Magnetic wells in dimension three

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Setting of the problem

- The configuration space is $\mathbb{R}^3$ with coordinates (q_1, q_2, q_3).
- The magnetic vector potential $\mathbf{A} = (A_1, A_2, A_3) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.
- The magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1).$$

Problem

The semiclassical analysis of the discrete spectrum of the magnetic Laplacian

$$\mathcal{L}_{\hbar,\mathbf{A}} := (-i\hbar\nabla_q - \mathbf{A}(q))^2.$$

This means that we will consider that $\hbar$ belongs to $(0, \hbar_0)$ with $\hbar_0$ small enough.

Self-adjointness and lower bounds

Define

$$b(q) := \|\mathbf{B}(q)\|.$$

Assumption 1

There exists a constant $C > 0$ such that

$$\|\nabla \mathbf{B}(q)\| \leqslant C\,(1 + b(q))\,,\ \forall q \in \mathbb{R}^3\,.$$

Under Assumption 1, we have [HelfferMohamed96]:

- the operator $\mathcal{L}_{\hbar,\mathbf{A}}$ is essentially self-adjoint on $L^2(\mathbb{R}^3)$;
- there exist $h_0 > 0$ and $C_0 > 0$ such that, for all $\hbar \in (0, h_0)$,

$$\hbar(1 - C_0\hbar^{\frac{1}{4}}) \int_{\mathbb{R}^3} b(q)|u(q)|^2 \mathrm{d}q \leqslant \langle \mathcal{L}_{\hbar,\mathbf{A}} u \mid u \rangle\,,\ \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^3)\,.$$

Bounds for the spectrum

Denote $b_0 := \inf_{q\in\mathbb{R}^3} b(q)$.

Lower bound for the spectrum

The bottom of the spectrum $\mathfrak{s}(\mathcal{L}_{\hbar,\mathbf{A}})$ is asymptotically above $\hbar b_0$: There exist $h_0 > 0$ and $C_0 > 0$ such that, for all $\hbar \in (0, h_0)$,

$$\mathfrak{s}(\mathcal{L}_{\hbar,\mathbf{A}}) \subset [\hbar b_0(1 - C_0\hbar^{\frac{1}{4}}), +\infty),$$

Denote $b_1 := \liminf_{|q|\to+\infty} b(q)$.

Lower bound for the essential spectrum

The bottom of the essential spectrum $\mathfrak{s}_{\mathrm{ess}}(\mathcal{L}_{\hbar,\mathbf{A}})$ is asymptotically above $\hbar b_1$: There exist $h_1 > 0$ and $C_1 > 0$ such that, for all $\hbar \in (0, h_1)$,

$$\mathfrak{s}_{\mathrm{ess}}(\mathcal{L}_{\hbar,\mathbf{A}}) \subset [\hbar b_1(1 - C_1\hbar^{\frac{1}{4}}), +\infty).$$

Main assumptions

Assumption 2

- The magnetic field does not vanish: $b_0 (:= \inf_{q\in\mathbb{R}^3} b(q)) > 0$;
- The magnetic field is confining: $b_0 < b_1 (:= \liminf_{|q|\to+\infty} b(q))$;
- There exists a point $q_0 \in \mathbb{R}^3$ and $\varepsilon > 0$, $\tilde{\beta}_0 \in (b_0, b_1)$ such that

$$\{b(q) \leqslant \tilde{\beta}_0\} \subset D(q_0, \varepsilon),$$

$D(q_0, \varepsilon)$ is the Euclidean ball centered at the origin and of radius ε.

Note that the last assumption is satisfied as soon as b admits a unique and non degenerate minimum.

Overview

- At the classical level, the Hamiltonian dynamics for a non-uniform magnetic field splits into three scales:
  - the cyclotron motion around field lines,
  - the center-guide oscillation along the field lines,
  - the oscillation within the space of field lines.

  Under our assumptions, we exhibit three semiclassical scales and their corresponding effective quantum Hamiltonians, by means of three microlocal normal forms *à la Birkhoff*.
- As a consequence, when the magnetic field admits a unique and non degenerate minimum, we are able to reduce the spectral analysis of the low-lying eigenvalues to a one-dimensional $\hbar$-pseudo-differential operator whose Weyl's symbol admits an asymptotic expansion in powers of $\hbar^{\frac{1}{2}}$.

Notation

Assume that b admits a unique and non degenerate minimum at q_0:

$$b(q_0) = b_0 := \inf_{q \in \mathbb{R}^3} b(q) > 0, \quad \mathrm{Hess}_{q_0} b > 0\,.$$

Denote

$$\sigma = \frac{\mathrm{Hess}_{q_0} b\,(\mathbf{B}, \mathbf{B})}{2 b_0^2}, \quad \theta = \sqrt{\frac{\det \mathrm{Hess}_{q_0} b}{\mathrm{Hess}_{q_0} b\,(\mathbf{B}, \mathbf{B})}}$$

Asymptotic description of the spectrum

Main Theorem 1

For all $c \in (0,3)$, the spectrum of $\mathcal{L}_{\hbar,\mathbf{A}}$ below $b_0\hbar + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}$ coincides modulo $\mathcal{O}(\hbar^\infty)$ with the spectrum of the operator $\mathcal{F}_\hbar$ acting on $L^2(\mathbb{R}_{x_2})$:

$$\mathcal{F}_\hbar = b_0\hbar + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} - \frac{\zeta}{2\theta}\hbar^2 + \hbar\left(\frac{\theta}{2}\mathcal{K}_\hbar + k^\star(\hbar^{\frac{1}{2}}, \mathcal{K}_\hbar)\right),$$

- $\mathcal{K}_\hbar = \hbar^2 D_{x_2}^2 + x_2^2$, ζ is some explicit constant,
- $k^\star \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $k^\star(\hbar^{\frac{1}{2}}, Z) = \mathcal{O}((\hbar + |Z|)^{\frac{3}{2}})$.

Remark

This description is reminiscent of the results *à la* Bohr-Sommerfeld of [Helffer-Robert84, HelfferSjostrand89] obtained in the case of one dimensional semiclassical operators.

Eigenvalue asymptotics

Main Theorem 2

Let $(\lambda_m(\hbar))_{m\geqslant 1}$ be the non decreasing sequence of the eigenvalues of $\mathcal{L}_{h,\mathbf{A}}$. For any $c \in (0,3)$, let

$$\mathsf{N}_{\hbar,c} := \{m \in \mathbb{N}^*; \quad \lambda_m(\hbar) \leqslant \hbar b_0 + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}\}.$$

Then:

the cardinal of $\mathsf{N}_{\hbar,c}$ is of order $\mathcal{O}(\hbar^{-\frac{1}{2}})$,

Eigenvalue asymptotics

and there exist $\upsilon_1, \upsilon_2 \in \mathbb{R}$ and $\hbar_0 > 0$ such that

$$\lambda_m(\hbar) = \hbar b_0 + \sigma^{\frac{1}{2}} \hbar^{\frac{3}{2}} + \left[\theta(m - \tfrac{1}{2}) - \frac{\zeta}{2\theta}\right]\hbar^2 + \upsilon_1(m - \tfrac{1}{2})\hbar^{\frac{5}{2}} + \upsilon_2(m - \tfrac{1}{2})^2\hbar^3 + \mathcal{O}(\hbar^{\frac{5}{2}}),$$

uniformly for $\hbar \in (0, \hbar_0)$ and $m \in \mathrm{N}_{\hbar,c}$.
In particular, the splitting between two consecutive eigenvalues satisfies

$$\lambda_{m+1}(\hbar) - \lambda_m(\hbar) = \theta\hbar^2 + \mathcal{O}(\hbar^{\frac{5}{2}}).$$

Remark

An upper bound of $\lambda_m(\hbar)$ for fixed $\hbar$-independent m with remainder in $\mathcal{O}(\hbar^{\frac{9}{4}})$ was obtained in [HelfferKordyukov13] through a quasimodes construction involving powers of $\hbar^{\frac{1}{4}}$.

- The phase space is

$$\mathbb{R}^6 = \{(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3\}$$

and we endow it with the canonical 2-form

$$\omega_0 = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 + \mathrm{d}p_3 \wedge \mathrm{d}q_3.$$

- The classical magnetic Hamiltonian, defined for $(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$H(q,p) = \|p - \mathbf{A}(q)\|^2.$$

- An important role will be played by the characteristic hypersurface

$$\Sigma = H^{-1}(0),$$

which is the submanifold defined by the parametrization:

$$\mathbb{R}^3 \ni q \mapsto j(q) := (q, \mathbf{A}(q)) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

- The vector potential $\mathbf{A} = (A_1, A_2, A_3) \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is associated (via the Euclidean structure) with the following 1-form

$$\alpha = A_1 \mathrm{d}q_1 + A_2 \mathrm{d}q_2 + A_3 \mathrm{d}q_3.$$

- Its exterior derivative $\mathrm{d}\alpha$ is a 2-form, called magnetic 2-form and expressed as

$$\begin{aligned}\mathrm{d}\alpha = (\partial_1 A_2 - \partial_2 A_1)\mathrm{d}q_1 \wedge \mathrm{d}q_2 \\ + (\partial_1 A_3 - \partial_3 A_1)\mathrm{d}q_1 \wedge \mathrm{d}q_3 + (\partial_2 A_3 - \partial_3 A_2)\mathrm{d}q_2 \wedge \mathrm{d}q_3 \,.\end{aligned}$$

It is identified with the magnetic vector field

$$\mathbf{B} = \nabla \times \mathbf{A} = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1).$$

We may notice the relation between

- the characteristic hypersurface Σ,
- the symplectic structure ω_0
- the magnetic 2-form $\mathrm{d}\alpha$:

$$j^*\omega_0 = \mathrm{d}\alpha\,,$$

where

$$j : \mathbb{R}^3 \ni q \mapsto (q, \mathbf{A}(q)) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

If $b_0 > 0$, then the restriction $j^*\omega_0$ of the canonical symplectic form ω_0 to Σ is

- in $2D$-case, non-degenerate (i.e. Σ is a symplectic submanifold);
- in $3D$-case, degenerate.

Localization

- For eigenvalues of $\mathcal{L}_{\hbar,\mathbf{A}}$ of order $\mathcal{O}(\hbar)$, the corresponding eigenfunctions are microlocalized in the semi-classical sense near the characteristic hypersurface Σ.
- We will be reduced to investigate the magnetic geometry locally in space near a point $q_0 \in \mathbb{R}^3$ belonging to the confinement region.
- We put

$$q_0 = 0.$$

Local coordinates

Claim

In a neighborhood of $(q_0, \mathbf{A}(q_0)) \in \Sigma$, there exist symplectic coordinates $(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3)$ such that, $\Sigma = \{x_1 = \xi_1 = \xi_3 = 0\}$.
Hence Σ is parametrized by (x_2, ξ_2, x_3).
Near Σ, in these new coordinates, the Hamiltonian H admits the expansion

$$\hat{H} = H^0 + \mathcal{O}(|x_1|^3 + |\xi_1|^3 + |\xi_3|^3),$$

where $\hat{H}$ denotes H in the coordinates $(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$, and with

$$H^0 = \xi_3^2 + b(x_2, \xi_2, x_3)(x_1^2 + \xi_1^2).$$

Remark

The direction of **B** considered as a vector field on Σ is $\frac{\partial}{\partial x_3}$:

$$\mathbf{B}(x_2, \xi_2, x_3) = b(x_2, \xi_2, x_3)\frac{\partial}{\partial x_3}.$$

About the proof

We first study the linearization near Σ, describing the transverse Hessian of the Hamiltonian H at Σ, and then apply the Weinstein symplectic neighborhood theorem.

The first Birkhoff normal form

Near Σ, in these new coordinates, the Hamiltonian H admits the expansion

$$\hat{H} = \xi_3^2 + b(x_2, \xi_2, x_3)(x_1^2 + \xi_1^2) + \mathcal{O}(|x_1|^3 + |\xi_1|^3 + |\xi_3|^3),$$

Theorem 1

If $\mathbf{B}(0) \neq 0$, there exists a neighborhood of $(0, \mathbf{A}(0))$ endowed with symplectic coordinates $(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3)$ in which $\Sigma = \{x_1 = \xi_1 = \xi_3 = 0\}$ and $(0, \mathbf{A}(0))$ has coordinates $0 \in \mathbb{R}^6$, and there exist an associated unitary Fourier integral operator $U_\hbar$ such that

$$U_\hbar^* \mathcal{L}_{\hbar,\mathbf{A}} U_\hbar = \mathcal{N}_\hbar + \mathcal{R}_\hbar,$$

where

$$\mathcal{N}_\hbar = \hbar^2 D_{x_3}^2 + \mathcal{I}_\hbar \operatorname{Op}_\hbar^w b(x_2, \xi_2, x_3) + \operatorname{Op}_\hbar^w f^\star(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3),$$

The first Birkhoff normal form

$$\mathcal{N}_\hbar = \hbar^2 D_{x_3}^2 + \mathcal{I}_\hbar \operatorname{Op}_\hbar^w b(x_2,\xi_2,x_3) + \operatorname{Op}_\hbar^w f^\star(\hbar,\mathcal{I}_\hbar,x_2,\xi_2,x_3,\xi_3),$$

(a) $\mathcal{I}_\hbar = \hbar^2 D_{x_1}^2 + x_1^2$,

(b) $f^\star(\hbar,Z,x_2,\xi_2,x_3,\xi_3)$ a smooth function, with compact support as small as we want with respect to Z and ξ_3 whose Taylor series with respect to $Z,\xi_3,\hbar$ is

$$\sum_{k\geqslant 3}\sum_{2\ell+2m+\beta=k} \hbar^\ell c^\star_{\ell,m,\beta}(x_2,\xi_2,x_3) Z^m \xi_3^\beta$$

and the operator $\operatorname{Op}_\hbar^w f^\star(\hbar,\mathcal{I}_\hbar,x_2,\xi_2,x_3,\xi_3)$ has to be understood as the Weyl quantization of an operator valued symbol,

(c) the remainder $\mathcal{R}_\hbar$ is a pseudo-differential operator such that, in a neighborhood of the origin, the Taylor series of its symbol with respect to $(x_1,\xi_1,\xi_3,\hbar)$ is 0.

Comparison of the spectra

$$\mathcal{N}_\hbar^\sharp = \mathrm{Op}_\hbar^w \left(N_\hbar^\sharp \right),$$

with

$$N_\hbar^\sharp = \xi_3^2 + \mathcal{I}_\hbar \underline{b}(x_2, \xi_2, x_3) + f^{\star,\sharp}(\hbar, \mathcal{I}_\hbar, x_2, \xi_2, x_3, \xi_3)$$

where

- $\underline{b}$ is a smooth extension of b away from $D(0, \varepsilon)$ such that

$$\{\underline{b}(q) \leqslant \tilde{\beta}_0\} \subset D(0, \varepsilon),$$

- $f^{\star,\sharp} = \chi(x_2, \xi_2, x_3) f^\star$, with χ is a smooth cutoff function being 1 in a neighborhood of $D(0, \varepsilon)$.

Corollary

Let $\beta_0 \in (b_0, \tilde{\beta}_0)$. If ε and the support of $f^\star$ are small enough, then the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar^\sharp$ below $\beta_0 \hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Spectral reduction

In order to investigate the spectrum of $\mathcal{L}_{\hbar,\mathbf{A}}$ near the low lying eigenvalues, we replace $\mathcal{I}_\hbar$ by $\hbar$:

$$\mathcal{N}_\hbar^{[1]} = \hbar^2 D_{x_3}^2 + \hbar \,\mathrm{Op}_\hbar^w\, b + \mathrm{Op}_\hbar^w\, f^\star(\hbar,\hbar,x_2,\xi_2,x_3,\xi_3),$$

and

$$\mathcal{N}_\hbar^{[1],\sharp} = \mathrm{Op}_\hbar^w\left(N_\hbar^{[1],\sharp}\right),$$

where $N_\hbar^{[1],\sharp} = \xi_3^2 + \hbar \underline{b}(x_2,\xi_2,x_3) + f^{\star,\sharp}(\hbar,\hbar,x_2,\xi_2,x_3,\xi_3)$.

Corollary

If ε and the support of $f^\star$ are small enough, then for all $c \in (0, \min(3b_0,\beta_0))$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{N}_\hbar^{[1],\sharp}$ below $c\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Preliminaries

Let us now state our results concerning the normal form of $\mathcal{N}_\hbar^{[1]}$ (or $\mathcal{N}_\hbar^{[1],\sharp}$) under the following assumption.

Notation

- $f = f(\mathbf{z})$ is a differentiable function $\Rightarrow T_{\mathbf{z}} f(\cdot)$ its tangent map at the point $\mathbf{z}$.
- f is twice differentiable $\Rightarrow T^2_{\mathbf{z}} f(\cdot,\cdot)$ the second derivative of f.

Assumption 3

$$T_0^2 b(\mathbf{B}(0), \mathbf{B}(0)) > 0.$$

If the function b admits a unique and positive minimum at 0 and that it is non degenerate, then Assumption 3 is satisfied.

Preliminaries

Under Assumption 3, we have $\partial_3 b(0,0,0) = 0$ and, in the coordinates (x_2, ξ_2, x_3) given in Theorem 1,

$$\partial_3^2 b(0,0,0) > 0\,.$$

By the implicit function theorem that, for small x_2, there exists a smooth function $(x_2, \xi_2) \mapsto s(x_2, \xi_2)$, $s(0,0) = 0$, such that

$$\partial_3 b(x_2, \xi_2, s(x_2, \xi_2)) = 0\,.$$

The point $s(x_2, \xi_2)$ is the unique (in a neighborhood of $(0,0,0)$) minimum of $x_3 \mapsto b(x_2, \xi_2, x_3)$. We define

$$\nu(x_2, \xi_2) := \left(\tfrac{1}{2}\partial_3^2 b(x_2, \xi_2, s(x_2, \xi_2))\right)^{1/4}.$$

Taylor expansion at $s(x_2, \xi_2)$

$$\mathcal{N}_\hbar^{[1]} = \hbar^2 D_{x_3}^2 + \hbar \operatorname{Op}_\hbar^w b + \operatorname{Op}_\hbar^w f^\star(\hbar, \hbar, x_2, \xi_2, x_3, \xi_3),$$

Theorem 2

Under Assumption 3, there exists a neighborhood $\mathcal{V}_0$ of 0 and a Fourier integral operator $V_\hbar$ which is microlocally unitary near $\mathcal{V}_0$ and such that

$$V_\hbar^* \mathcal{N}_\hbar^{[1]} V_\hbar =: \underline{\mathcal{N}}_\hbar^{[1]} = \operatorname{Op}_\hbar^w \left(\underline{N}_\hbar^{[1]} \right),$$

$$\underline{N}_\hbar^{[1]} = \nu^2(x_2, \xi_2) \left(\xi_3^2 + \hbar x_3^2 \right) + \hbar b(x_2, \xi_2, s(x_2, \xi_2)) + \underline{r}_\hbar$$

and $\underline{r}_\hbar$ is a semiclassical symbol such that

$$\underline{r}_\hbar = \mathcal{O}(\hbar x_3^3) + \mathcal{O}(\hbar \xi_3^2) + \mathcal{O}(\xi_3^3) + \mathcal{O}(\hbar^2).$$

Taylor expansion and comparison of the spectra

We have

$$V_\hbar^* \mathcal{N}_\hbar^{[1]} V_\hbar =: \underline{\mathcal{N}}_\hbar^{[1]} = \mathrm{Op}_\hbar^w \left(\underline{N}_\hbar^{[1]} \right),$$

$$\underline{N}_\hbar^{[1]} = \nu^2(x_2, \xi_2) \left(\xi_3^2 + \hbar x_3^2 \right) + \hbar b(x_2, \xi_2, s(x_2, \xi_2)) + \underline{r}_\hbar$$

We introduce

$$\underline{\mathcal{N}}_\hbar^{[1],\sharp} = \mathrm{Op}_\hbar^w \left(\underline{N}_\hbar^{[1],\sharp} \right),$$

$$\underline{N}_\hbar^{[1],\sharp} = \underline{\nu}^2(x_2, \xi_2) \left(\xi_3^2 + \hbar x_3^2 \right) + \hbar \underline{b}(x_2, \xi_2, s(x_2, \xi_2)) + \underline{r}_\hbar^\sharp,$$

where:

$\underline{r}_\hbar^\sharp = \chi(x_2, \xi_2, x_3, \xi_3) \underline{r}_\hbar$ with χ a cut-off function equal to 1 on $D(0, \varepsilon)$ with support in $D(0, 2\varepsilon)$,

$\underline{\nu}$ a smooth and constant (with a positive constant) extension of ν.

Taylor expansion and comparison of the spectra

Corollary

There exists a constant $\tilde{c} > 0$ such that, for any cut-off function χ equal to 1 on $D(0, \varepsilon)$ with support in $D(0, 2\varepsilon)$, we have:

(a) The spectra of $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ and $\mathcal{N}_\hbar^{[1],\sharp}$ below $(b_0 + \tilde{c}\varepsilon^2)\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

(b) For all $c \in (0, \min(3b_0, b_0 + \tilde{c}\varepsilon^2))$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ below $c\hbar$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Problem:

$$\underline{\mathcal{N}}_\hbar^{[1],\sharp} = \mathrm{Op}_\hbar^w \left(\underline{N}_\hbar^{[1],\sharp} \right),$$

$$\underline{N}_\hbar^{[1],\sharp} = \underline{\nu}^2(x_2, \xi_2) \left(\xi_3^2 + \hbar x_3^2 \right) + \hbar \underline{b}(x_2, \xi_2, s(x_2, \xi_2)) + \underline{r}_\hbar^\sharp,$$

is not an elliptic $\hbar$-pseudo-differential operator.

Change of semiclassical parameter

We let $h = \hbar^{\frac{1}{2}}$ and, if $A_\hbar$ is a semiclassical symbol on $T^*\mathbb{R}^2$, admitting a semiclassical expansion in $\hbar^{\frac{1}{2}}$, we write

$$\mathcal{A}_\hbar := \mathrm{Op}_\hbar^w A_\hbar = \mathrm{Op}_h^w \mathsf{A}_h =: \mathfrak{A}_h,$$

with

$$\mathsf{A}_h(x_2, \tilde{\xi}_2, x_3, \tilde{\xi}_3) = A_{h^2}(x_2, h\tilde{\xi}_2, x_3, h\tilde{\xi}_3).$$

Thus, $\mathcal{A}_\hbar$ and $\mathfrak{A}_h$ represent the same operator when $h = \hbar^{\frac{1}{2}}$, but the former is viewed as an $\hbar$-quantization of the symbol $A_\hbar$, while the latter is an h-pseudo-differential operator with symbol A_h. Notice that, if $A_\hbar$ belongs to some class $S(m)$, then $\mathsf{A}_h \in S(m)$ as well. This is of course not true the other way around.

The second Birkhoff normal form

Define an operator $\underline{\mathfrak{N}}_h^{[1],\sharp}$ to be the operator $\underline{\mathcal{N}}_\hbar^{[1],\sharp}$ (but written in the h-quantization):

$$\underline{\mathfrak{N}}_h^{[1],\sharp} = \mathrm{Op}_h^w\left(\underline{N}_h^{[1],\sharp}\right),$$

where

$$\underline{N}_h^{[1],\sharp} = h^2\underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2\mathcal{J}_h\underline{\nu}^2(x_2, h\tilde{\xi}_2) + \underline{r}_h^\sharp,$$

$$\mathcal{J}_h := \mathrm{Op}_h^w\left(\tilde{\xi}_3^2 + x_3^2\right)$$

Theorem

Under Assumption 3, there exists a unitary operator W_h such that

$$W_h^*\underline{\mathfrak{N}}_h^{[1],\sharp}W_h =: \mathfrak{M}_h = \mathrm{Op}_h^w\left(\mathsf{M}_h\right),$$

The second Birkhoff normal form

$$\underline{N}_h^{[1],\sharp} = h^2 \underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2 \mathcal{J}_h \underline{\nu}^2 (x_2, h\tilde{\xi}_2) + \underline{r}_h^\sharp,$$

$$\mathrm{M}_h = h^2 \underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2 \mathcal{J}_h \underline{\nu}^2 (x_2, h\tilde{\xi}_2) + h^2 g^\star(h, \mathcal{J}_h, x_2, h\tilde{\xi}_2) + h^2 \mathrm{R}_h + h^\infty S(1).$$

(a) $g^\star(h, Z, x_2, \xi_2)$ a smooth function, with compact support as small as we want with respect to Z and with compact support in (x_2, ξ_2), whose Taylor series with respect to Z, h is

$$\sum_{2m+2\ell \geqslant 3} c_{m,\ell}(x_2, \xi_2) Z^m h^\ell,$$

(b) the function R_h satisfies $\mathrm{R}_h(x_2, h\tilde{\xi}_2, x_3, \tilde{\xi}_3) = \mathcal{O}((x_3, \tilde{\xi}_3)^\infty)$.

Comparison of the spectra

Since W_h is exactly unitary, we get a direct comparison of the spectra.

$$\underline{\mathfrak{N}}_h^{[1],\sharp} = \mathrm{Op}_h^w\left(\underline{N}_h^{[1],\sharp}\right),$$

$$\underline{N}_h^{[1],\sharp} = h^2\underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2\mathcal{J}_h\underline{\nu}^2(x_2, h\tilde{\xi}_2) + \underline{r}_h^\sharp,$$

We introduce

$$\mathfrak{M}_h^\sharp = \mathrm{Op}_h^w\left(\mathsf{M}_h^\sharp\right),$$

$$\mathsf{M}_h^\sharp = h^2\underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^2\mathcal{J}_h\underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, \mathcal{J}_h, x_2, h\tilde{\xi}_2).$$

Corollary

If ε and the support of $g^\star$ are small enough, for all $\eta > 0$, the spectra of $\underline{\mathfrak{N}}_h^{[1],\sharp}$ and $\mathfrak{M}_h^\sharp$ below $b_0 h^2 + \mathcal{O}(h^{2+\eta})$ coincide modulo $\mathcal{O}(h^\infty)$.

Spectral reduction

We replace $\mathcal{J}_h$ by h:

$$\mathfrak{M}_h^{[1],\sharp} = \mathrm{Op}_h^w \left(\mathsf{M}_h^{[1],\sharp} \right),$$

with

$$\mathsf{M}_h^{[1],\sharp} = h^2 \underline{b}(x_2, h\tilde{\xi}_2, s(x_2, h\tilde{\xi}_2)) + h^3 \underline{\nu}^2(x_2, h\tilde{\xi}_2) + h^2 g^\star(h, h, x_2, h\tilde{\xi}_2).$$

Corollary

If ε and the support of $g^\star$ are small enough, we have

(a) For $c \in (0,3)$, the spectra of $\mathfrak{M}_h^\sharp$ and $\mathfrak{M}_h^{[1],\sharp}$ below $b_0 h^2 + c\sigma^{\frac{1}{2}} h^3$ coincide modulo $\mathcal{O}(h^\infty)$ (here $\sigma = \nu^4(0,0)$)).

(b) If $c \in (0,3)$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{M}_\hbar^{[1],\sharp} = \mathfrak{M}_h^{[1],\sharp}$ below $b_0\hbar + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

Preliminaries

Go back to $\hbar$: $\mathfrak{M}_h^{[1],\sharp}$ is written as

$$\mathcal{M}_\hbar^{[1],\sharp} = \mathrm{Op}_\hbar^w \left(\mathsf{M}_\hbar^{[1],\sharp} \right),$$

with

$$\mathsf{M}_\hbar^{[1],\sharp} = \hbar \underline{b}(x_2, \xi_2, s(x_2, \xi_2)) + \hbar^{\frac{3}{2}} \underline{\nu}^2(x_2, \xi_2) + \hbar g^\star(\hbar^{\frac{1}{2}}, \hbar^{\frac{1}{2}}, x_2, \xi_2).$$

We perform a last Birkhoff normal form for the operator $\mathcal{M}_\hbar^{[1],\sharp}$ as soon as $(x_2, \xi_2) \mapsto \underline{b}(x_2, \xi_2, s(x_2, \xi_2))$ admits a unique and non degenerate minimum at $(0,0) \Rightarrow$

Assumption 4

The function b admits a unique and positive minimum at 0 and it is non degenerate.

Taylor expansion

$\mathcal{M}_{\hbar}^{[1],\sharp} = \mathrm{Op}_{\hbar}^{w}\left(\mathsf{M}_{\hbar}^{[1],\sharp}\right)$, with

$$\mathsf{M}_{\hbar}^{[1],\sharp} = \hbar \underline{b}(x_2,\xi_2,s(x_2,\xi_2)) + \hbar^{\frac{3}{2}}\underline{\nu}^2(x_2,\xi_2) + \hbar g^{\star}(\hbar^{\frac{1}{2}},\hbar^{\frac{1}{2}},x_2,\xi_2).$$

By using a Taylor expansion, we get,

$$M_{\hbar}^{[1],\sharp} = \hbar b_0 + \frac{\hbar}{2}\mathrm{Hess}_{(0,0)}\underline{b}(x_2,\xi_2,s(x_2,\xi_2)) + \hbar^{\frac{3}{2}}\nu^2(0,0) + c x_2 \hbar^{\frac{3}{2}} + d\xi_2\hbar^{\frac{3}{2}} + \hbar\mathcal{O}((\hbar^{\frac{1}{2}},z_2)^3),$$

where $c = \partial_{x_2}\nu^2(0,0)$ and $d = \partial_{\xi_2}\nu^2(0,0)$.

Diagonalization of the Hessian

There exists a linear symplectic change of variables that diagonalizes the Hessian, so that, if $L_\hbar$ is the associated unitary transform,

$$L_\hbar^* \mathcal{M}_\hbar^{[1],\sharp} L_\hbar = \mathrm{Op}_\hbar^w \left(\hat{M}_\hbar^{[1],\sharp} \right),$$

with

$$\hat{M}_\hbar^{[1],\sharp} = \hbar b_0 + \frac{\hbar}{2}\theta(x_2^2+\xi_2^2) + \hbar^{\frac{3}{2}}\nu^2(0,0) + \hat{c}x_2\hbar^{\frac{3}{2}} + \hat{d}\xi_2\hbar^{\frac{3}{2}} + \hbar\mathcal{O}((\hbar^{\frac{1}{2}}, z_2)^3),$$

where

$$\theta = \sqrt{\det \mathrm{Hess}_{(0,0)} b(x_2,\xi_2,s(x_2,\xi_2))} == \sqrt{\frac{\det \mathrm{Hess}_{q_0} b}{\mathrm{Hess}_{q_0} b\,(\mathbf{B},\mathbf{B})}}\,.$$

Algebraic transformations

$$\theta(x_2^2+\xi_2^2)+\hat{c}x_2\hbar^{\frac12}+\hat{d}\xi_2\hbar^{\frac12}$$
$$=\theta\left(\left(x_2-\frac{\hat{c}\hbar^{\frac12}}{\theta}\right)^2+\left(\xi_2-\frac{\hat{d}\hbar^{\frac12}}{\theta}\right)^2\right)-\hbar\frac{\hat{c}^2+\hat{d}^2}{\theta}.$$

$$\hat{c}^2+\hat{d}^2=\|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2.$$

There exists a unitary transform $\hat{U}_{\hbar^{\frac12}}$, which is in fact an $\hbar$-FIO whose phase admits a Taylor expansion in powers of $\hbar^{\frac12}$:

$$\hat{U}^*_{\hbar^{\frac12}}L^*_\hbar\mathcal{M}_\hbar^{[1],\sharp}L_\hbar\hat{U}_{\hbar^{\frac12}}=:\underline{\mathcal{F}}_\hbar=\mathrm{Op}_\hbar^w\left(\underline{F}_\hbar\right),$$

where

$$\underline{F}_\hbar=\hbar b_0+\hbar^{\frac32}\nu^2(0,0)-\frac{\|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2}{2\theta}\hbar^2+\hbar\left(\frac{\theta}{2}|z_2|^2+\mathcal{O}((\hbar^{\frac12},z_2)^3)\right).$$

Birkhoff normal form

We have $\underline{\mathcal{F}}_\hbar = \mathrm{Op}^w_\hbar(\underline{F}_\hbar)$

$$\underline{F}_\hbar = \hbar b_0 + \hbar^{\frac{3}{2}}\nu^2(0,0) - \frac{\|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2}{2\theta}\hbar^2 + \hbar\left(\frac{\theta}{2}|z_2|^2 + \mathcal{O}((\hbar^{\frac{1}{2}}, z_2)^3)\right).$$

A Birkhoff normal form $\Rightarrow \mathcal{F}_\hbar$ acting on $L^2(\mathbb{R}_{x_2})$ given by

$$\mathcal{F}_\hbar = b_0\hbar + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} - \frac{\zeta}{2\theta}\hbar^2 + \hbar\left(\frac{\theta}{2}\mathcal{K}_\hbar + k^\star(\hbar^{\frac{1}{2}}, \mathcal{K}_\hbar)\right), \ \mathcal{K}_\hbar = \hbar^2 D_{x_2}^2 + x_2^2;$$

Here $k^\star \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ a compactly supported function with $k^\star(\hbar^{\frac{1}{2}}, Z) = \mathcal{O}((\hbar + |Z|)^{\frac{3}{2}})$,

$$\sigma = \nu^4(0,0) = \tfrac{1}{2}\partial_3^2 b(0,0,0) = \frac{\mathrm{Hess}_{q_0} b\,(\mathbf{B},\mathbf{B})}{2b_0^2},$$

and

$$\zeta = \|(\nabla_{x_2,\xi_2}\nu^2)(0,0)\|^2.$$

The third Birkhoff normal form

Theorem

Under Assumption 4, there exists a unitary $\hbar$-Fourier Integral Operator $Q_{\hbar^{\frac{1}{2}}}$ whose phase admits an expansion in powers of $\hbar^{\frac{1}{2}}$ such that

$$Q^*_{\hbar^{\frac{1}{2}}} \mathcal{M}_{\hbar}^{[1],\sharp} Q_{\hbar^{\frac{1}{2}}} = \mathcal{F}_{\hbar} + \mathcal{G}_{\hbar},$$

where

(a) *$\mathcal{F}_{\hbar}$ acting on $L^2(\mathbb{R}_{x_2})$ given by*

$$\mathcal{F}_{\hbar} = b_0\hbar + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} - \frac{\zeta}{2\theta}\hbar^2 + \hbar\left(\frac{\theta}{2}\mathcal{K}_{\hbar} + k^{\star}(\hbar^{\frac{1}{2}}, \mathcal{K}_{\hbar})\right), \ \mathcal{K}_{\hbar} = \hbar^2 D_{x_2}^2 + x_2^2;$$

(b) *the remainder is in the form $\mathcal{G}_{\hbar} = \mathrm{Op}_{\hbar}^{w}(G_{\hbar})$, with $G_{\hbar} = \hbar\mathcal{O}(|z_2|^{\infty})$.*

Comparison of the spectra

Corollary

If ε and the support of $k^\star$ are small enough, we have

(a) *For all $\eta \in \left(0, \frac{1}{2}\right)$, the spectra of $\mathcal{M}_\hbar^{[1],\sharp}$ and $\mathcal{F}_\hbar$ below $b_0\hbar + \mathcal{O}(\hbar^{1+\eta})$ coincide modulo $\mathcal{O}(\hbar^\infty)$.*

(b) *($\Leftrightarrow$ Main Theorem 1)*
For all $c \in (0,3)$, the spectra of $\mathcal{L}_{\hbar,\mathbf{A}}$ and $\mathcal{F}_\hbar$ below $b_0\hbar + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}$ coincide modulo $\mathcal{O}(\hbar^\infty)$.

The spectral analysis of

$$\mathcal{F}_\hbar = b_0\hbar + \sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}} - \frac{\zeta}{2\theta}\hbar^2 + \hbar\left(\frac{\theta}{2}\mathcal{K}_\hbar + k^\star(\hbar^{\frac{1}{2}}, \mathcal{K}_\hbar)\right), \ \mathcal{K}_\hbar = \hbar^2 D_{x_2}^2 + x_2^2;$$

is straightforward, and (b) implies Main Theorem 2.

Eigenvalue asymptotics

Main Theorem 2

Let $(\lambda_m(\hbar))_{m\geqslant 1}$ be the non decreasing sequence of the eigenvalues of $\mathcal{L}_{h,\mathbf{A}}$. For any $c\in(0,3)$, let

$$\mathsf{N}_{\hbar,c} := \{m\in\mathbb{N}^*;\quad \lambda_m(\hbar)\leqslant \hbar b_0 + c\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}\}.$$

Then the cardinal of $\mathsf{N}_{\hbar,c}$ is of order $\mathcal{O}(\hbar^{-\frac{1}{2}})$, and there exist $\upsilon_1,\upsilon_2\in\mathbb{R}$ and $\hbar_0>0$ such that

$$\lambda_m(\hbar)=\hbar b_0+\sigma^{\frac{1}{2}}\hbar^{\frac{3}{2}}+\left[\theta(m-\tfrac{1}{2})-\frac{\zeta}{2\theta}\right]\hbar^2 + \upsilon_1(m-\tfrac{1}{2})\hbar^{\frac{5}{2}}+\upsilon_2(m-\tfrac{1}{2})^2\hbar^3+\mathcal{O}(\hbar^{\frac{5}{2}}),$$

uniformly for $\hbar\in(0,\hbar_0)$ and $m\in\mathsf{N}_{\hbar,c}$.