APPROXIMATION OF INTEGRAL TYPE FUNCTIONAL OF MARKOV PROCESSES

Jakub Chorowski

Joint work with Randolf Altmeyer
Humboldt-Universität zu Berlin

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Occupation time functional

For a $d-$dimensional process $(X_t, 0 \leq t \leq 1)$ and function $f : \mathbb{R}^d \to \mathbb{R}$ we define the *occupation time* functional as

$$\Gamma_1(f) = \int_0^1 f(X_s)ds. \tag{1}$$

When X is observed at equidistant times k/n, $k = 0, ..., n-1$, it is natural to approximate $\Gamma_1(f)$ by a Riemann sum

$$\hat{\Gamma}_{1,n}(f) = \frac{1}{n}\sum_{k=1}^{n} f(X_{(k-1)/n}). \tag{2}$$

Strong L_2 error rates

The goal is to establish which features of the function f and process X are important for the estimation error. So far we know that the mean squared estimation error

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{1/2}$$

is upper bounded by

- $C(f)n^{-\frac{1+s}{2}}$ when X is a scalar diffusion and f Hölder continuous of order $s \in (0,1)$ (Kohatsu, Makhlouf, Ngo)
- $C\log(n)n^{-\frac{1+s}{2}}\|f\|_{H^s}$ when X is a Brownian motion and f Sobolev regular of order $s \in (0,1)$ (Altmeyer)
- $C(f)\left(\frac{\log(n)}{n}\right)^{1/2}$ when X is a Markov process (Ganychenko, Kulik).

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right] = \sum_{k,l=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{l-1}{n}}^{\frac{l}{n}} \mathbb{E}\big[(f(X_h) - f(X_{\frac{k-1}{n}}))(f(X_r) - f(X_{\frac{l-1}{n}})\big] drdh.$$

Let X be a Markov process with $P_t f(x) = \mathbb{E}\big[f(X_t)|X_0 = x\big]$ the transition operator. Assuming that X is strictly stationary and time-reversible (equivalently P_t is self-adjoint), the algebraic properties of the transition semigroup yield that the above display is bounded by

$$n^{-2}\langle (I - P_1)f, f\rangle_\mu + n^{-1}\langle (I - P_{1/n})f, f\rangle_\mu,$$

where μ is the stationary measure

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$$n^{-2}\langle (I - P_1)f, f\rangle_\mu + n^{-1}\langle (I - P_{1/n})f, f\rangle_\mu,$$

where μ is the stationary measure and

$$\langle f, g\rangle_\mu = \int fg \, d\mu, \quad \|f\|^2_{L^2(\mu)} = \langle f, f\rangle_\mu.$$

Since

$$\left| n^{-2}\langle (I - P_1)f, f\rangle_\mu \right| \leq n^{-2}\|(I - P_1)f\|_{L^2(\mu)}\|f\|_{L^2(\mu)}$$
$$\leq n^{-2}\|f\|^2_{L^2(\mu)},$$

the convergence rate is determined by the second term

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Calculating formally, we receive for the infinitesimal generator L that

$$\begin{aligned} n^{-1}\langle (I-P_{1/n})f,f\rangle_\mu &= n^{-1}\langle\Big(\int_0^{\frac{1}{n}} P_r dr\Big)(-L)f,f\rangle_\mu \\ &= n^{-1}\Big\|\Big(\int_0^{\frac{1}{n}} P_r dr\Big)^{\frac{1}{2}}(-L)^{1/2}f\Big\|^2_{L^2(\mu)} \\ &\le n^{-2}\Big\|(-L)^{1/2}f\Big\|^2_{L^2(\mu)}. \end{aligned}$$

Spectral measure

Definition

Consider an increasing family $(H_\lambda)_{\lambda \geq 0}$ of closed linear subspaces of the Hilbert space $L^2(\mu)$, which is right-continuous in the sense that $\bigcap_{\lambda' > \lambda} H_{\lambda'} = H_\lambda$. Furthermore, we require that $\bigcup_{\lambda \geq 0} H_\lambda$ is dense in $L^2(E, \mu)$. The *spectral measure* is the family $(E_\lambda)_{\lambda \geq 0}$ of orthogonal projections $E_\lambda : L^2(\mu) \to H_\lambda$.

Spectral measure - Example

Let $(e_k)_{k=0,\dots}$ be an orthonormal basis of $L^2(\mu)$. For $\lambda \geq 0$ define

$$H_\lambda = \mathrm{span}\{e_k \,:\, k \leq \lambda\}.$$

The orthogonal projection $E_\lambda : L^2(\mu) \to H_\lambda$ is of the form

$$E_\lambda f = \sum_{k=0}^{\lfloor \lambda \rfloor} \langle f, e_k \rangle_\mu e_k.$$

For any $f, g \in L^2(\mu)$ it holds

$$\langle E_\lambda f, g \rangle_\mu = \sum_{k=0}^{\lfloor \lambda \rfloor} \langle f, e_k \rangle_\mu \langle g, e_k \rangle_\mu.$$

Spectral measure continued

For any $f, g \in L^2(\mu)$ the map $\lambda \to \langle E_\lambda f, g\rangle_\mu$ is right-continuous and of bounded variation. Consequently, for any measurable function $\psi : [0, \infty) \to \mathbb{R}$ we may define the Stieltjes integral

$$\int_0^\infty \psi(\lambda) d\langle E_\lambda f, g\rangle_\mu.$$

By duality, the above defines a symmetric linear operator denoted by

$$\int_0^\infty \psi(\lambda) dE_\lambda = \Psi,$$

with domain

$$\mathrm{dom}\,(\Psi) = \left\{ f \in L^2(E, \mu) \,:\, \int_0^\infty \psi(\lambda)^2 d\langle E_\lambda f, f\rangle_\mu < \infty \right\}.$$

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Moreover, the operator norm (possibly infinite) of Ψ is given by

$$\|\Psi\| = \sup_{f \in \mathrm{dom}(\Psi)} \frac{\|\Psi f\|_{L^2(\mu)}}{\|f\|_{L^2(\mu)}} = \sup_{f \in \mathrm{dom}(\Psi)} \frac{\int_0^\infty \psi(\lambda)^2 d\langle E_\lambda f, f\rangle_\mu}{\int_0^\infty d\langle E_\lambda f, f\rangle_\mu}.$$

Fractional powers of the infinitesimal generator

Assumption

X is a stationary Markov process with the infinitesimal generator L being a non-positive, self-adjoint operator on the Hilbert space $L^2(\mu)$.

The spectral decomposition theorem for self-adjoint operators asserts that there exists a spectral measure $(E_\lambda)_{\lambda \geq 0}$ such that

$$-L = \int_0^\infty \lambda dE_\lambda.$$

Definition

For $s > 0$ let

$$(-L)^{s/2} = \int_0^\infty \lambda^{s/2} dE_\lambda,$$

with

$$\operatorname{dom}\left((-L)^{1/2}\right) = \left\{ f \in L^2(\mu) \, : \, \int_0^\infty \lambda d\langle E_\lambda f, f\rangle_\mu < \infty \right\}.$$

Main result

Theorem

There exists a constant $c < \infty$ such that for any function $f \in L^2(\mu)$ we have

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{1/2} \leq \frac{c}{\sqrt{n}}\|f\|_{L^2(\mu)}.$$

Furthermore, if $f \in \operatorname{dom}\left((-L)^{1/2}\right)$, it holds

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{1/2} \leq \frac{c}{n}\|(-L)^{1/2} f\|_{L^2(\mu)}.$$

Convergence rates for Bessel potential spaces

Corollary

There exists a constant $c < \infty$, such that for all $0 \leq s \leq 1$ and $f \in \operatorname{dom}\left((-L)^{s/2}\right)$ it holds

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{1/2} \leq cn^{-\frac{1+s}{2}} \|\left((-L)^{s/2}\right) f\|_{L^2(\mu)}.$$

Example: Ornstein-Uhlenbeck process

Let $(X_t, t \geq 0)$ be a stationary $d-$dimensional Ornstein-Uhlenbeck process, defined by the stochastic differential equation:

$$dX_t = -X_t dt + \sqrt{2} dW_t,$$

(where W_t is a standard $d-$dimensional Brownian motion) and with initial condition:

$$X_0 \stackrel{d}{=} \mathcal{N}(0, I) = \mu.$$

The infinitesimal generator L acts on twice differentiable functions by

$$Lf(x) = \Delta f(x) - x \cdot \nabla f(x).$$

Example: Ornstein-Uhlenbeck process continued

Let

$$H_k(x) = \frac{1}{\sqrt{k!}} \int_{\mathbb{R}} (x+iy)^k d\mu(y),\ x \in \mathbb{R},$$

be the one dimensional Hermite polynomial. $d-$dimensional tensor products

$$H_{\mathbf{k}}(x) = \prod_{k=1}^{d} H_{k_i}(x_i),\ \mathbf{k} = (k_1, ..., k_d) \in \mathbb{N}^d,\ x = (x_1, ..., x_d) \in \mathbb{R}^d$$

are the eigenfunctions of $-L$ with eigenvalues $\overline{\mathbf{k}} = \sum_{k=1}^{d} k_i$. The infinitesimal generator L acts on $f \in L^2(\mathbb{R}^d, \mu)$ by

$$Lf = -\sum_{\mathbf{k} \in \mathbb{N}^d} \langle f, H_{\mathbf{k}} \rangle_\mu \overline{\mathbf{k}} H_{\mathbf{k}}$$

with

$$\mathrm{dom}(L) = \left\{ f \in L^2(\mathbb{R}^d, \mu) \ :\ \sum_{\mathbf{k} \in \mathbb{N}^d} \langle f, H_{\mathbf{k}} \rangle_\mu^2 \overline{\mathbf{k}}^2 < \infty \right\}.$$

Example: Ornstein-Uhlenbeck process continued

Corollary

There exists a constant $c < \infty$*, such that for all* $0 \leq s \leq 1$ *and* $f \in \operatorname{dom}\left((-L)^{s/2}\right)$ *it holds*

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{1/2} \leq cn^{-\frac{1+s}{2}} \left\| \left((-L)^{s/2}\right) f \right\|_{L^2(\mu)}.$$

Example: Ornstein-Uhlenbeck process continued

Corollary

There exists a constant $c < \infty$*, such that for all* $0 \le s \le 1$ *and* $f \in \operatorname{dom}\left((-L)^{s/2}\right)$ *it holds*

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{1/2} \le cn^{-\frac{1+s}{2}} \|\left((-L)^{s/2}\right) f\|_{L^2(\mu)}.$$

Using integration by parts one gets

$$\|(-L)^{1/2} f\|_{L^2(\mu)}^2 = \langle f, (-L) f\rangle_\mu = \frac{1}{2}\|\nabla f\|_{L^2(\mu)}^2.$$

Example: Ornstein-Uhlenbeck process continued

Hence

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{\frac{1}{2}} \leq \frac{c}{n}\|\nabla f\|^2_{L^2(\mu)} \leq \frac{c}{n}\|f\|_{H^1},$$

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{\frac{1}{2}} \leq \frac{c}{\sqrt{n}}\|f\|_{L^2(\mu)} \leq \frac{c}{\sqrt{n}}\|f\|_{L^2}.$$

By interpolation we imply

Corollary

Let X be a stationary $d-$dimensional Ornstein-Uhlenbeck process. There exists a constant $C < \infty$ such that for any $0 \leq s \leq 1$ and $f \in H^s(\mathbb{R}^d)$ it holds

$$\mathbb{E}\left[\left|\Gamma_1(f) - \hat{\Gamma}_{1,n}(f)\right|^2\right]^{\frac{1}{2}} \leq Cn^{-\frac{1+s}{2}}\|f\|_{H^s}.$$

Other examples

- stationary scalar diffusions
- stationary diffusions with two reflecting boundaries
- Brownian motion?
- Markov processes with jumps?

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Future work:

- understand the assumption of stationarity (replace with absolutely continuous initial state)
- lower bounds