I. Simplicial and cosimplicial models for free loop spaces

A. Simplicial preliminaries

• Twisting functions

$K_\bullet$ simplicial set, $G_\bullet$ simplicial group

$\tau : K_\bullet \longrightarrow G_{\bullet-1}$ $(\bullet \geq 1)$ is a twisting function if $\forall x \in K_\bullet$:

$$d_i \tau(x) = \begin{cases} \tau(d_0 x)^{-1} \cdot \tau(d_1 x) & : i = 0 \\ \tau(d_{i+1}) & : i > 0 \end{cases}$$

$$s_i \tau(x) = \tau(s_{i+1} x) \quad \forall i \geq 0$$

$$\tau(s_0 x) = e.$$

Given a twisting function $\tau : K_\bullet \longrightarrow G_{\bullet-1}$ and a left action $\alpha : G_\bullet \times L_\bullet \longrightarrow L_\bullet$, the twisted Cartesian product (TCP) of $K_\bullet$ and $L_\bullet$ over τ

(The simplicial analogue of fiber bundles.)

$$K_\bullet \times_\tau L_\bullet$$

where $(K_\bullet \times_\tau L_\bullet)_n = K_n \times L_n$ and

$$d_i(x,y) = \begin{cases} (d_0 x, \tau(x) \cdot d_0 y) & : i = 0 \\ (d_i x, d_i y) & : i > 0 \end{cases}$$

$$s_i(x,y) = (s_i x, s_i y) \quad \forall i \geq 0.$$

Proposition: The projection $K_\bullet \times_\tau L_\bullet \longrightarrow K_\bullet$ is a simplicial map, which is a Kan fibration iff $L_\bullet$ is a Kan complex.

- **Kan classifying spaces and loop groups**

Theorem: $\exists$ adjunction

$$\mathbb{G} : \underline{sSet}_0 \rightleftarrows \underline{sGr} : \overline{W},$$

where: $\circ$ $(\mathbb{G}K)_n = \mathrm{Free}(K_{n+1}) / \langle s_0 X_n \rangle$ Nota: $\bar{x}$

$$d_i \bar{x} = \begin{cases} \overline{d_0 x}^{-1} \cdot \overline{d_1 x} & : i = 0 \\ \overline{d_{i+1} x} & : i > 0. \end{cases}$$

$$s_i \bar{x} = \overline{s_{i+1} x} \quad \forall i$$

$\circ$ $$(\overline{W}G_\bullet)_n = \begin{cases} \{(-)\} & : n = 0 \\ G_0 \times \cdots \times G_{n-1} & : n > 1 \end{cases}$$

$$d_i(a_0, \ldots, a_{n-1}) = \begin{cases} (a_0, \ldots, a_{n-2}) & : i = 0 \\ (a_0, \ldots, a_{n-i-1} \cdot d_0 a_{n-i}, \ldots, d_{i-1} a_{n-1}) & : 0 < i < n \\ (d_1 a_1, \ldots, d_{n-1} a_{n-1}) & : i = n \end{cases}$$

$$s_i(a_0, \ldots, a_{n-1}) = \begin{cases} (a_0, \ldots, a_{n-1}, e) & : i = 0 \\ (a_0, \ldots, a_{n-i-1}, e, s_0 a_{n-i}, \ldots, s_{i-1} a_{n-1}) & : 0 < i < n \\ (e, s_0 a_0, \ldots, s_{n-1} a_{n-1}) & : i = n \end{cases}$$

Proposition: The maps

universal twisting function

$$K_\bullet \xrightarrow{\tau_X} (\mathbb{G}K)_{\bullet-1} : x \longmapsto \bar{x}$$

and

couniversal twisting function

$$(\overline{W}G)_\bullet \xrightarrow{\nu_G} G_{\bullet-1} : (a_0, \dots, a_{n-1}) \longmapsto a_{n-1}$$

are twisting functions and mediate the adjunction above:

$$\underline{sSet}_0(K, \overline{W}G) \cong Tw(K, G) \cong \underline{sGr}(\mathbb{G}K, G).$$

$$f \longmapsto \nu_G \circ f$$

$$h \circ \tau_X \longleftarrow h$$

Remark: Homotopy classes of simplicial maps into $\overline{W}G$ classify TCP's with fiber G.

The universal simplicial G-bundle is

$$G \hookrightarrow \overline{W}G \underset{\nu_G}{\times} G \cong * \longrightarrow \overline{W}G$$

where G acts on itself on the left by translation.

Remark: $\eta_K : K \longrightarrow \overline{W}\mathbb{G}K : x \longmapsto (\overline{d_0^{n-1} x}, \overline{d_0^{n-2} x}, \dots, \bar{x})$ is a weak equivalence $\forall K$.

- The bar and cyclic bar constructions

Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal category (e.g., $\mathcal{V} = \underline{Top}$, $\underline{sSet}$, $\underline{Ch}$, ...). Let (A, μ, η) be a monoid in $\mathcal{V}$, endowed an augmentation $\varepsilon: A \longrightarrow I$.

The bar construction on A, denoted $\mathcal{B}_\bullet A$, is the simplicial object in $\mathcal{V}$ given by

$$\mathcal{B}_n A = A^{\otimes n}$$

$$d_i = \begin{cases} \varepsilon \otimes Id^{\otimes n-1} & : i=0 \\ Id^{\otimes i-1} \otimes \mu \otimes Id^{n-i-1} & : 0 < i < n \\ Id^{\otimes n-1} \otimes \varepsilon & : i=n \end{cases}$$

$$s_i = Id^{\otimes i} \otimes \eta \otimes Id^{\otimes n-i} \quad \forall i.$$

The cyclic bar construction on A is the simplicial object in $\mathcal{V}$ given by

$$\mathcal{Z}_n A = A^{\otimes n+1}$$

$$d_i = \begin{cases} (Id^{\otimes n-2} \otimes \mu)\, t & : i=0 \\ Id^{\otimes i-1} \otimes \mu \otimes Id^{\otimes n-i} & : 0 < i \leq n \end{cases}$$

where $t = (1\ n+1\ n \cdots 2) : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$.

Remark: $\mathcal{Z}_\bullet A$ is actually a cyclic object in $\mathcal{V}$, i.e., given by a functor $\Lambda^{op} \longrightarrow \mathcal{V}$.

Remark: $\exists$ natural simplicial map $\pi_A : \mathcal{Z}_\bullet A \longrightarrow \mathcal{B}_\bullet A$ given by $Id^{\otimes n} \otimes \varepsilon : A^{\otimes n+1} \longrightarrow A^{\otimes n}$.

Remark: The cases that interest us today are

$$\mathcal{B}_\cdot, \mathcal{Z}_\cdot : \underline{sGr} \longrightarrow \underline{bisSet}$$

and where

$$\mathcal{B}_\cdot, \mathcal{Z}_\cdot : \underline{TopGr} \longrightarrow s\underline{Top} \qquad \pi^{-1}(*) \cong \mathcal{C}_\cdot G.$$

and $|\mathcal{B}_\cdot G| = BG$.

- Artin-Mazur totalization

$$\mathrm{Tot}: \underline{bisSet} \longrightarrow \underline{sSet}$$

$$\mathrm{Tot}(K_{\cdot\cdot})_n = \{(x_0,\dots,x_n) \in \prod_{i=0}^{n} K_{i,n-i} \mid d_0^v x_i = d_{i+1}^h x_{i+1} \ \forall 0 \le i \le n\}$$

$$d_i(x_0,\dots,x_n) = (d_i^v x_0, \dots, d_1^v x_{i-1}, d_i^h x_{i+1}, \dots, d_i^h x_n)$$

$$s_i(x_0,\dots,x_n) = (s_i^v x_0, \dots, s_0^v x_i, s_i^h x_i, \dots, s_i^h x_n).$$

Theorem: [Cegarra-Remedios, 2005] For all $K_{\cdot\cdot}$, $\exists$ natural homotopy equivalence of spaces

$$|\mathrm{diag}\, K_{\cdot\cdot}| \xrightarrow{\simeq} |\mathrm{Tot}\, K_{\cdot\cdot}|.$$

Remark: $|\mathrm{diag}\, K_{\cdot\cdot}| \cong ||K_\cdot|_\cdot|$.

Exercise: $\mathrm{Tot}(\mathcal{C}_\cdot K_\cdot) \cong K_\cdot$ and $\mathrm{Tot}(\mathcal{B}_\cdot G_\cdot) \cong \overline{W} G_\cdot \ \forall G_\cdot$

B. The Burghelea-Fiedorowicz-Goodwillie model

[B-F; Topology, 1986], [G; Topology, 1985]
[Loday: Ch 6 & 7]

Theorem: Geometric realization induces a functor

$$|-|: \underline{CycTop} \longrightarrow S^1\text{-}\underline{Top}.$$

Idea of the proof: $\Delta^{op} \xhookrightarrow{\iota} \Lambda^{op}$ induces an adjunction

$$\begin{tikzcd} sTop \arrow[r, shift left, "Lan_\iota"] \arrow[r, phantom, "\perp"] & CycTop \arrow[l, shift left, "\iota^*"] \end{tikzcd}$$

where $Lan_\iota(X)_n = C_n \times X_n$ but faces and degeneracies "twisted" by the group elements. Moreover, $\exists$ natural homeomorphism $|Lan_\iota(X)| \xrightarrow{\cong} S^1 \times |X|$.

$$\begin{array}{c} S^1 \\ \| \\ |C_\bullet| \end{array}$$

Consequently, if Y is a cyclic space, then $|\iota^* Y|$ is an S^1-space with action

$$S^1 \times |\iota^* Y| \xrightarrow{\cong} |Lan_\iota(\iota^* Y)| \xrightarrow{|\varepsilon_Y|} |\iota^* Y|. \quad //$$

(Henceforth, drop ι^* from the notation.)

Theorem: $\exists$ natural S^1-equivariant map

$$|\mathcal{Z}_\bullet G| \xrightarrow{\simeq} \mathcal{L}BG$$

that is a homotopy equivalence, for all topological groups G.

Idea of the proof:

Take the transpose of $S^1 \times |\mathcal{Z}_\bullet G| \longrightarrow |\mathcal{Z}_\bullet G| \xrightarrow{|\pi|} |B_\bullet G|$.

$\Rightarrow \quad |\mathcal{Z}_\bullet G| \xrightarrow{r} \mathcal{L}|B_\bullet G|$, which fits into a commuting diagram:

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\bar{r}"'] & {|\mathcal{Z}_\bullet G|} \arrow[r] \arrow[d, "r"'] & BG \arrow[d, equal] \\ \Omega BG \arrow[r] & \mathcal{L}BG \arrow[r] & BG \end{tikzcd}$$

Suffices then to show that $\bar{r}$ is a homotopy equivalence, which is "classical". //

Remark: [Bökstedt-Hsiang-Madsen; Inventiones, 1993]
[H-Rognes, arXiv]

$\exists\ \lambda^{(r)}_{simp} : \mathcal{Z}_\bullet G \to \mathcal{Z}_\bullet G : (a_1, \dots, a_n, b) \mapsto (a_1, \dots, a_n, b(ab)^{r-1})$
$a = a_1 \cdots a_n$

$$\begin{tikzcd} {|\mathcal{Z}_\bullet G|} \arrow[r, "r"] \arrow[d, "|\lambda^{(r)}_{simp}|"'] & \mathcal{L}BG \arrow[d, "\lambda^{(r)}"] \\ {|\mathcal{Z}_\bullet G|} \arrow[r, "r"] & \mathcal{L}BG \end{tikzcd}$$

commutes up to homotopy.

C. The "Hochschild" model

Let $G_\bullet$ be a simplicial group.

Def^n: The simplicial Hochschild construction on $G_\bullet$, denoted $\mathbb{H}G_\bullet$, is the simplicial set

$$\mathbb{H}G_\bullet = \overline{W}G \times_{\nu_G} Ad(G),$$

where $Ad(G)$ denotes G endowed with the conjugation G-action.

Theorem: $\mathrm{Tot}(\mathscr{C}_\bullet G_\bullet \to \mathfrak{Z}_\bullet G \to \mathscr{B}_\bullet G_\bullet)$

$$\cong\ G_\bullet \longrightarrow \mathbb{H}G_\bullet \longrightarrow \overline{W}G_\bullet$$

(Proof by computation.)

Corollary: $|G_\bullet \longrightarrow \mathbb{H}G_\bullet \longrightarrow \overline{W}G_\bullet| \simeq |G| \to \mathcal{L}B|G_\bullet| \to B|G_\bullet|$

Proof: $|\mathbb{H}G_\bullet| = |\mathrm{Tot}\,\mathfrak{Z}_\bullet G| \xleftarrow{\simeq} |\mathrm{diag}\,\mathfrak{Z}_\bullet G_\bullet| \cong |\mathfrak{Z}_\bullet|G_\bullet||$
$\simeq \mathcal{L}B|G_\bullet|$.

previous section

//

Remark: $\lambda^{(r)}_{\mathrm{Hoch}} : \mathbb{H}G \longrightarrow \mathbb{H}G : (a_0, \ldots, a_{n-1}, b) \mapsto (a_0, \ldots, a_{n-1}, b^r)$
is a model of the rth power map.

D. The "coHochschild" model

Let $K_\bullet$ be any reduced simplicial set.

Defn: The coHochschild construction on K, denoted $\hat{\mathbb{H}}K$, is the simplicial set

$$\hat{\mathbb{H}}K = K \times_{\tau_K} \mathrm{Ad}(\mathbb{G}K).$$

Proposition: $\exists$ commuting diagram of Kan fibrations

$$\begin{tikzcd}
\mathbb{G}K \arrow[r, hook] \arrow[d, equal] & \hat{\mathbb{H}}K \arrow[r] \arrow[d, "\simeq"] & K \arrow[d, "\simeq"] \\
\mathbb{G}K \arrow[r, hook] & \mathbb{H}\mathbb{G}K \arrow[r] & \overline{W}\mathbb{G}K.
\end{tikzcd}$$

← much smaller simplicial model!

Corollary: $|\mathbb{G}K \to \hat{\mathbb{H}}K \to K| \simeq \Omega|K| \to \mathcal{L}|K| \to |K|$.

Remark: $\lambda^{(r)}_{\text{cohoch}} : \hat{\mathbb{H}}K \to \hat{\mathbb{H}}K : (x,a) \mapsto (x, a^r)$
is a model of $\lambda^{(r)} : \mathcal{L}|K| \to \mathcal{L}|K|$.

E. The Jones model [Jones; Inventiones, 1987]

Let X be a topological space.

Def^n: The Jones cocyclic free loop model is the cosimplicial space $\mathcal{J}^\bullet(X)$, where

$$\mathcal{J}^n(X) = X^{n+1}$$

$$d^i(x_0, \dots, x_{n-1}) = \begin{cases} (x_0, \dots, x_i, x_i, \dots, x_{n-1}) : 0 \leq i \leq n-1 \\ (x_0, x_1, \dots, x_{n-1}, x_0) : i = n \end{cases}$$

$$s^i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_i, x_{i+2}, \dots, x_{n-1}).$$

Remark: $\mathcal{J}^\bullet(X)$ admits an obvious cyclic structure as well. Consequently, $\mathrm{Tot}\,\mathcal{J}^\bullet(X) = \mathrm{Map}_{\Delta}(\Delta^\bullet, \mathcal{J}^\bullet(X))$ is an S^1-space, by an argument dual to that for cyclic spaces.

Theorem: $\exists$ S^1-equivariant homeomorphism
$$\mathrm{Tot}\,\mathcal{J}^\bullet(X) \xrightarrow{\cong} \mathcal{L}X.$$

Proof sketch: Observe first that for every cyclic set $K_\bullet$, $\exists$ S^1-equivariant homeomorphism

$$\mathrm{Tot}(X^{K_\bullet}) \xrightarrow{\cong} \mathrm{Map}(|K|, X)$$

for any topological space X, where $(X^K)^\bullet$ is the cocyclic space given by $(X^K)^n = \mathrm{Map}(K_n, X)$.

Let $C_\bullet$ denote the cyclic set with $C_n = \mathbb{Z}/{n+1}\mathbb{Z}$. (*)
Then $X^{C_\bullet} = \mathcal{J}^\bullet(X)$ and $|C_\bullet| \cong S^1$, so we can conclude. //

(*) This is nothing but the usual simplicial model of S^1 with exactly two non-degenerate simplices, in levels 0 and 1.

Question: $\exists ?\ \mathfrak{X}^{(r)}_{\mathrm{Jones}} : \mathcal{J}^\bullet(X) \to \mathcal{J}^\bullet(X)$ model for $\mathfrak{X}^{(r)}_{\mathrm{top}}$?

Remark: Can generalize $\mathcal{J}^\bullet$ to

$$\mathcal{J}^\bullet : \mathrm{Comon} \longrightarrow \mathcal{V}^{\Delta}$$

for any symmetric monoidal category $\mathcal{V}$.

II. Chain complex models

A. Preliminaries

- Twisting cochains $\Bbbk$ - commutative ring, $\otimes = \otimes_{\Bbbk}$

Henceforth, simply "algebra" and "coalgebra".

Let A be a connected, augmented dg $\Bbbk$-algebra and C a 1-connected, coaugmented dg $\Bbbk$-coalgebra.

A twisting cochain from C to A is a $\Bbbk$-linear map $t: C_* \longrightarrow A_{*-1}$ of degree -1 such that

$$dt + td = \mu(t \otimes t)\Delta.$$

If M is a left A-module with action $\lambda: A \otimes M \to M$ and N is a right C-comodule with coaction $\rho: N \to N \otimes C$, then $\forall$ twisting cochain $t: C \to A$, we can construct the twisted tensor product of N and M over t:

$$N \otimes_t M = (N \otimes M, d_t)$$

$$d_t = d \otimes Id + Id \otimes d + (Id \otimes \lambda)(Id \otimes t \otimes Id)(\rho \otimes Id).$$

<u>Example</u>: $A \xhookrightarrow{Id \otimes \eta} A \otimes_t C \xrightarrow{\varepsilon \otimes Id} C$

- ## The bar/cobar adjunction

Theorem: $\exists$ adjunction

$$\Omega : \underline{\mathrm{Coalg}} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \underline{\mathrm{Alg}} : \mathcal{B}$$

where: • $\Omega C = (T(s^{-1}C_{>0}), d_\Omega)$

free associative algebra — built from diff'l and comult on C

$TV = \bigoplus_{n\geq 0} V^{\otimes n} \ni v_1|\cdots|v_n$

• $\mathcal{B}A = (T(sA_{>0}), d_{\mathcal{B}})$.

cofree coassoc. coalg — built from diff'l and mult on A.

Proposition: The $\Bbbk$-linear maps $t_\Omega : C \to \Omega C : c \mapsto s^{-1}c$ and
$t_{\mathcal{B}} : \mathcal{B}A \longrightarrow A : sa_1|\cdots|sa_n \mapsto \begin{cases} a_1 & : n=1 \\ 0 & : \text{else} \end{cases}$
are twisting cochains and mediate the cobar/bar-adjunction:

$$\begin{array}{ccccc} \alpha_t & \longleftarrow & t & \longmapsto & \beta_t \\ \underline{\mathrm{Alg}}(\Omega C, A) & \cong & Tw(C,A) & \cong & \mathrm{Coalg}(C, \mathcal{B}A) . \\ f & \longmapsto & f t_\Omega & & \\ & & t_{\mathcal{B}} g & \longleftarrow & g \end{array}$$

Remark: $C \xrightarrow{\simeq} \mathcal{B}\Omega C$ and $\Omega\mathcal{B}A \xrightarrow{\simeq} A$.

Remark: $\mathcal{B}A = |\mathcal{B}_\bullet A|$. (Similarly, $\Omega C \simeq \mathrm{Tot}\, \Omega^\bullet C$.)

Add remark (✩) HERE.

Topological significance

[Szczarba]

Rmk: G simpl gp $\Rightarrow C_*G$ dg Hopf algebra.

Rmk: $\exists$ also $C_*\overline{W}GX \to \mathcal{B}C_*GX$.

<u>Theorem</u>: [Adams, Baues, HPST] If K is a 1-connected simplicial set, then $\exists$ quasi-iso of dga's $\alpha_K : \Omega C_*K \xrightarrow{\simeq} C_*GK$, given by a tw. cochain $t_K : C_*K \to C_*GK$.

Moreover, ΩC_*K admits a natural Hopf algebra structure such that α_X is a sh coalgebra map, whence the power map on ΩC_*K is a model for the simpl power map.

(✩) <u>Remark</u>: (H,μ,Δ) Hopf algebra $\Rightarrow \exists$ power maps

$$\begin{tikzcd} H \arrow[r, "\Delta^{(r)}"] \arrow[rr, bend right, "\lambda^{(r)}"'] & H^{\otimes r} \arrow[r, "\mu^{(r)}"] & H \end{tikzcd}$$

Since $\Omega, \mathcal{B}$ induce
$$\Omega : \mathrm{CocommCoalg} \to \mathrm{Hopf},$$
$$\mathcal{B} : \mathrm{CommAlg} \to \mathrm{Hopf}$$

ΩC admits power maps if C cocommutative
and
$\mathcal{B}A$ admits power maps if A commutative.

B. The Hochschild complex

<u>Def^n</u>: The Hochschild complex is the functor
$$\mathcal{H} : \mathrm{Alg} \to \underline{\mathrm{Ch}}$$
defined on objects by $\mathcal{H}A = |\mathcal{Z}_\bullet A| \cong (A \otimes Ts\bar{A}, d_{\mathcal{H}})$

where $d_{\mathcal{H}}(b \otimes sa_1|\cdots|sa_n) = ba_1 \otimes sa_2|\cdots|sa_n$
$\pm \sum a_1 \otimes sa_1|\cdots|s(a_i a_{i+1})|\cdots|sa_n$
$\pm a_n b \otimes sa_1|\cdots|sa_{n-1}$.
+ linear-type terms.

Rmk: $\exists$ "fiber bundle" $A \hookrightarrow \mathcal{H}A \longrightarrow BA$.

Defn: The Hochschild homology of a dga A is

$$HH_*(A) := H_*(\mathcal{H}A)$$

and the Hochschild cohomology of A with coefficients in $A^\vee$ is

$$HH^*(A) := H^*(\mathrm{Hom}(\mathcal{H}A, \Bbbk)).$$

Topological significance

Theorem: [Goodwillie, Burghelea-Fiedorowicz, Jones]

X pointed, connected $\Rightarrow HH_*(S_*\Omega X) \cong H_*\mathcal{L}X$
X 1-connected $\Rightarrow HH_*(S^*X) \cong H^*\mathcal{L}X$.
(as graded $\Bbbk$-modules)

Sketch of proof: Eilenberg-Zilber equivalence

$$S_*Y \otimes S_*Z \xrightarrow{\simeq} S_*(Y \times Z)$$

$\Rightarrow$ natural chain equivalence

$$\mathcal{H}(S_*\Omega X) \xrightarrow{\simeq} S_*(|\mathfrak{Z}_\bullet \Omega X|)$$

In the cochain case, the E-Z map induces

(EMSS argument) $\quad \mathcal{H}(S^*X) \xrightarrow{\simeq} S^*(|\mathfrak{Z}^\bullet X|)$

which is a quasi-iso if X is 1-connected. //

Remark: $X = |K|$, where K is reduced
$\Rightarrow \exists\ \mathcal{H}(C_*\mathbb{G}K) \xleftarrow{\simeq} \cdot \xrightarrow{\simeq} C_*(\mathbb{H}K)$
by the proof above, since $\exists\ |\mathcal{J}^\bullet|\mathbb{G}K|| \xrightarrow{\simeq} |\mathbb{H}K|$.

The result above has been strengthened in various ways, so that more structure is taken into account.

Theorem: [Ndombol-Thomas, 2001 and 2002]
If $\Bbbk$ is a field, and X is 1-connected, then $\mathcal{H}(S^*X)$ admits an A_∞-algebra such that
$$HH_*(S^*X) \cong H^*(\mathcal{L}X;\Bbbk) \text{ as gr algebras.}$$

Theorem: [Menichi, 2001] If X is path connected, then
$$HH^*(S_*\Omega X) \cong H^*(\mathcal{L}X;\Bbbk)$$
as graded algebras, $\forall\ \Bbbk$.

Remark: It is also possible to include models of power maps into this picture, under various hypotheses on X.

C. The coHochschild complex [Doi], [Idrissi], [HPS]

Defn: The coHochschild complex is the functor
$$\widehat{\mathcal{H}} : \underline{\mathrm{Coalg}} \longrightarrow \underline{\mathrm{Ch}}$$
defined by $\widehat{\mathcal{H}}(C) = \mathrm{Tot}\ \mathcal{J}^\bullet(C) = (Ts^{-1}C_{>0} \otimes C, d_{\widehat{\mathcal{H}}})$

where: $d_{\hat{\mathcal{H}}}\left(s^{-1}c_1|\cdots|s^{-1}c_n \otimes \bar{c}\right)$

$$= \sum_i \pm s^{-1}c_1|\cdots|s^{-1}c_{i_j}|s^{-1}c_i{}^{j}|\cdots|s^{-1}c_n \otimes \bar{c}$$

$$\pm\ s^{-1}c_1|\cdots|s^{-1}c_n|s^{-1}\bar{c}_j \otimes \bar{c}^{j}$$

$$\pm\ s^{-1}\bar{c}^{j}|s^{-1}c_1|\cdots|s^{-1}c_n \otimes \bar{c}_j$$

$$+ \text{ linear terms}$$

<u>Remark</u>: $\exists$ "fiber bundle" $\Omega C \hookrightarrow \hat{\mathcal{H}}C \longrightarrow C$.

<u>Def^n</u>: The coHochschild homology of a dg coalgebra C is

$$\widehat{HH}_*(C) := H_*(\hat{\mathcal{H}}C).$$

Hochschild vs coHochschild

<u>Proposition</u>: For every twisting cochain $t: C \to A$, $\exists$ natural
[HPS] commuting diagram of chain maps.

$$\begin{tikzcd}
\Omega C \arrow[r, hook] \arrow[d, "\alpha_t"] & \hat{\mathcal{H}}C \arrow[r] \arrow[d, "\gamma_t"] & C \arrow[d, "\beta_t"] \\
A \arrow[r, hook] & \mathcal{H}A \arrow[r] & \mathcal{B}A
\end{tikzcd}$$

Moreover: α_t q-iso $\Leftrightarrow \beta_t$ q-iso $\Leftrightarrow \gamma_t$ q-iso.

Topological relevance

<u>Theorem</u>: If K is a 1-reduced simplicial set, then
[HPS] $\exists$ natural commuting diagram of chain maps

$$\begin{tikzcd}
\Omega C_* K \arrow[r, hook] \arrow[d, "\alpha_K", "\simeq"'] & \widehat{\mathcal{H}} C_* K \arrow[r] \arrow[d, "\theta_K", "\simeq"'] & C_* K \arrow[d, "="] \\
C_* G K \arrow[r] & C_* \widehat{H} K \arrow[r] & C_* K
\end{tikzcd}$$

where • α_K is an algebra map and an sh coalgebra map

and

• θ_K is an sh coalgebra map.

<u>Corollary</u>: Let X be a 1-connected space, and let K be a 1-reduced simplicial set such that $X \simeq |K|$.
Then

$$H^*(\mathcal{L}X; \Bbbk) \cong H^*(\widehat{\mathcal{H}} C_* K; \Bbbk)$$

as algebras, $\forall$ commutative rings $\Bbbk$.

<u>Remarks</u>: 1) $t_K : C_* K \to C_* G K \Rightarrow \widehat{\mathcal{H}}(C_* K) \xrightarrow{\simeq} \mathcal{H}(C_* G K)$

much smaller model with which to compute $H_* \mathcal{L}|K|$.

2) Have generalized this construction to model coincidence spaces and associated power maps
$\Rightarrow$ model for $TC(X; p)$, Hodge decomposition.

D. The cyclic complex [Loday]

Def^n: $\mathcal{C}: \underline{Alg}_R \longrightarrow \underline{Ch}_R$ - the cyclic complex functor.

Let $\mathbb{Z}[v]$ denote the polynomial algebra on a generator v of degree 2. Then:

$$\mathcal{C}(A,d) = (\mathbb{Z}[v] \otimes A \otimes T s\bar{A}, d_{\mathcal{C}}), \text{ where}$$

$$d_{\mathcal{C}}(v^k \otimes a \otimes sb_1|\cdots|sb_n) = v^k \otimes d_{\mathcal{H}}(a \otimes sb_1|\cdots|sb_n) + v^{k-1} \otimes N(sa|sb_1|\cdots|sb_n) \otimes 1$$

NB: $\exists\ (\underset{\overset{\shortparallel}{H_* BS^1}}{\mathbb{Z}[v]},0) \hookrightarrow \mathcal{C}(A,d)$

where $N: TV \longrightarrow TV : v_1|\cdots|v_n \longmapsto \sum_{1\leq j\leq n} \pm v_j|\cdots|v_n|v_1|\cdots|v_{j-1}$ is the norm operator.

- $HC_*(A,d) := H_*(\mathcal{C}(A,d))$ - the cyclic homology of (A,d)

- $\exists$ exact sequence of complexes

$$0 \longrightarrow \mathcal{H}A \overset{j}{\hookrightarrow} \mathcal{C}A \overset{q}{\longrightarrow} s^2\mathcal{C}A \longrightarrow 0$$

where

$$j(a \otimes sb_1|\cdots|sb_n) = 1 \otimes a \otimes sb_1|\cdots|sb_n$$

$$q(v^k \otimes a \otimes sb_1|\cdots|sb_n) = \begin{cases} s^2(v^{k-1} \otimes a \otimes sb_1|\cdots|sb_n) & : k \geq 1 \\ 0 & : k = 0 \end{cases}$$

The induced long exact sequence in homology is the Connes exact sequence relating Hochschild and cyclic homology

Topological relevance

Theorem: [Goodwillie, 1985], [Burghelea - Fiedorowicz, 1986], [Jones, 1987] (Coefficients in any comm ring.)

X pointed, connected $\Rightarrow$ $HC_*(S_*\Omega X) \cong H_*^{S^1}\mathcal{L}X$

X 1-connected $\Rightarrow$ $HC^-_{-*}(S^*X) \cong H^*_{S^1}\mathcal{L}X$

$\Big\}$ as H_*BS^1-modules

where $H_*^{S^1}\mathcal{L}X := H_*(ES^1 \times_{S^1} \mathcal{L}X)$.

Rmk: $\exists$ fibration $\mathcal{L}X \hookrightarrow ES^1 \times_{S^1} \mathcal{L}X \longrightarrow BS^1$

the homotopy orbit space $(\mathcal{L}X)_{hS^1}$

E. The cocyclic complex

Defn: $\hat{\mathscr{C}} : \mathrm{Coalg}_R \longrightarrow \mathrm{Ch}_R$ - the cocyclic complex functor

$\hat{\mathscr{C}}(C,d) = (\mathbb{Z}[v] \otimes \Omega C \otimes C, d_{\hat{\mathscr{C}}})$, where

$$d_{\hat{\mathscr{C}}}(v^k \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes 1) = v^k \otimes d_\Omega(s^{-1}c_1|\cdots|s^{-1}c_n) \otimes 1$$
$$+ \sum_{1\leq j\leq n} \pm v^{k-1} \otimes s^{-1}c_{j+1}|\cdots|s^{-1}c_{j-1} \otimes c_j$$

while

$$d_{\hat{\mathscr{C}}}(v^k \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes c) = v^k \otimes d_{\hat{\mathscr{H}}}(s^{-1}c_1|\cdots|s^{-1}c_n \otimes c).$$

- $\widehat{HC}_*(C,d) := H_*(\hat{\mathcal{C}}(C,d))$ - the cocyclic homology of (C,d).

- $\exists$ short exact sequence of chain complexes

$$0 \to \hat{\mathcal{H}}C \xrightarrow{j} \hat{\mathcal{C}}C \xrightarrow{q} s^2\hat{\mathcal{C}}C \to 0$$

where: $j(s^{-1}c_1|\cdots|s^{-1}c_n \otimes c) = 1 \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes c$

$$q(v^k \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes c) = \begin{cases} s^2(v^{k-1} \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes c) & : k \geq 1 \\ 0 & : k = 0 \end{cases}$$

The induced long exact sequence is analogous to the Connes long exact sequence and relates coHochschild and cocyclic homology

Relation with the cyclic complex

Theorem [HPS] For every dg algebra map $\varphi : \Omega C \to A$, $\exists$ chain map $\Phi : \hat{\mathcal{C}}C \to \mathcal{C}A$ such that

$$\begin{tikzcd} \hat{\mathcal{H}}C \arrow[r, hook] \arrow[d, "\hat{\varphi}"'] & \hat{\mathcal{C}}C \arrow[d, "\Phi"] \\ \mathcal{H}A \arrow[r, hook] & \mathcal{C}A \end{tikzcd}$$

commutes

Recall earlier theorem.

Moreover, φ quasi-isomorphism $\Rightarrow$ Φ quasi-iso.
Dual result for dg coalg map $C \to BA$ holds as well.

Topological relevance

As a corollary of the preceding theorem and of the topological relevance of $\mathcal{C}(S_*\Omega X)$, we have ...

Theorem: Given $C \in \underline{\text{Coalg}}$ and $g: \Omega C \longrightarrow S_* \Omega X$, there is an isomorphism

$$\widehat{HC}_*(C, d) \cong H_*^{S^1}(\mathcal{L}X) \text{ of graded ab gps.}$$

Example $X \simeq |K| \Rightarrow$ can take $C = C_* K$ and get relatively small, tractable model for computing $H_*^{S^1}(\mathcal{L}X)$.

Question: How to capture more algebraic structure?

Theorem: [H, 2012] Let Y be any (left) S^1-space. Then the tensor product $H_* BS^1 \otimes C\mathcal{U}^* Y$ admits a dg coalg structure such that there is a quasi-isomorphism

$$H_* BS^1 \tilde{\otimes} C\mathcal{U}_* Y \longrightarrow C\mathcal{U}_*(ES^1 \underset{S^1}{\times} Y)$$

preserving comultiplication up to (strong) homotopy.

Applying this theorem to $Y = \mathcal{L}X$ and using methods of acyclic models, can prove ...

Theorem: If $X \simeq |K|$, then $\hat{\mathcal{C}}(C_*K)$ admits a comult. such that $\hat{\mathcal{C}}(C_*K) \simeq C\mathcal{U}_*(ES^1 \times_{S^1} \mathcal{L}X)$ as dg coalgebras (up to strong homotopy).

In particular, $H^*(\mathrm{Hom}(\hat{\mathcal{C}}(C_*K), \Bbbk)) \cong H^*(ES^1 \times_{S^1} \mathcal{L}X, \Bbbk)$ as algebras, $\forall$ comm. ring $\Bbbk$.

F. Hochschild cohomology revisited

$$HH^*(A) = H^*(\mathrm{Hom}(\mathcal{H}A, \Bbbk)) \cong H^*(\mathrm{Hom}(T s\bar{A}, A^{\#}), \tilde{d}_{\mathcal{H}})$$

but

$HH^*(A,A) = H^*(\mathrm{Hom}(Ts\bar{A}, A), \tilde{d}_{\mathcal{H}})$, both special cases of

$HH^*(A,M) = H^*(\mathrm{Hom}(Ts\bar{A}, M), \tilde{d}_{\mathcal{H}})$, where

A bimodule

$$(\tilde{d}_{\mathcal{H}} f)(sa_1|\cdots|sa_n) = a_1 \cdot f(sa_2|\cdots|sa_n) \pm f(sa_1|\cdots|sa_{n-1}) \cdot a_n \pm f d_{\mathcal{B}}(sa_1|\cdots|sa_n)$$

Theorem: $HH^*(A,A)$ admits a natural Gerstenhaber algebra structure. [Gerstenhaber],

If A is a Frobenius algebra, then $HH^*(A,A)$ is even a Batalin-Vilkoviskiy algebra. [Menichi]