

Hypoocoercivity and diffusion limit of a finite volume scheme for linear kinetic equations

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Linear kinetic equation

For $t \geq 0$, $x \in \mathbb{T}$ and $v \in \mathbb{R}$,

$$\begin{cases} \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = \mathcal{Q}(f), \\ f(0, x, v) = f_0(x, v) \geq 0. \end{cases}$$

Mass preserving collision operators :

$$\mathcal{Q}_{FP}(f)(v) = \partial_v (\partial_v f + vf),$$

or

$$\mathcal{Q}_{BGK}(f)(v) = \rho \mathcal{M}(v) - m_0 f(v),$$

where \mathcal{M} is the Maxwellian, and ρ the local mass of f , namely

$$\rho = \int_{\mathbb{R}} f(v) \, dv.$$

Hypoocoercivity

Exponential return to equilibrium : there exists an appropriate functional space \mathcal{X} and constants $\kappa > 0$, $C \geq 1$ such that

$$\|f(t) - m_f \mathcal{M}\|_{\mathcal{X}} \leq C \|f(0) - m_f \mathcal{M}\|_{\mathcal{X}} e^{-\kappa t},$$

where $m_f = \int \int f \, dx \, dv$.

A very brief history :

- **Hérau** (2006)
- **Villani** (2009)
- **Dolbeault, Mouhot, Schmeiser** (2015)

Diffusive limit

Parabolic scaling : $t \rightarrow t/\varepsilon^2$, $x \rightarrow x/\varepsilon$

$$\Rightarrow \varepsilon \frac{\partial f^\varepsilon}{\partial t} + v \frac{\partial f^\varepsilon}{\partial x} = \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon).$$

Moments equations :

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x J^\varepsilon = 0, \\ \varepsilon^2 \partial_t J^\varepsilon + m_2 \partial_x \rho^\varepsilon + \partial_x S^\varepsilon = -J^\varepsilon, \end{cases}$$

where

$$\rho^\varepsilon := \int_{\mathbb{R}} f^\varepsilon \, dv, \quad J^\varepsilon := \frac{1}{\varepsilon} \int_{\mathbb{R}} f^\varepsilon v \, dv, \quad S^\varepsilon := \int_{\mathbb{R}} (v^2 - m_2) f^\varepsilon \, dv.$$

Diffusive limit $\varepsilon \rightarrow 0$: $S^\varepsilon \rightarrow 0$ and $J^\varepsilon \rightarrow -m_2 \partial_x \rho$, with ρ solution of

$$\frac{\partial \rho}{\partial t} - \partial_x (m_2 \partial_x \rho) = 0.$$

- **Degond, Goudon, Poupaud** (2000) : linear kinetic equations.
- **Mellet, Mischler, Mouhot** (2011) : fractional diffusion limits.

Asymptotic Preserving schemes

AP schemes for kinetic equations :

- **Klar** (1998) : linear BGK equation for semiconductors,
- **Jin, Pareschi, Toscani** (2000) : relaxation models,
- **Jin** (2010), **Dimarco, Pareschi** (2014) : review papers,
- ...

Proofs of AP properties :

- **Lemou, Mieussens** (2008) : micro-macro approach,
- **Liu, Mieussens** (2010) : detailed stability analysis,
- **Crouseilles, Hivert, Lemou** (2016) : anomalous diffusion, time discretization,
- **Dujardin, Héreau, Lafitte** (2018) : hypocoercivity for full discrete FP equation,
- ...

Aims

Develop numerical schemes for linear kinetic equations which

- mimic continuous behaviour (energy estimates, entropy–entropy dissipation inequalities, hypocoercivity),

- are stable for all ε and have the correct diffusion limit.

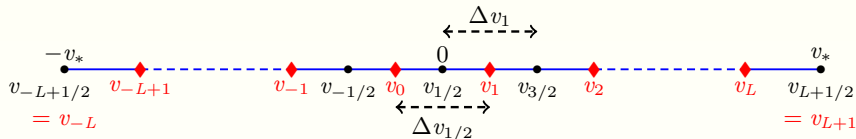
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Mesh

- Discretization of the bounded velocity domain $[-v_*, v_*]$ into $2L$ cells :

$$\mathcal{V}_j = (v_{j-\frac{1}{2}}, v_{j+\frac{1}{2}}), \quad j \in \mathcal{J} = \{-L+1, \dots, L\}.$$



Dual mesh : $\mathcal{V}_{j+\frac{1}{2}}^* = (v_j, v_{j+1}), \quad j \in \mathcal{J}^* = \{-L, \dots, L\}.$

- Discretization of the spatial domain \mathbb{T} into N subintervals :

$$\mathcal{X}_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \quad i \in \mathcal{I} = \mathbb{Z}/N\mathbb{Z}.$$

- Time step $\Delta t > 0$, $t^n = n\Delta t$ for all $n \geq 0$.

Discrete Maxwellians

BGK case. We define cell values $\mathcal{M}^\Delta = (\mathcal{M}_j)_{j \in \mathcal{J}}$ such that

$$\left\{ \begin{array}{l} \mathcal{M}_j > 0, \quad \mathcal{M}_j = \mathcal{M}_{-j+1}, \quad \forall j = 1, \dots, L, \\ \sum_{j \in \mathcal{J}} \mathcal{M}_j \Delta v_j = 1, \\ 0 < \underline{m}_2 \leq m_2^{\Delta v} \leq \overline{m}_2, \quad m_4^{\Delta v} \leq \overline{m}_4. \end{array} \right.$$

Fokker Planck case. We define interface values $(\mathcal{M}_{j+1/2}^*)_{j \in \mathcal{J}^*} \in \mathbb{R}^{\mathcal{J}^*}$ such that

$$\left\{ \begin{array}{l} \mathcal{M}_{j+1/2}^* = \mathcal{M}_{-j+1/2}^*, \quad \forall j \in \mathcal{J}^*, \\ \mathcal{M}_{L+1/2}^* = \mathcal{M}_{-L+1/2}^* = 0, \\ \mathcal{M}_j := \frac{\mathcal{M}_{j-1/2}^* - \mathcal{M}_{j+1/2}^*}{v_j \Delta v_j} > 0, \quad \forall j \in \mathcal{J}; \\ \sum_{j \in \mathcal{J}} \mathcal{M}_j \Delta v_j = 1, \\ 0 < \underline{m}_2 \leq m_2^{\Delta v} \leq \overline{m}_2, \quad m_4^{\Delta v} \leq \overline{m}_4. \end{array} \right.$$

Fully discrete Fokker-Planck equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

$$\varepsilon \Delta x_i \Delta v_j (f_{ij}^{n+1} - f_{ij}^n) + \Delta t \left(\mathcal{F}_{i+\frac{1}{2},j}^{n+1} - \mathcal{F}_{i-\frac{1}{2},j}^{n+1} \right) = \frac{\Delta t}{\varepsilon} \left(\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} - \mathcal{G}_{i,j-\frac{1}{2}}^{n+1} \right),$$

where the **free transport** fluxes $\mathcal{F}_{i+\frac{1}{2},j}$ and the **Fokker-Planck** fluxes $\mathcal{G}_{i,j+\frac{1}{2}}$ are :

$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} \approx \int_{\mathcal{V}_j} v f(t^{n+1}, x_{i+\frac{1}{2}}, v) dv,$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} \approx \int_{\mathcal{X}_i} \left(\partial_v f + v_{j+\frac{1}{2}} f \right) (t^{n+1}, x, v_{j+\frac{1}{2}}) dx.$$

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$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} \approx \int_{\mathcal{X}_i} \left(\partial_v f + v_{j+\frac{1}{2}} f \right) (t^{n+1}, x, v_{j+\frac{1}{2}}) dx.$$

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$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} \approx \int_{x_i} \mathcal{M} \partial_v \left(\frac{f}{\mathcal{M}} \right) (t^{n+1}, x, v_{j+\frac{1}{2}}) dx.$$

Fully discrete Fokker-Planck equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

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$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} = \mathcal{M}_{j+\frac{1}{2}}^* \left(\frac{f_{i,j+1}^{n+1}}{\mathcal{M}_{j+1}} - \frac{f_{ij}^{n+1}}{\mathcal{M}_j} \right) \frac{1}{\Delta v_{j+\frac{1}{2}}} \Delta x_i, \quad \forall j \in \mathcal{J}^* \setminus \{-L, L\}, \quad \forall i \in \mathcal{I}.$$

Fully discrete Fokker-Planck equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

$$\varepsilon \Delta x_i \Delta v_j (f_{ij}^{n+1} - f_{ij}^n) + \Delta t \left(\mathcal{F}_{i+\frac{1}{2},j}^{n+1} - \mathcal{F}_{i-\frac{1}{2},j}^{n+1} \right) = \frac{\Delta t}{\varepsilon} \left(\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} - \mathcal{G}_{i,j-\frac{1}{2}}^{n+1} \right),$$

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$$\mathcal{G}_{i,j+\frac{1}{2}}^{n+1} = \mathcal{M}_{j+\frac{1}{2}}^* \left(\frac{f_{i,j+1}^{n+1}}{\mathcal{M}_{j+1}} - \frac{f_{ij}^{n+1}}{\mathcal{M}_j} \right) \frac{1}{\Delta v_{j+\frac{1}{2}}} \Delta x_i, \quad \forall j \in \mathcal{J}^* \setminus \{-L, L\}, \quad \forall i \in \mathcal{I},$$

$$\mathcal{G}_{i,-L+\frac{1}{2}}^{n+1} = \mathcal{G}_{i,L+\frac{1}{2}}^{n+1} = 0, \quad \forall i \in \mathcal{I}.$$

Fully discrete BGK equation

For all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq 0$,

$$\varepsilon \Delta x_i \Delta v_j (f_{ij}^{n+1} - f_{ij}^n) + \Delta t \left(\mathcal{F}_{i+\frac{1}{2},j}^{n+1} - \mathcal{F}_{i-\frac{1}{2},j}^{n+1} \right) = \frac{\Delta t}{\varepsilon} \Delta x_i \Delta v_j \left(\rho_i^{n+1} \mathcal{M}_j - f_{ij}^{n+1} \right),$$

where

$$\mathcal{F}_{i+\frac{1}{2},j}^{n+1} = v_j \frac{f_{i+1,j}^{n+1} + f_{ij}^{n+1}}{2} \Delta v_j, \quad \forall j \in \mathcal{J}, \quad \forall i \in \mathcal{I},$$

and

$$\rho_i^{n+1} := \sum_{j \in \mathcal{J}} \Delta v_j f_{ij}^{n+1} \quad \forall i \in \mathcal{I}.$$

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Discrete “entropy” estimate

Proposition

Let $(f_{ij}^n)_{i \in \mathcal{I}, j \in \mathcal{J}, n \in \mathbb{N}}$ solve the scheme either in Fokker Planck or BGK case. Then for all $\varepsilon > 0$, and for every $n \geq 0$,

$$\frac{\|f^{n+1}\|_{2,\gamma}^2 - \|f^n\|_{2,\gamma}^2}{2\Delta t} + \frac{1}{\varepsilon^2} \|f^{n+1} - \rho^{n+1} \mathcal{M}^\Delta\|_{2,\gamma}^2 \leq 0.$$

In particular one has

$$\max \left(\sup_{n \geq 0} \|f^n\|_{2,\gamma}^2, \frac{2}{\varepsilon^2} \sum_{n=1}^{\infty} \Delta t \|f^n - \rho^n \mathcal{M}^\Delta\|_{2,\gamma}^2 \right) \leq \|f^0\|_{2,\gamma}^2.$$

Consequence : existence and uniqueness of the discrete solution.

Proof

Multiplying equation by f/\mathcal{M} , and integrating in x and v yields

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2(dx dv)}^2 + \frac{1}{2\varepsilon} \iint \partial_x \left(v \frac{f^2}{\mathcal{M}} \right) dx dv = \frac{1}{\varepsilon^2} \iint \mathcal{Q}(f) \frac{f}{\mathcal{M}} dx dv.$$

Proof

Multiplying equation by f/\mathcal{M} , and integrating in x and v yields

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2 + 0 = \frac{1}{\varepsilon^2} \iint \mathcal{Q}(f) \frac{f}{\mathcal{M}} \mathrm{d}v \mathrm{d}v.$$

BGK case : Using $\int \mathcal{Q}_{BGK}(f) \mathrm{d}v = 0$,

$$\Rightarrow \iint \mathcal{Q}_{BGK}(f) \frac{f}{\mathcal{M}} = \iint \mathcal{Q}_{BGK}(f) \left(\frac{f}{\mathcal{M}} - \rho \right) = -\|f - \rho\mathcal{M}\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2.$$

Fokker Planck case : Using **Gaussian Poincaré inequality**,

$$\iint \mathcal{Q}_{FP}(f) \frac{f}{\mathcal{M}} = - \left\| \partial_v \left(\frac{f}{\mathcal{M}} \right) \right\|_{L^2(\mathrm{d}x \mathrm{d}\mathcal{M})}^2 \leq -\|f - \rho\mathcal{M}\|_{L^2(\mathrm{d}x \mathrm{d}\gamma)}^2.$$

Discrete moments estimates

$$J_i^n := \frac{1}{\varepsilon} \sum_{j \in \mathcal{J}} \Delta v_j v_j f_{ij}^n, \quad S_i^n := \sum_{j \in \mathcal{J}} \Delta v_j (v_j^2 - m_2^{\Delta v}) f_{ij}^n, \quad \forall i \in \mathcal{I}, \quad \forall n \geq 0.$$

Lemma

The moments $(\rho_i^n)_{i \in \mathcal{I}}$, $(J_i^n)_{i \in \mathcal{I}}$ and $(S_i^n)_{i \in \mathcal{I}}$ satisfy :

$$\|\rho^n\|_2 \leq \|f^n\|_{2,\gamma},$$

$$\varepsilon \|J^n\|_2 \leq (m_2^{\Delta v})^{1/2} \|f^n\|_{2,\gamma},$$

$$\varepsilon \|J^n\|_2 \leq (m_2^{\Delta v})^{1/2} \|f^n - \rho^n \mathcal{M}^\Delta\|_{2,\gamma},$$

$$\|S^n\|_2 \leq (m_4^{\Delta v} - (m_2^{\Delta v})^2)^{1/2} \|f^n - \rho^n \mathcal{M}^\Delta\|_{2,\gamma}.$$

In particular, there is a constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \|\rho^n\|_2^2, \quad \varepsilon^2 \sup_{n \in \mathbb{N}} \|J^n\|_2^2, \quad \sum_{n=0}^{\infty} \Delta t \|J^n\|_2^2, \quad \frac{1}{\varepsilon^2} \sum_{n=0}^{\infty} \Delta t \|S^n\|_2^2 \leq C \|f_0\|_{2,\gamma}^2.$$

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Discrete moments equations

Lemma

Let us consider a solution to the scheme. Then the discrete moments satisfy the following equations. For all $i \in \mathcal{I}$, $n \geq 0$,

$$\Delta x_i (\rho_i^{n+1} - \rho_i^n) + \Delta t (J_{i+\frac{1}{2}}^{n+1} - J_{i-\frac{1}{2}}^{n+1}) = 0,$$

$$\begin{aligned} \varepsilon^2 \Delta x_i (J_i^{n+1} - J_i^n) + \Delta t (S_{i+\frac{1}{2}}^{n+1} - S_{i-\frac{1}{2}}^{n+1}) + \Delta t m_2^{\Delta v} (\rho_{i+\frac{1}{2}}^{n+1} - \rho_{i-\frac{1}{2}}^{n+1}) \\ = -\Delta t \Delta x_i J_i^{n+1}, \end{aligned}$$

where

$$Q_{i+\frac{1}{2}} = \frac{Q_i + Q_{i+1}}{2}, \quad X = \rho, J, S.$$

Asymptotic-preserving property

Theorem

Let $f_\varepsilon^n = (f_{ij}^n)_{i \in \mathcal{I}, j \in \mathcal{J}}$ for $n \in \mathbb{N}$ be a solution of the scheme. Then there is $\rho^n = (\rho_i^n)_{i \in \mathcal{I}}$ for all $n \geq 0$ such that when $\varepsilon \rightarrow 0$ one has

$$f_\varepsilon^n \longrightarrow \rho^n \mathcal{M}^\Delta \quad \text{in } \mathbb{R}^{\mathcal{I} \times \mathcal{J}}, \quad \text{for all } n \geq 1,$$

and the limit ρ satisfies the following scheme for the heat equation

$$\Delta x_i \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{m_2 \Delta v}{2} \left((D_x \rho)_{i+1}^{n+1} - (D_x \rho)_{i-1}^{n+1} \right), \quad \forall i \in \mathcal{I},$$

with initial data $\rho_i^0 = \sum_{j \in \mathcal{J}} \Delta v_j f_{ij}^0$.

Proof

- A priori estimates \Rightarrow up to a subsequence $\varepsilon_k \rightarrow 0$,

$$\begin{aligned}\rho_{\varepsilon_k}^n &\rightarrow \rho^n \quad \forall n \geq 0, \\ f_{\varepsilon_k}^n &\rightarrow (\rho_i^n \mathcal{M}_j)_{i \in \mathcal{I}, j \in \mathcal{J}} \quad \forall n \geq 1.\end{aligned}$$

- Combining moments equations implies

$$\begin{aligned}\Delta x_i (\rho_i^{n+1} - \rho_i^n) &= m_2^{\Delta v} \Delta t \left((D_x \rho^{n+1})_{i+1/2} - (D_x \rho^{n+1})_{i-1/2} \right) \\ &+ \Delta t \left((D_x S^{n+1})_{i+1/2} - (D_x S^{n+1})_{i-1/2} \right) + \varepsilon^2 (J_{i+\frac{1}{2}}^{n+1} - J_{i-\frac{1}{2}}^{n+1}) - \varepsilon^2 (J_{i+\frac{1}{2}}^n - J_{i-\frac{1}{2}}^n).\end{aligned}$$

Proof

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- A priori estimates \Rightarrow up to a subsequence $\varepsilon_k \rightarrow 0$,

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- The limit scheme has a unique solution \Rightarrow the whole sequence converges.

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Main result

Theorem

Assume that the number of points N in the space discretization is odd.

Then there are constants $C \geq 1$ and $\kappa > 0$ such that for all $\varepsilon \in (0, 1)$, all $\Delta t \leq \Delta t_{\max}$ and all initial data $(f_{ij}^0)_{i \in \mathcal{I}, j \in \mathcal{J}}$, the solution $(f_{ij}^n)_{i \in \mathcal{I}, j \in \mathcal{J}, n \in \mathbb{N}}$ of the scheme satisfies

$$\|f^n - m_f \mathcal{M}^\Delta\|_{2,\gamma} \leq C \|f^0 - m_f \mathcal{M}^\Delta\|_{2,\gamma} e^{-\kappa t^n},$$

where $m_f = \sum_{(i,j) \in \mathcal{I} \times \mathcal{J}} \Delta x_i \Delta v_j f_{ij}^0$.

Moreover, the constants C and κ do not depend on the size of the discretization, and $\Delta t_{\max} > 0$ can be chosen arbitrarily.

Modified entropy

Up to changing f by $f - m_f \mathcal{M}$, we can assume that $m_f = 0$.

Modified entropy (Dolbeault, Mouhot, Schmeiser, 2015)

$$H(f^\varepsilon(t)) = \frac{1}{2} \|f^\varepsilon(t)\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2 + \eta \varepsilon^2 \langle J^\varepsilon, \partial_x \phi^\varepsilon \rangle_{L_x^2},$$

where $\phi^\varepsilon(t, x)$ is the solution of the Poisson equation

$$-\partial_{xx}^2 \phi^\varepsilon = \rho^\varepsilon, \quad \int_{\mathbb{T}} \phi^\varepsilon \mathrm{d}x = 0,$$

and $\eta > 0$ is a small parameter to be chosen.

Outline of proof

① Entropy–dissipation estimate :

$$\begin{aligned} \frac{d}{dt} H(f^\varepsilon) + K_\eta (\|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2 + \|\rho^\varepsilon \mathcal{M}\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2) &\leq 0, \\ \implies \frac{d}{dt} H(f^\varepsilon) + \frac{K_\eta}{2} \|f^\varepsilon\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2 &\leq 0. \end{aligned}$$

② Control of the entropy :

$$\frac{c_\eta}{2} \|f^\varepsilon\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2 \leq H(f^\varepsilon) \leq \frac{C_\eta}{2} \|f^\varepsilon\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}^2.$$

\implies Exponential decay of $H(t)$, and then of $\|f^\varepsilon(t)\|_{L^2(\mathrm{d}x\mathrm{d}\gamma)}$.

Adaptation to the discrete framework

Modified discrete entropy

$$\begin{aligned} \mathbb{H}(f^n) &:= \frac{1}{2} \|f^n\|_{2,\gamma}^2 + \eta \varepsilon^2 \sum_{i \in \mathcal{I}} \Delta x_i J_i^n (D_x \phi)_i^n \\ &\quad + \frac{\eta \varepsilon^2}{2} \sum_{i \in \mathcal{I}} \Delta x_i \frac{((D_x \phi)_i^n - (D_x \phi)_i^{n-1})^2}{\Delta t}, \end{aligned}$$

where $(\phi_i^n)_{i \in \mathcal{I}}$ is the solution of the discrete Poisson equation

$$\begin{aligned} - \frac{(D_x \phi)_{i+1}^n - (D_x \phi)_{i-1}^n}{2} &= \Delta x_i \rho_i^n, \quad \forall i \in \mathcal{I}, \\ \sum_{i \in \mathcal{I}} \Delta x_i \phi_i^n &= 0, \end{aligned}$$

with

$$(D_x \phi)_i = \frac{\phi_{i+1} - \phi_{i-1}}{\Delta x_i}.$$

N odd \Rightarrow existence and uniqueness of $(\phi_i^n)_{i \in \mathcal{I}}$.
 \Rightarrow discrete Poincaré inequality on \mathbb{T} .

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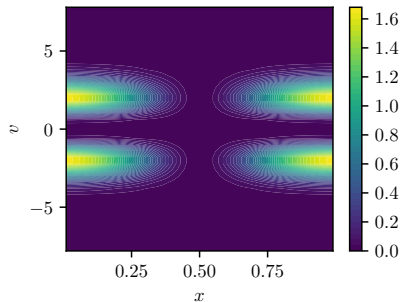
Data

Discretization parameters.

- $v_* = 8$, $L = 20$ (\Rightarrow 40 cells in velocity),
- $N = 51$ cells in the torus,
- $\Delta t = 0.1$.

Initial data.

$$f_0(x, v) := \frac{1}{\sqrt{2\pi}} v^4 e^{-v^2/2} \frac{1 + \cos(4\pi x)}{2}, \quad \forall x \in \mathbb{T}, v \in [-v_*, v_*].$$



Diffusive limit (Fokker-Planck case)

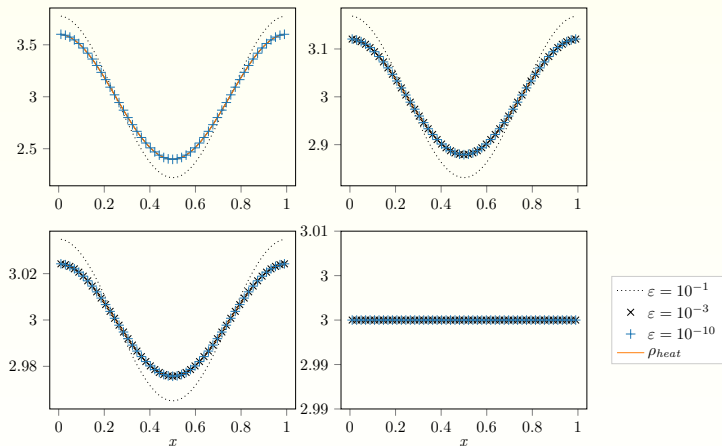


FIGURE – Comparison of the approximate solution of the heat equation (solid line) with the approximate densities obtained with the kinetic scheme for different ε , at times $t = 0.1, 0.2, 0.3$ and 10 .

Diffusive limit (BGK case)

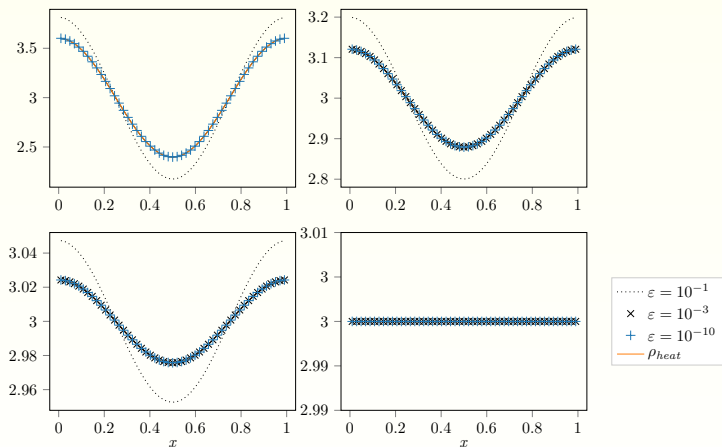


FIGURE – Comparison of the approximate solution of the heat equation (solid line) with the approximate densities obtained with the kinetic scheme for different ϵ , at times $t = 0.1, 0.2, 0.3$ and 10 .

Trend to equilibrium

Fokker-Planck case, $\varepsilon = 1$.

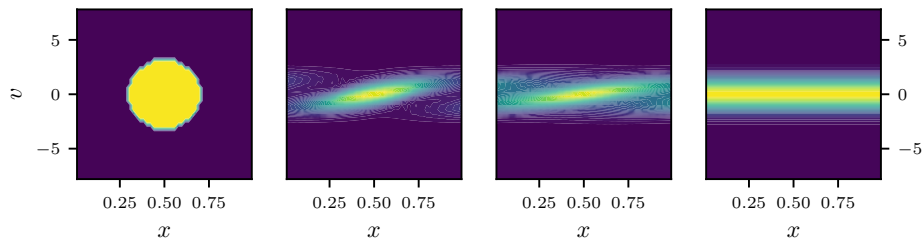


FIGURE – Snapshot of the particle distribution function $f^\varepsilon(t, x, v)$ in the (x, v) -phase-plane, at times $t = 0, 0.3, 0.6$, and 30 .

Trend to equilibrium

Fokker-Planck case, $\varepsilon = 1$.

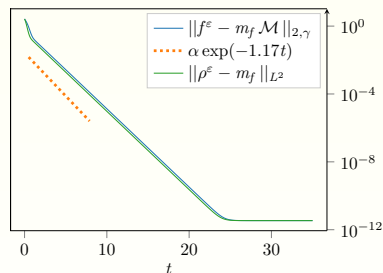
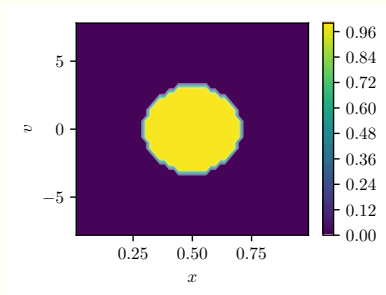


FIGURE – Left : Initial data in the (x, v) -phase-plane. Right : Time evolution of the weighted L^2 norm of the difference between f^ε and the global equilibrium.

Dependency of the hypocoercivity rate wrt ε

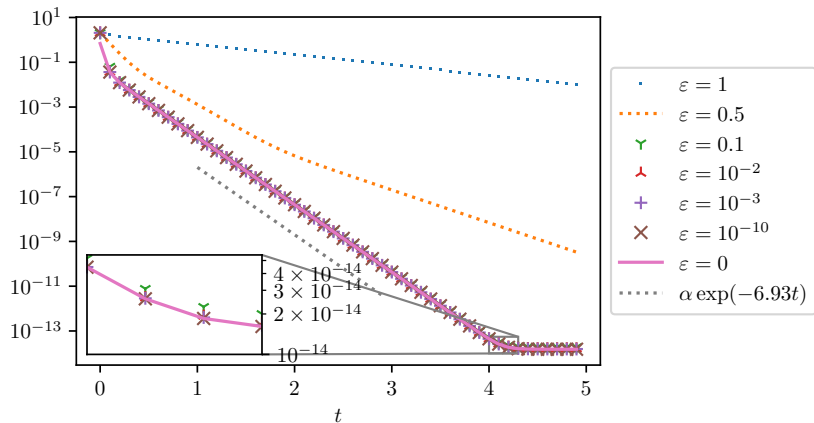


FIGURE – Comparison of the rate of convergence of $\|f - m_f \mathcal{M}\|_{2,\gamma}$ for different values of ε , in the **Fokker-Planck** case, with f_0 uniformly distributed in the (x, v) -phase-plane.

Conclusion

- Numerical scheme for a 1D linear kinetic equation.
- Proof of Asymptotic-Preserving property in the diffusion limit.
- Adaptation of hypocoercivity method to the discrete setting.
- Exponential return to equilibrium, uniformly in ε .

Outlook.

- Extension to more general linear collision operators?
- Coupling with Poisson equation?

Thank you for your attention !