Intensity estimation of the **homogeneous** Poisson point process

Jean-François Coeurjolly

Joint work with Marianne Clausel and Jérôme Lelong

Stein intensity estimation of the homogeneous Poisson point process

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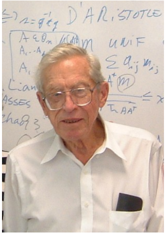
Back to the initial ideas of Charles Stein



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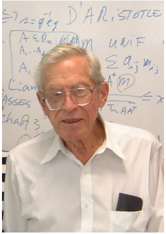

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Everything started in the following context

- Let $X \sim \mathcal{N}(\theta, \sigma^2 \mathbf{I}_d)$
  where I_d is the d-dimensional identity matrix.
- Objective : estimate θ based on a **single** (for simplicity) observation X.

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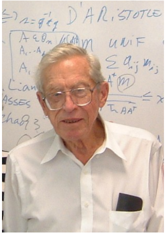

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- $\widehat{\theta}^{mle} = X$ min. $\text{MSE}(\widehat{\theta}) = \text{E}\left(\|\widehat{\theta} - \theta\|^2\right)$ among unbiased estimators.

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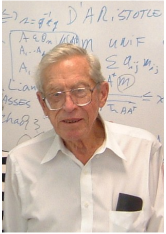

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- $\widehat{\theta}^{mle} = X$ min. $\mathrm{MSE}(\widehat{\theta}) = \mathrm{E}\left(\|\widehat{\theta} - \theta\|^2\right)$ among unbiased estimators.
- e.g. Stein (1956), James-Stein (1961)

$$\widehat{\theta}_{JS} = X(1 - (d-2)/\|X\|^2) \Rightarrow \mathrm{MSE}(\widehat{\theta}_{JS}) \leq \mathrm{MSE}(\widehat{\theta}^{mle}) \text{ when } d \geq 3$$

- Stein (1981) key-ingredients for the class : $\widehat{\theta} = X + g(X)$, $g : \mathbb{R}^d \to \mathbb{R}^d$.

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1. Using $\mathrm{E}[Zh(Z)] = \mathrm{E}[h'(Z)],\ Z \sim N(0,1)$

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2. Now choose $g = \sigma^2 \nabla \log f$. Then using the **well-known** fact that

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MSE of $\widehat{\theta} = X + g(X)$ ($X \sim \mathcal{N}(\theta, \sigma^2 \mathbf{I}_d$, σ^2 known)

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$$\text{for } h : \mathbb{R} \to \mathbb{R}, \qquad 2(\log h)'' + (\log(h)')^2 = 4\frac{(\sqrt{h})''}{\sqrt{h}}$$

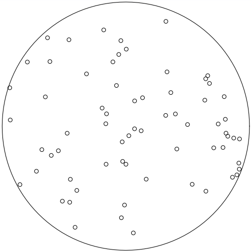
we get

$$\mathrm{MSE}(\widehat{\theta}) = \mathrm{MSE}(\widehat{\theta}^{mle}) + 4\sigma^2 \mathrm{E}\left(\frac{\nabla\nabla\sqrt{f(X)}}{\sqrt{f(X)}}\right) \leq \mathrm{MSE}(\widehat{\theta}^{mle}) \text{ if } \nabla\nabla\sqrt{f} \leq 0.$$

- X : homogeneous Poisson point process on $\mathbb{R}^d$ with intensity parameter θ.
- Assume X is observed on $W = B(0,1)$,

$$\widehat{\theta}^{mle} = N(W)/|W|$$

(here $\theta = 20$, $\widehat{\theta} = 67/\pi \simeq 21.3$).



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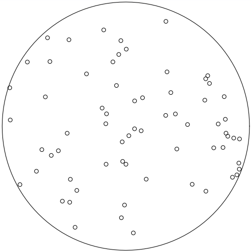

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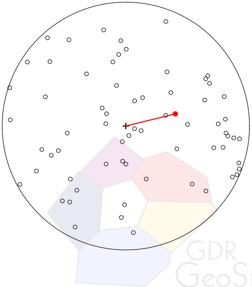
Objectives :

- Mimic Stein's technique, derive a Stein estimator of θ. [extension Privault-Réveillac (2009), $d = 1$]
- Example of Stein estimator :



$$\widehat{\theta} = \frac{N(W)}{\pi} - \frac{8}{\pi}(1 - d_k^2),$$

where d_k is the distance of the kth closest point of X to 0. (ex : ● is the 15th closest point)

What do we need ?

Theorem [F "point process functional"]

Let $\widehat{\theta} = \widehat{\theta}^{mle} + \frac{1}{|W|}\nabla \log(\mathbf{F})$ is such that $\nabla \log(\mathbf{F}) \in \mathrm{Dom}(\overline{\mathbf{D}}^{\pi})$ then

$$\mathrm{MSE}(\widehat{\theta}) = \mathrm{MSE}(\widehat{\theta}^{mle}) + \frac{4}{|W|^2}\mathrm{E}\left(\frac{\nabla\nabla\sqrt{F}}{\sqrt{F}}\right). \qquad (1)$$

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Let $\widehat{\theta} = \widehat{\theta}^{mle} + \frac{1}{|W|}\nabla \log(\mathbf{F})$ is such that $\nabla \log(\mathbf{F}) \in \mathrm{Dom}(\overline{\mathbf{D}}^{\pi})$ then

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Proof :

$$\begin{aligned}
\mathrm{MSE}(\widehat{\theta}) &= \mathrm{E}\left[\left(\widehat{\theta}^{mle} + \frac{1}{|W|}\nabla \log F - \theta\right)^2\right] \\
&= \mathrm{MSE}(\widehat{\theta}^{mle}) + \frac{1}{|W|^2}\left(\mathrm{E}[(\nabla \log F)^2] + 2\mathrm{E}[(\nabla \log \mathbf{F})(\mathbf{N}(\mathbf{W}) - \theta|\mathbf{W}|)]\right) \\
&= \mathrm{MSE}(\widehat{\theta}^{mle}) + \frac{1}{|W|^2}\left(\mathrm{E}[(\nabla \log F)^2] + 2\mathrm{E}[\nabla\nabla \log \mathbf{F}]\right) \quad \text{[IbP]} \\
&= (1) \quad \text{[Chain Rule]}
\end{aligned}$$

(Malliavin) Derivatives of Poisson functionals

- Given $N(W) = n$, we denote $X_1, \ldots, X_n$ the n points in W.
- $\mathcal{S}$: space of Poisson functionals F defined on Ω by

$$F = f_0 \mathbf{1}(N(W) = 0) + \sum_{n \geq 1} \mathbf{1}(N(W) = n) f_n(X_1, \ldots, X_n) ,$$

$f_0 \in \mathbb{R}$, $f_n \in L^1(W^n)$ is a symmetric function.

- Let $\mathcal{S}' = \Big\{ F \in \mathcal{S} : \exists C > 0 \text{ s.t. } \forall n \geq 1, f_n \in \mathcal{C}^1(W^n, \mathbb{R})$ and $\|f_n\|_{L^\infty(W^n,\mathbb{R})} + \sum_{i=1}^n \|\nabla_{x_i} f_n\|_{L^\infty(W^n,\mathbb{R}^d)} \leq C^n \Big\} \subset \mathcal{S}$
- Differential operator : let $F \in \mathcal{S}'$ and $\pi : W^2 \to \mathbb{R}^d$

$$D_x^\pi F = - \sum_{n \geq 1} \mathbf{1}(N(W) = n) \sum_{i=1}^n (\nabla_{x_i} f_n)(X_1, \ldots, X_n) \pi(X_i, x) ,$$

where $\nabla_{x_i} f_n$ gradient of $x_i \mapsto f_n(\ldots, x_i, \ldots)$

Malliavin derivatives (2)

Lemma [chain rules]

For any $x \in W$, for all $F, G \in \mathcal{S}'$, $g \in C_b^1(\mathbb{R})$ we have

$$D_x^\pi(FG) = (D_x^\pi F)G + F(D_x^\pi G) \qquad \text{and} \qquad D_x^\pi g(F) = g'(F) D_x^\pi F\ .$$

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To get an IbP type formula, we need to introduce $\mathrm{Dom}(D^\pi)$ of $\mathcal{S}'$ as

$$\mathrm{Dom}(D^\pi) = \Big\{ F \in \mathcal{S}' : \forall n \geq 1 \text{ and } z_1, \dots, z_n \in \mathbb{R}^d \quad \mathbf{f_{n+1}}\big|_{z_{n+1} \in \partial W}(z_1, \dots, z_{n+1}) = \mathbf{f_n}(z_1, \dots, z_n),\ \mathbf{f_1}\big|_{\mathbf{z} \in \partial \mathbf{W}}(z) = \mathbf{0} \Big\}\ , \qquad (2)$$

Remark : **compatibility conditions** important to derive a correct Stein estimator.

Integration by parts formula

Theorem

Let $F \in \mathrm{Dom}(\overline{D}^{\pi})$, $V : \mathbb{R}^d \to \mathbb{R}$, $V \in \mathcal{C}^1(W, \mathbb{R}^d)$

$$\mathrm{E}\underbrace{\left[\int_W D^{\pi}_x F \cdot V(x)\mathrm{d}x\right]}_{:=\nabla^{\pi,V} F} = \mathrm{E}\left[F\left(\sum_{u\in\mathbf{X}_W} \nabla\cdot\mathcal{V}(u) - \theta\int_W \nabla\cdot\mathcal{V}(u)\mathrm{d}u\right)\right]$$

where $\mathcal{V} : W \to \mathbb{R}^d$ is defined by $\mathcal{V}(u) = \int_W V(x)\boldsymbol{\pi}(u,x)\mathrm{d}x$.

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Main application : let $\boldsymbol{\pi}(u,x) = u^{\top}V(x)$, we can find some V (omit details) such that $\mathcal{V}(u) = u/d$ and $\nabla \cdot \mathcal{V}(u) = 1$. Then

$$\nabla F = \nabla^{\pi,V} F = -\frac{1}{d}\sum_{n \geq 1} \mathbf{1}(N(W) = n) \sum_{i=1}^{n} \nabla_{x_i} f_n(X_1, \ldots, X_n) \cdot X_i$$

$$\Rightarrow \qquad \mathrm{E}[\nabla F] = \mathrm{E}\left[F(N(W) - \theta|W|)\right].$$

Non-uniqueness of the integration by parts formula

It is natural and easier to define a Stein estimator which is isotropic.
With

$$\nabla \log F = -\frac{1}{d} \sum_{n \geq 1} \mathbf{1}(N(W) = n) \sum_{i=1}^{n} \nabla_{x_i} (\log f_n)(X_1, \ldots, X_n) \cdot X_i$$

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$\log F$ is isotropic $\Rightarrow \nabla \log F$ is isotropic (and so will be $\widehat{\theta}$).
If we consider $V(x) = (d|W|)^{-1/2} \mathbf{1}(x \in W) 1^\top$ and $\pi(y, x) = y^\top V(x)$ then $\operatorname{div} \mathcal{V}(y) = 1$. In that case, the gradient operator reduces to

$$\nabla \log F = -\sum_{n\geq 1} \mathbf{1}(N(W) = n) \sum_{i=1}^{n} (\operatorname{div}_{x_i} \log f_n)(X_1, \dots, X_n) \times \overline{X_i}$$

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and we **still** have

$$\mathrm{E}[\nabla \log F] = \mathrm{E}\left[\log F(N(W) - \theta |W|)\right]$$

1. but we lose the isotropic characteristic.
2. this can induce some discontinuity problems when computing $\nabla \log F$ and $\nabla \nabla \log F$ ...

Example for $W = B_d(0,1)$

- For $1 \le k \le n$, $x_{(k),n}$ kth closest (wrt $\|\cdot\|$) point of $\{x_1, \dots, x_n\}$ to 0
- $X_{(k)}$ kth closest point to 0 of the PPP X (defined on $\mathbb{R}^d$)
- Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $\varphi > 0$, $\varphi'(1) = 0$

$$F_k = \mathbf{1}(N(W) < k) + \sum_{n \ge k} \mathbf{1}(N(W) = n)\, \varphi(\|X_{(k),n}\|^2)^2.$$

$$Gain(\widehat{\theta}_k) = 1 - \mathrm{MSE}(\widehat{\theta}_k)/\mathrm{MSE}(\widehat{\theta}^{mle})$$

Theorem

$\zeta_k = \nabla \log(F_k) \in \mathrm{Dom}(\overline{\mathbf{D}}^{\pi})$ and

$$\widehat{\theta}_k = \widehat{\theta}^{mle} - \frac{4d}{|W|} \frac{\varphi'(\|X_{(k)}\|^2)}{\varphi(\|X_{(k)}\|^2)} \quad \text{and} \quad Gain(\widehat{\theta}_k) = \mathrm{E}[\ \mathcal{G}(\|X_{(k)}\|^2)\ ]$$

where

$$\mathcal{G}(t) = -\frac{16}{d^2\theta|W|} \frac{t\,(\varphi'(t) + t\varphi''(t))}{\varphi(t)}$$

Two specific examples ...

- "Linear" function : $0 < \gamma < 1$, $\kappa > 0$

$$\varphi(t) = (1-t)(\chi_{[0,1-\gamma]} * \psi)(t) + \kappa,$$

where χ= characteristic function, ψ Schwarz function given by

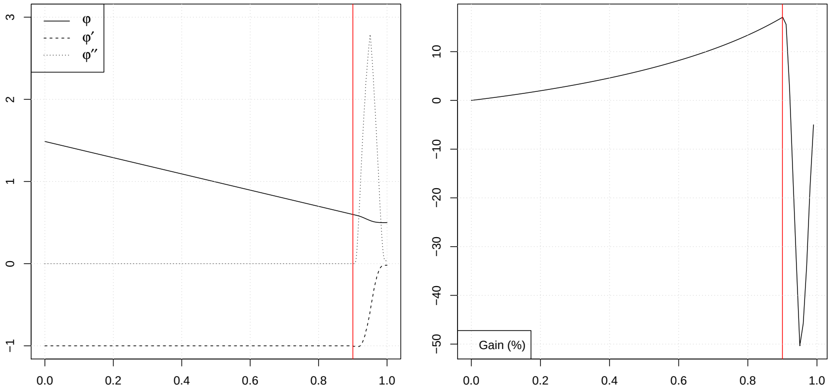
$$\psi(t) = c e^{-1/(1-|t|)} \text{ with } c \text{ such that } \int_0^1 \psi(t) ṭ = 1.$$

- Exponential function : $\gamma \in \mathbb{R}$, $\kappa \geq 2$

$$\varphi(t) = e^{\gamma(1-t)^{\kappa}} \mathbf{1}(t \leq 1).$$

Linear function : Plots of $\varphi, \varphi', \varphi'', \mathcal{G}$

($\kappa = 0.1$; $\gamma = 0.05$)



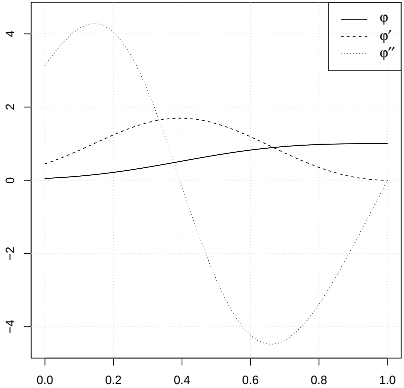
Interesting choice, since for $t < 1 - 2\gamma$:

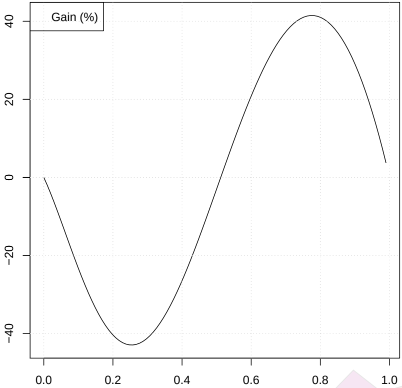
$$\varphi(t) \simeq 1 - t + \kappa,\ \varphi'(t) \simeq -1,\ \varphi''(t) \simeq 0,\ \mathcal{G}(t) \simeq ct/(1 + \kappa - t) > 0$$

Seems encouraging, but the "interesting" positive values of $\mathcal{G}$ are too close to strong negative values ! (too dangerous)

Exp. function : Plots of $\varphi, \varphi', \varphi'', \mathcal{G}$ ($\kappa = 3$; $\gamma = -3$)

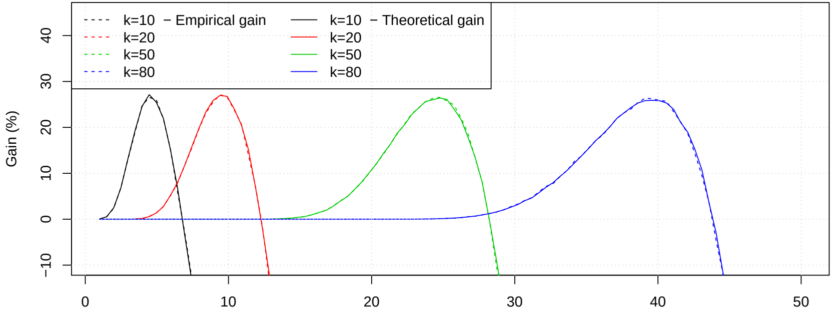
$$\widehat{\theta}^{stein} = \widehat{\theta}^{mle} - \frac{4d}{|W|}\left\{\gamma\kappa\left(1 - \|X_{(k)}\|^2\right)^{\kappa-1}\right\}\mathbf{1}(\|X_{(k)}\| \leq 1)$$



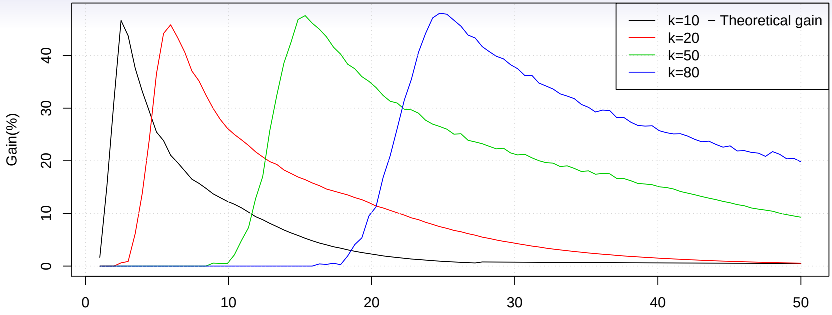


$\Rightarrow$ $\mathcal{G}(t)$ is not positive everywhere but when t is large (i.e. when $\|X_{(k)}\|$ is large, i.e. when k is large), then $\mathcal{G}(\cdot)$ is positive and can reach high values.

Focus on the exp. function and on $\mathbb{E}[\mathcal{G}(\|X_{(k)}\|^2)]$



- $m = 50000$ replications of $PPP(\overset{\theta}{B}(0,1),\theta)$, $d = 2$.
- Empirical and Monte-Carlo approximations of theoretical gains, for different parameters k, κ, γ.
- General comments :
  1. The IbP formula is empirically checked.
  2. The parameters k, κ, γ and θ are strongly connected. A bad choice can lead to **negative** gains.



- $m = 50000$ replications of $PPP(\overset{\theta}{B}(0,1),\theta),\ d = 2$.
- Monte-Carlo approximations of theoretical gains for different values of k. The parameters κ and γ optimize $\text{Gain}(\widehat{\theta_k})$ for each value of θ.
- General comments :
  1. For any k, if we optimize in terms of κ and γ, the gain becomes always positive.
  2. Still, if we want interesting values of gains, k needs to be optimized.

- Simulation based on $m = 50000$ replications.
- For each value of θ, d

$$(k^\star, \gamma^\star, \kappa^\star) = \text{argmax}_{(k,\gamma,\kappa)} \text{Gain}(\widehat{\theta_k}) = \text{argmax}_{(k,\gamma,\kappa)} \text{E}[\mathcal{G}(\|X_{(k)}\|^2)].$$

	MLE			STEIN				Gain (%)
	mean	sd	mse	$k^\star$	mean	sd	mse	
$\theta = 5,\ d = 1$	5	1.6	2.52	11	4.4	1.0	1.44	43.0
$d = 2$	5	1.3	1.58	18	4.6	0.8	0.86	45.6
$d = 3$	5	1.1	1.19	22	4.6	0.7	0.64	46.1
$\theta = 10,\ d = 1$	10	2.2	5.03	22	9.2	1.4	2.73	45.8
$d = 2$	10	1.8	3.18	34	9.4	1.2	1.72	46.0
$d = 3$	10	1.5	2.37	44	9.5	1.0	1.27	46.3
$\theta = 20,\ d = 1$	20	3.1	9.91	42	18.8	2.0	5.31	46.4
$d = 2$	20	2.5	6.38	66	19.1	1.6	3.41	46.5
$d = 3$	20	2.2	4.72	84	19.1	1.3	2.47	47.5
$\theta = 40,\ d = 1$	40	4.5	20.09	84	38.5	2.9	10.61	47.2
$d = 2$	40	3.6	12.79	125	38.6	2.2	6.78	46.9
$d = 3$	40	3.1	9.58	169	38.8	1.9	4.95	48.3

Data-driven est. : replace θ by $\widehat{\theta}_{mle}$ in the optimization

- Simulation based on $m = 5000$ replications.
- For each value of θ, d, let $\Theta(\theta,\rho) = \left[\theta - \rho\sqrt{\theta/|W|}, \theta + \rho\sqrt{\theta/|W|}\right]$. Then, we suggest define $\kappa^\star$, $\gamma^\star$ as the maximum of

$$\int_{\Theta(\widehat{\theta}_{MLE},\rho)} \mathrm{Gain}(\widehat{\theta}_k)\mathrm{d}\theta = \frac{16}{d^2|W|}\mathrm{E}\int_{\Theta(\widehat{\theta}_{MLE},\rho)} \frac{\mathcal{G}(Y_{(k)})}{\theta}\mathrm{d}\theta. \quad (3)$$

	Gain (%)			
	$\rho = 0$	$\rho = 1$	$\rho = 1.6449$	$\rho = 1.96$
$\theta = 5,\ d = 1$	48.8	47.9	36.4	30.1
$d = 2$	38.6	42.4	37.1	31.4
$d = 3$	39.4	42.6	37.0	31.7
$\theta = 10,\ d = 1$	40.3	43.8	36.7	30.1
$d = 2$	36.2	38.8	33.7	27.9
$d = 3$	31.6	36.6	32.0	28.3
$\theta = 20,\ d = 1$	37.3	38.6	34.5	28.0
$d = 2$	27.3	33.1	31.0	26.5
$d = 3$	20.8	28.6	28.1	23.8
$\theta = 40,\ d = 1$	22.3	30.8	29.2	23.9
$d = 2$	16.3	24.0	28.2	24.4
$d = 3$	12.7	19.0	24.5	22.0

A few more comments

- Even if the results are done under the Poisson assumption, if the simulated model
  - is clustered (e.g. Thomas, LGCP) the empirical gains (compared to $N(W)/|W|$) are significant .
  - is regular the empirical gains seems to be close to zero (not really worse than $N(W)/|W|$)

Perspectives

1. Deriving a general IbP formula for inhomogeneous Poisson point processes or Cox point processes seems reasonable.
2. Exploit the IbP for other statistical methodologies or explicit moment calculations

Comparison with Privault-Réveillac's est. when $d = 1$

- Assume $\mathbf{X}$ is observed on $\widetilde{W} = [0, 2]$.
- Let X_1 be the closest point of $\mathbf{X}$ to 0, then $\widehat{\theta}_{pr}$ is defined for some $\kappa > 0$ by

$$\widehat{\theta}_{pr} = \widehat{\theta}_{mle} + \frac{2}{\kappa}\mathbf{1}(N(\widetilde{W}) = 0) + \frac{2X_1}{2(1+\kappa) - X_1}\mathbf{1}(0 < X_1 \leq 2).$$

  Note that $X_1 \sim \mathcal{E}(\theta)$.
- The gain writes

$$\mathrm{Gain}(\widehat{\theta}_{pr}) = \frac{2}{\theta\kappa^2}\exp(-2\theta) - \frac{2}{\theta}\mathrm{E}\left(\frac{X_1}{2(1+\kappa) - X_1}\mathbf{1}(X_1 \leq 2)\right).$$

Gain optimized in κ in terms of θ.