } K_{D'}, K_{D''} \text{ are children of } K_D \text{ and } p_{D'} = p_{D''} = p_D;$$

  - p-**enrichment**

    $$e_{D'} \leq e_D \quad \text{if } K_{D'} = K_D \text{ and } p_{D'} \geq p_D.$$

- **Global monotonicity:**

$$\mathrm{E}_{\widetilde{\mathcal{D}}}(v, f) \leq \mathrm{E}_{\mathcal{D}}(v, f) \qquad \text{if} \quad \mathcal{D} \leq \widetilde{\mathcal{D}}.$$

- **Cardinality and Conformity:**
  - The cardinality of $\mathcal{D}$ is defined as $\#\mathcal{D} := \sum_{D \in \mathcal{D}} d_D$ (d_D local dimension).
  - The set $\mathbb{D}$ of all hp-partitions contains the subset $\mathbb{D}^c$ of the 'conforming' partitions. We assume that for any $\mathcal{D} \in \mathbb{D}$ there exists a conforming partition $\mathcal{C}(\mathcal{D})$ such that

    $$\mathcal{D} \leq \mathcal{C}(\mathcal{D}).$$

Outline

Setting for hp-Adaptivity

Basic hp-Adaptive Algorithm

hp-Adaptive Approximation

Practical Realizations of the hp-Adaptive Algorithm

Saturation Properties

References

Basic Modules

Let $V = H_0^1(\Omega)$ and $F = L^2(\Omega)$. We assume availability of the following routines, which realize the two fundamental steps of the algorithm.

- $[\mathcal{D}, f_{\mathcal{D}}] :=$ **hp-NEARBEST**(ε, v, f)

  The routine **hp-NEARBEST** takes as input $\varepsilon > 0$ and $(v, f) \in V \times F$, and outputs $\mathcal{D} \in \mathbb{D}$ as well as $f_{\mathcal{D}}$, such that

  (i) $\mathrm{E}_{\mathcal{D}}(v, f)^{\frac{1}{2}} \leq \varepsilon$;

  (ii) $\#\mathcal{D} \leq B\#\widehat{\mathcal{D}}$ for any $\widehat{\mathcal{D}} \in \mathbb{D}$ with $\mathrm{E}_{\widehat{\mathcal{D}}}(v, f)^{\frac{1}{2}} \leq b\varepsilon$, for some constants $0 < b \leq 1 \leq B$.

  This routine may be implemented via Binev's algorithm (see below).

- $[\bar{\mathcal{D}}, \bar{u}] :=$ **PDE**$(\varepsilon, \mathcal{D}, f_{\mathcal{D}})$

  The routine **PDE** takes as input $\varepsilon > 0$, $\mathcal{D} \in \mathbb{D}^c$, and data $f_{\mathcal{D}} \in F_{\mathcal{D}}$. It outputs $\bar{\mathcal{D}} \in \mathbb{D}^c$ with $\mathcal{D} \leq \bar{\mathcal{D}}$ and $\bar{u} \in V_{\bar{\mathcal{D}}}^c$ such that $\|u(f_{\mathcal{D}}) - \bar{u}\|_V \leq \varepsilon$.

Assumptions on Global Error Functional

We assume the existence of constants $C_1, C_2 > 0$ with

$$C_1 C_2 < b,$$

such that the following properties hold:

- **Continuity of the solution upon data:**

$$\|u(f) - u(f_{\mathcal{D}})\|_V \leq C_1 \inf_{w \in V} \mathrm{E}_{\mathcal{D}}(w, f)^{\frac{1}{2}} \qquad \forall \mathcal{D} \in \mathbb{D}, \forall\, f \in F,$$

- **Lipschitz continuity of $\mathrm{E}_{\mathcal{D}}^{\frac{1}{2}}$ upon state:**

$$\sup_{f \in F} |\, \mathrm{E}_{\mathcal{D}}(w, f)^{\frac{1}{2}} - \mathrm{E}_{\mathcal{D}}(v, f)^{\frac{1}{2}} \,| \leq C_2 \|w - v\|_V \qquad \forall \mathcal{D} \in \mathbb{D}, \forall v, w \in V.$$

Verifying the Assumptions on Global Error Functional

Error and Oscillation: Let

$$\mathsf{e}_D(v,f) := |v - P^1_{p_D} v|^2_{H^1(K_D)} + \frac{1}{\kappa}\mathrm{osc}_D(f)^2 \qquad \forall D \in \mathcal{D},$$

with

$$\mathrm{osc}_D(f)^2 = \|p_D^{-1} h_D (f - P^0_{p_D-1} f)\|^2_{L^2(K_D)}$$

- **Continuity of the solution upon data:**

$$|u(f) - u(f_{\mathcal{D}})|_{H^1(\Omega)} = \|f - f_{\mathcal{D}}\|_{H^{-1}(\Omega)} \le C\,\mathrm{osc}_{\mathcal{D}}(f) \le \underbrace{C\kappa^{\frac{1}{2}}}_{=C_1} \mathrm{E}_{\mathcal{D}}(w,f)^{\frac{1}{2}} \quad \forall w \in V.$$

- **Lipschitz continuity of $\mathrm{E}_{\mathcal{D}}$ upon state:**

$$\left| \mathrm{E}_{\mathcal{D}}(w,f)^{\frac{1}{2}} - \mathrm{E}_{\mathcal{D}}(v,f)^{\frac{1}{2}} \right| \le |w - v|_{H^1(\Omega)} \quad \Rightarrow \quad C_2 = 1.$$

- **Constant b:** $\quad C_1 C_2 = C\kappa^{\frac{1}{2}} C_2 < b = \frac{1}{2}$ for κ sufficiently small.

Basic hp-AFEM

- **hp-AFEM:**

$\mathbf{hp}$-**AFEM**$(\bar{u}_0, f, \varepsilon_0)$

% Input: $(\bar{u}_0, f) \in V \times F$, $\varepsilon_0 > 0$ with $\|u(f) - \bar{u}_0\|_V \leq \varepsilon_0$.

% Parameters: $\mu \in (0,1)$ such that $C_1 C_2 < b(1-\mu)$, and $\omega \in (\frac{C_2}{b}, \frac{1-\mu}{C_1})$.

`for` $i = 1, 2, \ldots$ `do`

$[\mathcal{D}_i, f_{\mathcal{D}_i}]$:=**hp-NEARBEST**$(\omega\varepsilon_{i-1}, \bar{u}_{i-1}, f)$

$[\bar{\mathcal{D}}_i, \bar{u}_i]$:= **PDE**$(\mu\varepsilon_{i-1}, \mathcal{C}(\mathcal{D}_i), f_{\mathcal{D}_i})$

$\varepsilon_i := (\mu + C_1\omega)\varepsilon_{i-1}$

`end do`

- **Error Reduction:** Note that $\varepsilon_i = (\mu + C_1\omega)^i \varepsilon_0$ where $\mu + C_1\omega < 1$.

- **Coarsening:** Tolerance for $\bar{u}_i$ within **PDE** is $\tau_i := \mu\varepsilon_{i-1}$ and the subsequent input tolerance of **hp-NEARBEST** is $\omega\varepsilon_i = \omega(\mu + C_1\omega)\varepsilon_{i-1} > \omega\tau_i$. Since in our applications $C_2 = 1$ and $\omega > C_2/b \geq 1$, we see that $\omega\varepsilon_i > \tau_i = \mu\varepsilon_{i-1}$.

Theorem: Convergence and Instance Optimality

Assume there exist constants $C_1, C_2 > 0$ with $C_1 C_2 < b$ for which one has

- Continuity of the solution upon data:

$$\|u(f) - u(f_{\mathcal{D}})\|_V \leq C_1 \inf_{w \in V} \mathrm{E}_{\mathcal{D}}(w, f)^{\frac{1}{2}} \qquad \forall \mathcal{D} \in \mathbb{D}, \forall f \in F,$$

- Lipschitz continuity of $\mathrm{E}_{\mathcal{D}}$ upon state:

$$\sup_{f \in F} |\, \mathrm{E}_{\mathcal{D}}(w, f)^{\frac{1}{2}} - \mathrm{E}_{\mathcal{D}}(v, f)^{\frac{1}{2}} \,| \leq C_2 \|w - v\|_V \qquad \forall \mathcal{D} \in \mathbb{D}, \forall v, w \in V.$$

Then, for the sequences $(\bar{u}_i)$, $(\mathcal{D}_i)$ produced in **hp-AFEM**, it holds that

$$\|u - \bar{u}_i\|_V \leq \varepsilon_i \quad \forall i \geq 0, \qquad \mathrm{E}_{\mathcal{D}_i}(u, f)^{\frac{1}{2}} \leq (\omega + C_2)\varepsilon_{i-1} \quad \forall i \geq 1,$$

and

$$\#\mathcal{D}_i \leq B \#\mathcal{D} \quad \text{for any } \mathcal{D} \in \mathbb{D} \text{ with } \mathrm{E}_{\mathcal{D}}(u, f)^{\frac{1}{2}} \leq (b\omega - C_2)\varepsilon_{i-1},$$

where $u = u(f)$.

Proof of Theorem (Convergence and Instance Optimality)

- The bound $\|u - \bar{u}_0\|_V \leq \varepsilon_0$ is valid by assumption.

- For $i \geq 1$ the tolerances of **hp-NEARBEST**and **PDE** and continuity of the solution upon data yield

$$\begin{aligned}\|u - \bar{u}_i\|_V &\leq \|u(f_{\mathcal{D}_i}) - \bar{u}_i\|_V + \|u - u(f_{\mathcal{D}_i})\|_V \\ &\leq \mu\varepsilon_{i-1} + C_1\, \mathrm{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, f)^{\frac{1}{2}} \leq (\mu + C_1\omega)\varepsilon_{i-1} = \varepsilon_i.\end{aligned}$$

- Using this and Lipschitz continuity of $\mathrm{E}_{\mathcal{D}}$

$$\mathrm{E}_{\mathcal{D}_i}(u, f)^{\frac{1}{2}} \leq \mathrm{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, f)^{\frac{1}{2}} + C_2\|u - \bar{u}_{i-1}\|_V \leq (\omega + C_2)\varepsilon_{i-1} \qquad \forall i \geq 1.$$

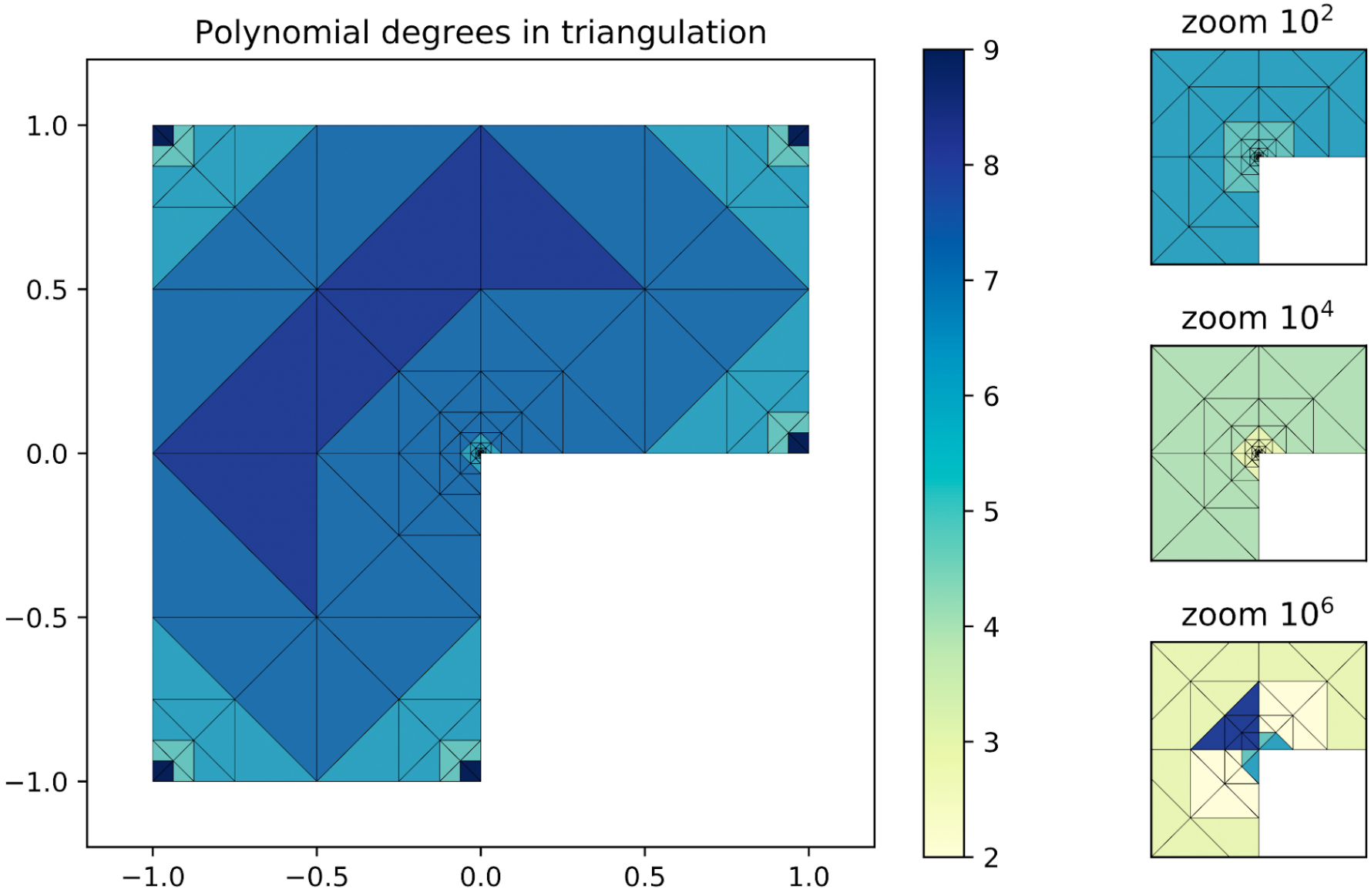
- Let $\mathcal{D} \in \mathbb{D}$ with $\mathrm{E}_{\mathcal{D}}(u, f)^{\frac{1}{2}} \leq (b\omega - C_2)\varepsilon_{i-1}$. Lipschitz continuity of $\mathrm{E}_{\mathcal{D}}^{\frac{1}{2}}$ leads to

$$\mathrm{E}_{\mathcal{D}}(\bar{u}_{i-1}, f)^{\frac{1}{2}} \leq \mathrm{E}_{\mathcal{D}}(u, f)^{\frac{1}{2}} + C_2\|u - \bar{u}_{i-1}\|_V \leq b\omega\varepsilon_{i-1} \quad \Rightarrow \quad \#\mathcal{D}_i \leq B\#\mathcal{D}$$

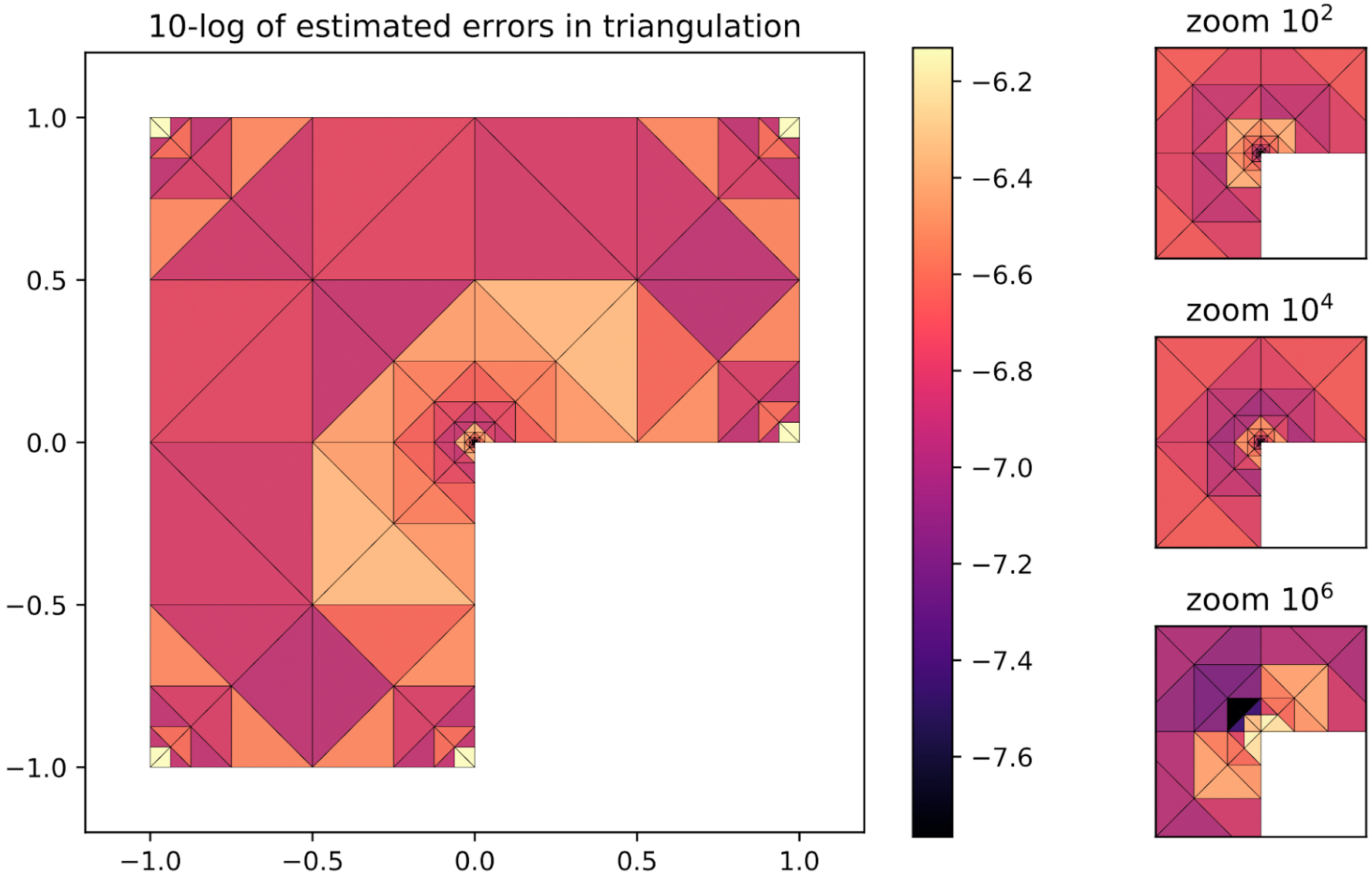
because of the optimality property of **hp-NEARBEST**.

Numerical Experiments: Degrees of Freedom (J. Westerdiep '17)

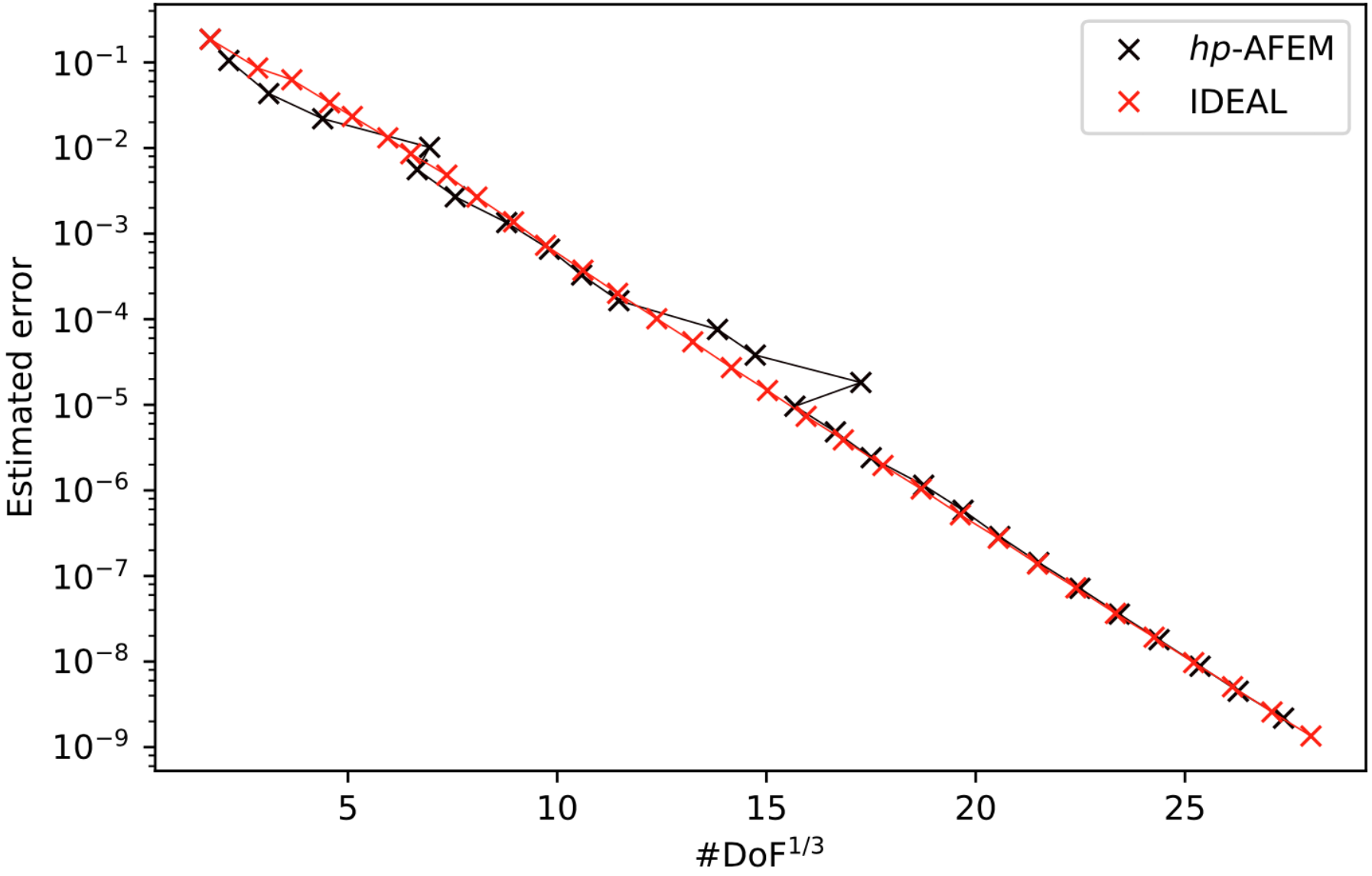
Model Problem: $-\Delta u = 1$ in L-shape domain Ω and $u = 0$ on $\partial\Omega$.



Numerical Experiments: Error Indicators (J. Westerdiep '17)



Numerical Experiments: hp-AFEM vs Ideal Error Decay (J. Westerdiep)



Outline

Setting for hp-Adaptivity

Basic hp-Adaptive Algorithm

hp-Adaptive Approximation

Practical Realizations of the hp-Adaptive Algorithm

Saturation Properties

References

Adaptive hp-Approximation: hp-NEARBEST(Binev)

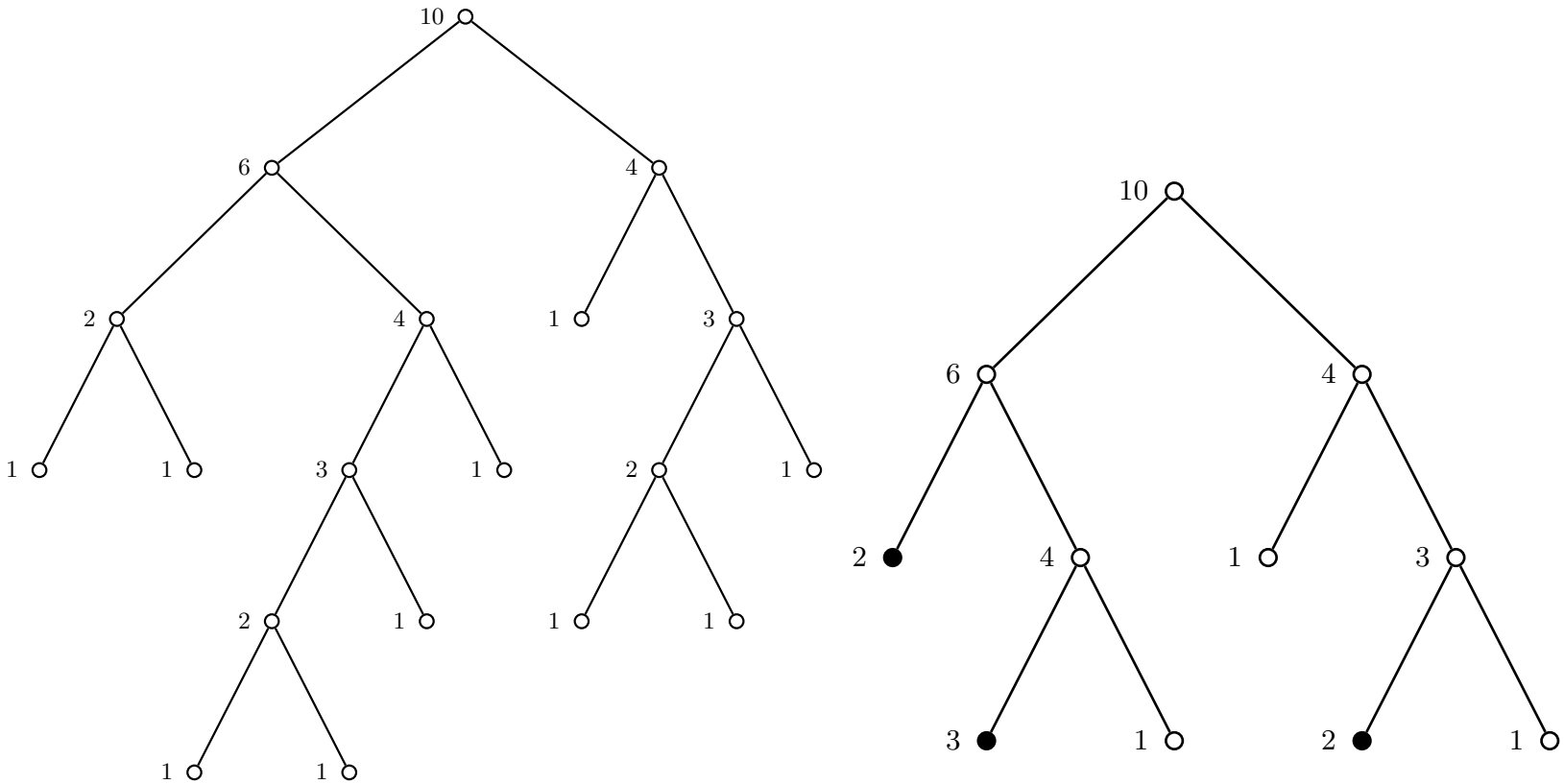
- **Ghost h-tree $\mathcal{T}$:** h-tree associated with $v \in L^2(\Omega)$ by Binev-DeVore (2004).
- **Admissible dimension:** Given $K \in \mathcal{T}$, the dimension $d(K, \mathcal{T})$ is

$$d(K, \mathcal{T}) = \#\mathcal{L}(\mathcal{T}(K)),$$

  where $\mathcal{T}(K)$ is the subtree of $\mathcal{T}$ emanating from K. This quantity depends on both T and the underlying tree $\mathcal{T}$.
- **Local hp-error functionals:** Let $e_{K,d}$ be the local error indicator on $K \in \mathcal{T}$ with polynomial space of dimension d.
  The local hp-error functional $e_K(\mathcal{T})$ reads
  - $e_K(\mathcal{T}) := e_{K,1}$ provided $K \in \mathcal{L}(\mathcal{T})$ is a leaf;
  - $e_K(\mathcal{T}) := \min\left\{e_{K'}(\mathcal{T}) + e_{K''}(\mathcal{T}), e_{K,d(K,\mathcal{T})}\right\}$ otherwise.
- **Subordinate hp-tree $\mathcal{P}$:** This tree is obtained from the h-tree upon eliminating the subtree $\mathcal{T}(K)$ whenever increasing the polynomial dimension in K from 1 to $d(K, \mathcal{T})$ reduces the error, i.e.

$$e_K(\mathcal{T}) = e_{K,d(K,\mathcal{T})}.$$

The Ghost h-Tree and Subordinate hp-Tree (Binev)



Ghost h-tree $\mathcal{T}$ (left) with 10 leaves ($\#\mathcal{L}(\mathcal{T}) = 10$); the root K of $\mathcal{T}$ thus has an admissible dimension (polynomial degree for $n = 1$) $d(K, \mathcal{T}) = 10$. The subordinate hp-tree $\mathcal{P}$ (right) results from $\mathcal{T}$ upon trimming 3 subtrees and raising the polynomial degrees of the interior nodes of $\mathcal{T}$, now leaves of $\mathcal{P}$, to $d(K, \mathcal{T}) = 2, 3, 2$ respectively.

Instance Optimality of hp-NEARBEST

- **Theorem (Binev):** The subordinate hp-tree $\mathcal{P}_N$ with cardinality

$$\#\mathcal{P}_N = \sum_{K\in\mathcal{L}(\mathcal{P}_N)} d(K,\mathcal{T}_N) = \#\mathcal{L}(\mathcal{T}_N) = N$$

gives a partition $\mathcal{D}_N$ with $\#\mathcal{D}_N = N$ and near-best hp-approximation over $\mathcal{D}_N$ in the sense that the global error functional satisfies

$$\mathrm{E}_{\mathcal{D}_N}(v) = \sum_{K\in\mathcal{L}(\mathcal{P}_N)} e_{K,d(K,\mathcal{T}_N)} \le \frac{2N}{N-n+1}\sigma_n(v) \quad \forall\, n \le N,$$

where σ_n is the best hp-error for v with n total degrees of freedom.

- **Interpretation:** Choosing $B > 1$, $n = \frac{N}{B}$ and $b = \left(\frac{1}{2}(1-\frac{1}{B})\right)^{\frac{1}{2}}$ implies
  - $\mathrm{E}_{\mathcal{D}_N}(v)^{\frac{1}{2}} \le \varepsilon$
  - $\#\mathcal{D}_N \le B\#\mathcal{D}$ for all $\mathcal{D}\in\mathbb{D}$ such that $\mathrm{E}_{\mathcal{D}}(v)^{\frac{1}{2}} \le b\varepsilon$.

Remark. The cost for constructing $\mathcal{D}_N$ is bounded by $\mathcal{O}\big(\sum_{K\in\mathcal{T}_N} d(K,\mathcal{T}_N)\big)$, and varies from $\mathcal{O}(N\log N)$ for well balanced trees to $\mathcal{O}(N^2)$ for highly unbalanced trees.

Outline

Setting for hp-Adaptivity

Basic hp-Adaptive Algorithm

hp-Adaptive Approximation

Practical Realizations of the hp-Adaptive Algorithm

Saturation Properties

References

A practical hp-adaptive algorithm

- **PDE:** We assume that **PDE** exploits the work already carried out within **hp-AFEM**. It takes as input a partition $\mathcal{D} \in \mathbb{D}^c$, data $f_{\mathcal{D}} \in F_{\mathcal{D}}$, and a desired error reduction factor $\varrho \in (0,1]$, and produces a conforming partition $\bar{\mathcal{D}} = \bar{\mathcal{D}}(\mathcal{D}, \varrho) \in \mathbb{D}^c$ with $\bar{\mathcal{D}} \geq \mathcal{D}$ and a function $\bar{u} \in V^c_{\bar{\mathcal{D}}}$ such that

$$\|u(f_{\mathcal{D}}) - \bar{u}\|_V \leq \varrho \inf_{v \in V^c_{\bar{\mathcal{D}}}} \|u(f_{\mathcal{D}}) - v\|_V .$$

  Note that $f_{\mathcal{D}}$ is piecewise polynomial and data oscillation vanishes.

- **Local vs Global:** To bound the right-hand side, we prove

$$\inf_{w \in V^c_{\bar{\mathcal{D}}}} \|v - w\|_V \leq C_{3,\mathcal{D}} \inf_{f \in F} \mathrm{E}_{\mathcal{D}}(v, f)^{\frac{1}{2}} \qquad \forall \mathcal{D} \in \mathbb{D}, \forall v \in V.$$

  The infimum on the right-hand side reads as the error in the broken H^1 semi-norm of the best approximation to v from $V_{\mathcal{D}} = \prod_{D \in \mathcal{D}} V_D$.
  Constant $C_{3,\mathcal{D}}$ depends logaritmically on the polynomial degree for dimension $n = 2$ (this extends an h-type result by Veeser (2012)).

- **PDE:** This routine may be implemented by the usual loop

$$\textbf{SOLVE} \rightarrow \textbf{ESTIMATE} \rightarrow \textbf{MARK} \rightarrow \textbf{GROW}$$

where

GROW may be either an h-refinement or a p-enrichment.

Since the task of producing a near-best hp-partition is assigned to **hp-NEARBEST**, in principle even a *uniform* refinement is allowed.

Error Indicators

- **Model problem:** Recall the model elliptic problem in Ω

$$-\Delta u = f \quad \text{in } \Omega; \qquad u = 0 \quad \text{in } \partial\Omega.$$

- **Residual-based error Indicator:** choose as a local error indicator in **PDE**

$$\eta_D^2(u_{\mathcal{D}}, f_{\mathcal{D}}) := \frac{|K_D|}{p_D^2} \|f_{\mathcal{D}} + \Delta u_{\mathcal{D}}\|^2_{L^2(K_D)} + \sum_{\{e \in \mathcal{E}(\mathcal{D}) : e \subset \partial K_D \cap \Omega\}} \frac{|e|}{2p_{e,\mathcal{D}}} \|[\![\nabla u_{\mathcal{D}} \cdot \mathbf{n}_e]\!]\|^2_{L^2(e)}.$$

[Melenk and Wohlmuth (2001)].

- **Constants:** This introduces a factor $\simeq \|p_{\mathcal{D}}\|_\infty^{-2-2\varepsilon}$ in the efficiency estimate, leading to a convergence analysis that is not p-robust.

Error Reduction

- **Proposition.** The number $M = M(\varrho)$ of iterations that are required by $\mathbf{PDE}$ to reduce the error by a factor ρ, i.e.

$$\|u(f_{\mathcal{D}}) - u_{\bar{\mathcal{D}}}(f_{\mathcal{D}})\|_V \leq \varrho \inf_{v \in V_{\mathcal{D}}^c} \|u(f_{\mathcal{D}}) - v\|_V$$

  is at most proportional to $\log \varrho^{-1}$, and the output partition $\bar{\mathcal{D}}$ satisfies $\#\bar{\mathcal{D}} \lesssim \#\mathcal{D}$, the constants being both independent of $\mathcal{D} \in \mathbb{D}^c$, and $f_{\mathcal{D}} \in F_{\mathcal{D}}$.

- **Warning:** Since the Melenk-Wohlmuth's estimator (2001) is not p-robust, the number $M = M(\varrho)$ of iterations behaves indeed like

$$M \approx \log \varrho^{-1} \, \|p_{\mathcal{D}}\|_\infty^{2+\varepsilon},$$

  for any $\varepsilon > 0$.

- **Remark:** In 1D, $\mathbf{hp}$-**AFEM** applied to a general elliptic self-adjoint problem

$$-(\mu u_x)_x + \sigma u = f + g_x$$

  is p-robust, thanks to the explicit evaluation of the residual.

Towards p-Robust Estimators: Equilibrated Flux Estimators

- **p-robust convergence:** This can be achieved for **hp-AFEM** in 2D upon resorting to equilibrated flux estimators.

- **Equilibrated flux estimator:** We introduce the following standard notation:

  - Given a partition $\mathcal{D}$ made of triangles K, with vertices $a \in \mathcal{A}_{\mathcal{D}}$, denote by ω_a the star (or patch) of elements containing a.

  - For any such vertex, define the local energy space

$$H^1_*(\omega_a) := \begin{cases} \{v \in H^1(\omega_a) \colon \langle v, 1\rangle_{\omega_a} = 0\} & a \in \mathcal{A}^{\text{int}}_{\mathcal{D}}, \\ \{v \in H^1(\omega_a) \colon v = 0 \text{ on } \partial\omega_a \cap \partial\Omega\} & a \in \mathcal{A}^{\text{bdry}}_{\mathcal{D}}. \end{cases}$$

  - Define the global and local residuals for the Galerkin solution $u_{\mathcal{D}} \in V_{\mathcal{D}}$

$$r(v) := \langle f, v\rangle_\Omega - \langle \nabla u_{\mathcal{D}}, \nabla v\rangle_\Omega, \qquad r_a(v) = r(\phi_a v).$$

    where ϕ_a is the piecewise linear hat function centered at a.

p-Robust A Posteriori Estimates

- **Upper and lower bounds:**

$$\|\nabla(u-u_{\mathcal{D}})\|_{\Omega}^2 \leq 3 \sum_{a\in\mathcal{A}_{\mathcal{D}}} \|r_a\|_{H_*^1(\omega_a)'}^2, \quad \|r_a\|_{H_*^1(\omega_a)'} \lesssim \|\nabla(u-u_{\mathcal{D}})\|_{\omega_a} \quad \forall a \in \mathcal{A}_{\mathcal{D}}.$$

- **p-robust equivalence:**

$$\|r_a\|_{H_*^1(\omega_a)'} \simeq \|\boldsymbol{\sigma}_a\|_{\omega_a}$$

where $\boldsymbol{\sigma}_a$ is a suitable equilibrated flux for $u_{\mathcal{D}}$ that satisfies

$$\boldsymbol{\sigma}_a \in \mathcal{RT}(\mathcal{D}_a): \quad \langle \nabla\cdot\boldsymbol{\sigma}_a, 1\rangle_T = \langle f, 1\rangle_T \quad \forall T \subset \omega_a;$$

[Braess, Pillwein and Schöberl (2009)].

- **Computability.** A particular equilibrated flux $\boldsymbol{\sigma}_a$ can be efficiently computed by solving a discrete saddle-point problem in the space $\mathcal{RT}(\mathcal{D}_a)$.
[Ern and Vohralík (2015)]

Note: For simplicity, we assume polynomial data (no oscillation).

- **Upper and lower bound for discrete functions:** If $\widetilde{\mathcal{D}} \geq \mathcal{D}$ yields $V_{\mathcal{D}} \subset V_{\widetilde{\mathcal{D}}}$, then
$$\|\nabla(u_{\widetilde{\mathcal{D}}} - u_{\mathcal{D}})\|_{\Omega}^2 \leq 3 \sum_{a \in \mathcal{A}_{\mathcal{D}}} \|r_a\|^2_{(H^1_*(\omega_a) \cap V_{\widetilde{\mathcal{D}}}(\omega_a))'},$$
and
$$\|r_a\|_{(H^1_*(\omega_a) \cap V_{\widetilde{\mathcal{D}}}(\omega_a))'} \lesssim \|\nabla(u_{\widetilde{\mathcal{D}}} - u_{\mathcal{D}})\|_{\omega_a} \qquad \forall a \in \mathcal{A}_{\mathcal{D}}.$$
- **Marking:** Suppose we apply a *star-based Dörfler marking*, and that for any marked star we can find a local space $V_{\widetilde{\mathcal{D}}}(\omega_a) \supset V_{\mathcal{D}}(\omega_a)$ for which it holds
$$\|r_a\|_{H^1_*(\omega_a)'} \lesssim \|r_a\|_{(H^1_*(\omega_a) \cap V_{\widetilde{\mathcal{D}}}(\omega_a))'} \qquad \text{uniformly in } p.$$
- **Saturation property:** Then, we immediately obtain the p-robust saturation property
$$\|\nabla(u - u_{\mathcal{D}})\|_{\Omega}^2 \lesssim \|\nabla(u_{\widetilde{\mathcal{D}}} - u_{\mathcal{D}})\|_{\Omega}^2$$
- **Contraction property:** This implies the following bound with $\varrho < 1$
$$\|\nabla(u - u_{\widetilde{\mathcal{D}}})\|_{\Omega} \leq \varrho \|\nabla(u - u_{\mathcal{D}})\|_{\Omega}$$

Outline

Setting for hp-Adaptivity

Basic hp-Adaptive Algorithm

hp-Adaptive Approximation

Practical Realizations of the hp-Adaptive Algorithm

Saturation Properties

References

Reducing the saturation problem from a star to a triangle

Theorem (saturation property on a triangle)

Suppose that in a reference triangle $\hat{T}$ (with edges e_1, e_2, e_3, and with $\psi \in \mathbb{P}_1(\hat{T})$ defined by $\psi(e_1 \cap e_2) = 1$, $\psi(e_3) = 0$), one has for a suitable $q = q(p) > p$

- $\|\psi g\|_{(H^1_{0,e_1\cup e_2}(\hat{T}))'} \lesssim \|\psi g\|_{(H^1_{0,e_1\cup e_2}(\hat{T})\cap\mathbb{P}_q)'} \quad \forall g \in \mathbb{P}_{p-1}(\hat{T})$,
- $\|v \mapsto \int_{e_1} gv\|_{(H^1_{0,e_2}(\hat{T})')} \lesssim \|v \mapsto \int_{e_1} gv\|_{(H^1_{0,e_2}(\hat{T})\cap\mathbb{P}_q)'} \quad \forall g \in \mathbb{P}_p(e_1)$,
- $\|v \mapsto \int_{\partial\hat{T}} gv\|_{(H^1_*(\hat{T})')} \lesssim \|v \mapsto \int_{\partial\hat{T}} gv\|_{(H^1_*(\hat{T})\cap\mathbb{P}_q)'}$
  $\forall g$ satisfying $\int_{\partial T} g = 0$ and $g_{|e_i} \in \mathbb{P}_p(e_i)$, $i = 1, 2, 3$.

Then, if the larger space $V_{\widetilde{\mathcal{D}}}(\omega_a)$ is built from $V_{\mathcal{D}}(\omega_a)$ by increasing the polynomial degree from p to q, the following desired bound is valid

$$\|r_a\|_{(H^1_*(\omega_a)')} \lesssim \|r_a\|_{(H^1_*(\omega_a)\cap V_{\widetilde{\mathcal{D}}}(\omega_a))'}.$$

Numerical investigation

Checking the first assumption

$$\|\psi g\|_{(H^1_{0,e_1\cup e_2}(\hat{T}))'} \lesssim \|\psi g\|_{(H^1_{0,e_1\cup e_2}(\hat{T})\cap\mathbb{P}_q(\hat{T}))'} \quad \forall g \in \mathbb{P}_{p-1}(\hat{T})$$

is equivalent to

$$C^{(1)}_{p,q} =: \sup_{g\in\mathbb{P}_{p-1}(\hat{T})} \frac{\|\nabla u(g)\|_{0,\hat{T}}}{\|\nabla u_q(g)\|_{0,\hat{T}}} \leq C^{(1)} < \infty,$$

where $u = u(g) \in H^1_{0,e_1\cup e_2}(\hat{T})$ solves

$$-\Delta u = \psi g \qquad \text{in } \hat{T}$$

and $u_q = u_q(g) \in H^1_{0,e_1\cup e_2}(\hat{T}) \cap \mathbb{P}_q(\hat{T})$ is the corresponding Galerkin projection.

Numerical results

Numerical experiments indicate that $q = p + \text{const}$ does not work

p	$q = p + 4$	$C^{(1)}_{p,q}$
4	8	1.0072781599
12	16	1.1590636448
28	32	1.6580856832
60	64	2.7635533362

whereas $q = \lambda p$ for any fixed $\lambda \in (0, 1)$ suffices:

p	$q = 2p$	$C^{(1)}_{p,q}$
4	8	1.0072781599
8	16	1.0007015438
16	32	1.0001682680
32	64	1.0000689675

p	$q = \frac{8}{7}p$	$C^{(1)}_{p,q}$
14	16	10.109454650
28	32	1.6580859228
56	64	1.3327470997
112	128	1.0735108345

Similar results hold for the other constants $C^{(2)}_{p,q}$ and $C^{(3)}_{p,q}$.

Numerical results

Numerical experiments indicate that $q = p + \text{const}$ does not work

p	$q = p + 4$	$C^{(1)}_{p,q}$
4	8	1.0072781599
12	16	1.1590636448
28	32	1.6580856832
60	64	2.7635533362

whereas $q = \lambda p$ for any fixed $\lambda \in (0, 1)$ suffices:

p	$q = 2p$	$C^{(1)}_{p,q}$
4	8	1.0072781599
8	16	1.0007015438
16	32	1.0001682680
32	64	1.0000689675

p	$q = \frac{8}{7}p$	$C^{(1)}_{p,q}$
14	16	10.109454650
28	32	1.6580859228
56	64	1.3327470997
112	128	1.0735108345

Similar results hold for the other constants $C^{(2)}_{p,q}$ and $C^{(3)}_{p,q}$.

A Prototypal Problem: Saturation on the Square

Set $\hat{R} = (-1,1)^2$. Given $g \in \mathbb{P}_p(\hat{R})$, let $u = u(g) \in H_0^1(\hat{R})$ be the solution of

$$\int_{\hat{R}} \nabla u \cdot \nabla v = \int_{\hat{R}} g\, v \qquad \forall v \in H_0^1(\hat{R}),$$

and let $u_q = u_q(g) \in V_q := H_0^1(\hat{R}) \cap \mathbb{P}_q(\hat{R})$ be the solution of

$$\int_{\hat{R}} \nabla u_q \cdot \nabla v = \int_{\hat{R}} g\, v \qquad \forall v \in H_0^1(\hat{R}) \cap \mathbb{P}_q(\hat{R}).$$

Theorem (saturation property for the square)

There exists a constant Λ such that for all $\lambda > 1$, any mapping $p \mapsto q = q(p)$

$$q(p) > \max\{\lambda p, p+4\}$$

yields

$$\|\nabla u\|_{0,\hat{R}} \le \Lambda \frac{\lambda}{\lambda - 1} \|\nabla u_{q(p)}\|_{0,\hat{R}} \qquad \text{uniformly in } g \text{ and } p.$$

Outline

Setting for hp-Adaptivity

Basic hp-Adaptive Algorithm

hp-Adaptive Approximation

Practical Realizations of the hp-Adaptive Algorithm

Saturation Properties

References

References

- P. Binev, *Tree approximation for hp-adaptivity*, SIAM J. Numer. Anal. (submitted).
- C. Canuto, R.N. Nochetto, R. Stevenson, M. Verani, *High-Order Adaptive Galerkin Methods*, pp. 51-72 in: Lecture Notes in Computational Science and Engineering 106 (R.M. Kirby, M. Berzins, and J.S. Hesthaven, Eds.), Springer 2015.
- C. Canuto, R.H. Nochetto, R. Stevenson, and M. Verani, *Convergence and optimality of hp-AFEM*, Numer. Math. 135 (2017), 1073–1119.
- C. Canuto, R. Stevenson, R.H. Nochetto, and M. Verani, *On p-robust saturation for hp-AFEM*, Computers Math. Appl. 73, 9 (2017), 2004–2022.
- C. Canuto, R. Stevenson, R.H. Nochetto, and M. Verani, *A saturation property for the spectral-Galerkin approximation of a Dirichlet problem in a square*, (submitted).