An application for area and Rice's formulas: Real roots of random multidimensional polynomials.

José R. León
Universidad Central de Venezuela.
Joint work with D. Armitano, J-M. Azaïs and F. Dalmao

The study of the roots of random polynomials is among the most important and popular topics in mathematics and in some areas of physics. In spite of its rich history it is an extremely active field. The case of algebraic polynomials $P_d(t) = \sum_{j=1}^{d} a_j t^j$ with independent identically distributed coefficients was studied and was completely understood during the 70s.

The case of systems of polynomial equations seems to be considerably harder and has received much less attention. The results in this direction are confined to the Shub-Smale model and some other invariant distributions. Recently, F. Dalmao obtained the asymptotic variance and a CLT for the number of zeros as the degree d goes to infinity in the one dimensional case.
We present a generalization of Dalmao result to the case of Kostlan-Shub-Smale random systems with m equations and m variables. Our study is deeply inspired in the recent work of Estrade & León.

We pass now to define a Shub-Smale system. Consider a square system $\mathbf{P} = 0$ of m polynomial equations in m variables. We assume that the equations have the same degree $d > 1$. Let $\mathbf{P} = (P_1, \dots, P_m)$, we can write each polynomial P_ℓ in the form

$$P_\ell(t) = \sum_{|\boldsymbol{j}| \leq d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}},$$

where

1. d is the common degree of the polynomial P_ℓ,
2. $\boldsymbol{j} = (j_1, \dots, j_m) \in \mathbb{N}^m$ is a multi-index of nonnegative integers and $|\boldsymbol{j}| = \sum_{k=1}^m j_k$,
3. $a_{\boldsymbol{j}}^{(\ell)} = a_{j_1 \dots j_m}^{(\ell)} \in \mathbb{R}$ is one of the coefficients,
4. $t = (t_1, \dots, t_m)$ is a point in $\mathbb{R}^m$ and $t^{\boldsymbol{j}} = \prod_{k=1}^m t_k^{j_k}$.

We say that **P** has the Kostlan-Shub-Smale (KSS for short) distribution if the coefficients $a_{\boldsymbol{j}}^{(\ell)}$ are independent mean zero gaussian r.v. with variances

$$\mathbf{Var}\left(a_{\boldsymbol{j}}^{(\ell)}\right) = \binom{d}{\boldsymbol{j}} = \frac{d!}{j_1! \dots j_m!(d - |\boldsymbol{j}|)!}.$$

We are interested in the number of real roots of **P** that we denote by $N_d^{\mathbf{P}}$. Shub and Smale proved that $\mathbb{E}(N_d^{\mathbf{P}}) = d^{m/2}$.

Our main result is the following.

Theorem

Let **P** *be a KSS random polynomial system of dimension m and degree d. Then,*

$$\lim_{d\to\infty} \frac{\mathbf{Var}(N_d^{\mathbf{P}})}{d^{m/2}} = V_\infty^2,$$

where $0 < V_\infty^2 < \infty$ *is given below. Moreover*

$$\zeta_d := \frac{N_d^{\mathbf{P}} - d^{m/2}}{d^{m/4}}$$

converges in distribution towards a centered Gaussian random variable with variance V_∞^2 *as d tends to* ∞*.*

First we homogenize the polynomials. More precisely, we multiply the monomial in P_ℓ corresponding to the index $\boldsymbol{j}$ by $t_0^{d-|\boldsymbol{j}|}$. Let $\mathbf{Y} = (Y_1, \dots, Y_m)$ denote the resulting vector of m homogeneous polynomials in $m+1$ real variables with common degree $d > 1$. We have,

$$Y_\ell(t) = \sum_{|\boldsymbol{j}|=d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}}, \quad \ell = 1, \dots, m,$$

where this time $\boldsymbol{j} = (j_0, \dots, j_m) \in \mathbb{N}^{m+1}$; $a_{\boldsymbol{j}}^{(\ell)} = a_{j_0 \cdots j_m}^{(\ell)} \in \mathbb{R}$; $t = (t_0, \dots, t_m) \in \mathbb{R}^{m+1}$; $|\boldsymbol{j}| = \sum_{k=0}^m j_k$ and $t^{\boldsymbol{j}} = \prod_{k=0}^m t_k^{j_k}$.

Since $\mathbf{Y}$ is homogeneous, its roots consists of lines through 0 in $\mathbb{R}^{m+1}$. Each root of $\mathbf{P}$ correspond exactly to two (opposite) roots of $\mathbf{Y}$ on the unit sphere S^m of $\mathbb{R}^{m+1}$. That is, denoting by $N_d^{\mathbf{P}}$ the number of roots of $\mathbf{P}$ on $\mathbb{R}^m$ and by $N_d^{\mathbf{Y}}$ the number of roots of $\mathbf{Y}$ on S^m, we have $N_d^{\mathbf{P}} = N_d^{\mathbf{Y}}/2$. We work with $\mathbf{Y}$. Multinomial formula shows that for all $s, t \in \mathbb{R}^{m+1}$ we have

$$\mathbb{E}(Y_\ell(s) Y_{\ell'}(t)) =< s, t >^d \delta_{\ell\ell'},$$

where $< \cdot, \cdot >$ is the usual inner product in $\mathbb{R}^{m+1}$. As a consequence we see that the distribution of the system $\mathbf{Y}$ is invariant under the action of the orthogonal group in $\mathbb{R}^{m+1}$.

<u>STEP 0:</u> **Integral formula**

We obtain an integral expression for the normalized variance via Rice Formula. Such a formula gives a expression for the factorial moment.

$$\mathbf{Var}\left(N_d^{\mathbf{P}}\right) = \mathbf{Var}\left(\frac{N_d^{\mathbf{Y}}}{2}\right) = \frac{1}{4}\mathbb{E}(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}}-1)) - d^m + \frac{d^{m/2}}{2}.$$

The following expression can be found in Section 12.1.2 of Azaïs-Wschebor book. Below $||\cdot||_{(k)}$ and ξ_j, η_j denote the norm in $\mathbb{R}^k$ and independent standard Gaussians respectively.

$$\mathbb{E}(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}}-1)) = \frac{d^m}{(2\pi)^m}\iint_{(S^m)^2}\frac{\Delta_d(\langle s,t\rangle)}{(1-\langle s,t\rangle^{2d})^{m/2}}\lambda_m(ds)\lambda_m(dt),$$

where

$$\Delta_d(x) = (1 - x^{2d-2})^{\frac{m-1}{2}} (\sigma_d^4(x) - \tau_d^2(x))^{1/2}$$
$$\cdot \prod_{k=1}^{m-1} \mathbb{E}\left[\|\xi_k\|_{(k)} \|\eta_k + \frac{x^{d-1}}{(1 - x^{2d-2})^{1/2}} \xi_k\|_{(k)} \right]$$
$$\cdot \mathbb{E}\left[\|\xi_m\|_{(m)} \|\eta_m + \frac{\tau_d(x)}{(\sigma_d^4(x) - \tau_d^2(x))^{1/2}} \xi_m\|_{(m)} \right]; \quad (1)$$

with

$$\sigma_d^2(x) = 1 - \frac{dx^{2d-2}(1-x^2)}{1-x^{2d}};$$
$$\tau_d(x) = x^{d-2} \left[1 - \frac{d(1-x^2)}{1-x^{2d}} \right].$$

STEP 1: **Reduction to a simple integral.** Let us introduce

$$\begin{aligned} p_d(x) &= (1-\cos^{2d}(x))^{-m/2}; \qquad (2)\\ q_d(x) &= (1-\cos^{2d-2}(x))^{\frac{m-1}{2}};\\ r_d(x) &= (\sigma_d^4(\cos(x)) - \tau_d^2(\cos(x)))^{1/2}\\ M_{k,d}(x) &= \mathbb{E}\left[\|\xi_k\|_{(k)}\|\eta_k + \frac{\cos^{d-1}(x)}{(1-\cos^{2d-2}(x))^{1/2}}\xi_k\|_{(k)}\right], k=1,..,m-1\\ M_{m,d}(x) &= \mathbb{E}\left[\|\xi_m\|_{(m)}\|\eta_m + \frac{\tau_d(\cos(x))}{r_d(x)}\xi_m\|_{(m)}\right]. \end{aligned}$$

and also

$$m_{k,j} = \mathbb{E}\left(\|\xi_k\|_{(k)}^j\right) = 2^{j/2}\frac{\Gamma((j+k)/2)}{\Gamma(k/2)}.$$

Lemma

Let the above notations prevail. We have

$$\mathbb{E}(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}}-1)) = \frac{d^m \lambda_m^2(S^m)}{(2\pi)^m w_{m-1}} \cdot I_m, \tag{3}$$

with

$$I_m = \int_0^{\frac{\pi}{2}} \sin^{m-1}(\phi_0) p_d(\phi_0) q_d(\phi_0) r_d(\phi_0) \prod_{k=1}^{m} M_{k,d}(\phi_0) d\phi_0. \tag{4}$$

And w_m is the Wallis' integral of m order.

Proof. By the invariance of the law of **Y** under isometries, we can fix s at, say, $s = e_0 = (1, 0, \dots, 0)$. Equation (10) reduces to

$$\mathbb{E}(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}} - 1)) = \frac{d^m}{(2\pi)^m}\lambda_m(S^m)\int_{S^m}\frac{\Delta_d(\langle e_0, t\rangle)}{(1-\langle e_0, t\rangle^{2d})^{m/2}}\lambda_m(dt).$$

Note that $\langle e_0, t\rangle = t_0$, that is, the integrand depends only on one of the coordinates of t. Thus, it is convenient to we write the integral in terms of spherical coordinates. Then, $\langle e_0, t\rangle = t_0 = \cos(\phi_0)$ and denoting J_m the Jacobian of the change of variables to spherical coordinates we have

$$\mathbb{E}(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}}-1)) = \frac{d^m}{(2\pi)^m}\lambda_m(S^m) \cdot \int_{[0,\pi)^{m-2}\times[0,2\pi)} J_m \, d\phi_1 \dots d\phi_{m-1} \cdot \int_0^{\pi} \sin^{m-1}(\phi_0) \cdot \frac{\Delta_d(\cos(\phi_0))}{(1-\cos^{2d}(\phi_0))^{m/2}} d\phi_0.$$

The integrand is symmetric w.r.t. $\phi_0 = \pi/2$
$\mathbb{E}(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}}-1))$

$$= \frac{d^m \lambda_m^2(S^m)}{(2\pi)^m w_{m-1}} \int_0^{\pi/2} \sin^{m-1}(\phi_0) \cdot \frac{\Delta_d(\cos(\phi_0))}{(1-\cos^{2d}(\phi_0))^{m/2}} d\phi_0.$$

With the notations introduced in (2) we can write
$\Delta_d(\cos(\phi_0)) = q_d(\phi_0) r_d(\phi_0) \prod_{k=1}^m M_{d,k}(\phi_0)$. The result follows.

STEP 3: We use the change of variable $\phi_0 = \frac{t}{\sqrt{d}}$ and subtract the expectation term for getting

$$\frac{\frac{1}{4}\mathbb{E}(N_d^{\mathbf{Y}}(N_d^{\mathbf{Y}}-1)) - d^m}{d^{m/2}} = \frac{\lambda_m^2(S^m)}{4(2\pi)^m w_{m-1}} \int_0^{\sqrt{d}\frac{\pi}{2}} d^{\frac{m-1}{2}} \sin^{m-1}\left(\frac{t}{\sqrt{d}}\right) \cdot \left[p_d(t)q_d(t)r_d(t)\prod_{k=1}^m M_{k,d} - \prod_{k=1}^m m_{k,1}^2\right] dt.$$

Taking limit when $d \to \infty$ we have

$$V_\infty^2 = \frac{1}{2} + \frac{\lambda_m^2(S^m)}{4(2\pi)^m w_{m-1}}$$
$$\cdot \int_0^\infty t^{m-1} \left[\frac{\sigma^4(t) - \tau^2(t)}{1 - e^{-t^2}}\right]^{1/2} \left[\prod_{k=1}^m M_k(t) - \prod_{k=1}^m m_{k,1}^2\right] dt < \infty.$$

It is important to point out the reason of the above convergence. All is based on the fact that when $d \to \infty$,

$$< e_0, \mathbf{t} >^d = (\cos(\phi_0))^d = (\cos(\frac{t}{\sqrt{d}}))^d \to e^{-\frac{t^2}{2}}.$$

In the third equality we have used the suggested change of variable.

We need to introduce the Hermite polynomials $H_n(x)$ defined by $H_0(x) = 1$ and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$. The multi-dimensional versions are, for multi-indexes $\boldsymbol{\alpha} = (\alpha_\ell) \in \mathbb{N}^m$ and $\boldsymbol{\beta} = (\beta_{\ell,k}) \in \mathbb{N}^{m^2}$, and vectors $\mathbf{x} = (x_\ell) \in \mathbb{N}^m$ and $\mathbf{x}' = (x_{\ell,k}) \in \mathbb{N}^{m^2}$,

$$\mathbf{H}_{\boldsymbol{\alpha}}(y) = \prod_{\ell=1}^{m} H_{\alpha_\ell}(y_\ell), \quad \mathbf{H}_{\boldsymbol{\beta}}(x') = \prod_{\ell,k=1}^{m} H_{\beta_{\ell,k}}(x'_{\ell,k}).$$

From now on we consider the normalize the vector

$$\overline{\mathbf{Y}'}(t) := \frac{\mathbf{Y}(t)}{\sqrt{d}},$$

Note that each component of $\overline{\mathbf{Y}'}(t)$ have unit variance.

Proposition

We have, in the L^2 sense, that

$$\zeta_d := \frac{N_d^{\mathbf{Y}} - 2d^{m/2}}{2d^{m/4}} = \sum_{q=1}^{\infty} I_{q,d},$$

where

$$I_{q,d} = d^{m/4} \int_{S^m} f_q(\mathbf{X}(t), \overline{\mathbf{Y}'}(t))\lambda_m(dt),$$

with

$$f_q(x, x') = \sum_{|\alpha|+|\beta|=q} b_\alpha f_\beta \mathbf{H}_\alpha(x)\mathbf{H}_\beta(x').$$

This proposition is consequence of the following lemma. Consider the approximation

$$N_\varepsilon = \int_{S^m} |\det(\mathbf{X}'(t))|\delta_\varepsilon(\mathbf{X}(t))\lambda_m(dt), \tag{5}$$

being $\delta_\varepsilon = \prod_{\ell=1}^{m} \frac{1}{2\varepsilon}\mathbf{1}_{|X_\ell(t)|<\varepsilon}$ and $\mathbf{X}'$ the matrix which columns are the gradients, relative to the sphere, of the polynomials X_ℓ.

Lemma

For $v \in \mathbb{R}^m$, let $N_d^{\mathbf{X}}(v)$ denote the number of roots of the equation $\mathbf{X}(t) = v$. Then,

1. $N_d^{\mathbf{X}}(v) \in L^2$ *for all* v.
2. $N_\varepsilon \to N_d^{\mathbf{X}}$ *almost surely and in the* L^2 *sense as* $\varepsilon \to 0$.
3. *The random variable* $N_d^{\mathbf{X}}$ *admits a Hermite's expansion.*

This convergence allows us getting a Hermite's expansion. In first place we have $\delta_\varepsilon(\mathbf{y}) = \prod_{j=1}^{m} \delta_\varepsilon(y_j)$ with $\delta_\varepsilon(\cdot)$ an approximated Dirac's delta. Let us consider the Hermite orthogonal basis for the Gaussian $N(0, I_{m^2 \times m^2})$. We have

$$\delta_\varepsilon(\mathbf{x}) = \sum_{\varpi=0}^{\infty} \sum_{|\boldsymbol{\alpha}|=\varpi} b_{\boldsymbol{\alpha}}^{\varepsilon} \mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{x}),$$

$$|\det(\mathbf{x}')| = \sum_{\kappa=0}^{\infty} \sum_{|\boldsymbol{\beta}|=\kappa} f_{\boldsymbol{\beta}} \mathbf{H}_{\boldsymbol{\beta}}(\mathbf{x}').$$

Furthermore, we have that there exists $\lim_{\varepsilon\to 0} b^{\varepsilon}_{\boldsymbol{\alpha}} = b_{\boldsymbol{\alpha}}$. It holds $b_{\boldsymbol{\alpha}} = \prod_{k=1}^{m} b_{\alpha_k}$ with $b_{\alpha_k} = \frac{1}{\alpha_k!}\varphi(0)H_k(0)$ and the $f_{\boldsymbol{\beta}}$ result the Hermite coefficients of $f : \mathbb{R}^{m^2} \to \mathbb{R}$ such that $f(\mathbf{x}') = |\det(\mathbf{x}')|$. Now, taking limit into the chaos and regrouping terms we get that

$$N_d^{\mathbf{X}}(0) = \sum_{q=0}^{\infty} d^{m/4} \int_{S^m} \sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=q} b_{\alpha} f_{\beta} \mathbf{H}_{\alpha}(\mathbf{X}(t))\mathbf{H}_{\beta}(\overline{\mathbf{Y}'}(t))\lambda_m(dt).$$

It is important to point out that given that $|\det(\mathbf{x}')|$ is an even function and $H_{2k+1}(0) = 0$, only the even coefficients remain.

By Theorem 11.8.3 in Peccati and Taqqu book, in order to get a CLT we need to check

Point 1. The existence of the limit of the variance of $I_{q,d}$ for each q.

Point 2. That the tail $\sum_{q=Q+1}^{\infty} \mathbf{Var}(I_{q,d})$ is negligible as $Q \to \infty$.

Point 3. That each $I_{q,d}$ converges in distribution towards a Normal random variable.

We will prove Point 1 only. Point 2 is a consequence of the convergence of the variance and Point 3 is obtained by showing that the contractions tend to zero.

Now

$$\zeta_d = \frac{d^{m/4}}{2} \sum_{q=1}^{\infty} \sum_{|\alpha|+|\beta|=q} b_\alpha f_\beta \int_{S^m} H_\alpha(\mathbf{Y}(t)) H_\beta(\overline{\mathbf{Y}}'(t)) d\lambda_m(t)$$

$$:= \frac{d^{m/4}}{2} \sum_{q=1}^{\infty} \int_{S^m} g_q(\mathbf{Y}(t), \overline{\mathbf{Y}}'(t)) d\lambda_m(t).$$

We will compute the variance of each term corresponding to a fixed q.

Thus

$$\mathrm{Var}\left(\int_{S^m} g_q(\mathbf{Y}(t), \overline{\mathbf{Y}}'(t))d\lambda_m(t)\right)$$
$$= \int_{S^m}\int_{S^m} \sum_{|\alpha|+|\beta|=|\alpha'|+|\beta'|=q} b_\alpha f_\beta b_{\alpha'} f_{\beta'}$$
$$\times \mathbb{E}[H_\alpha(\mathbf{Y}(t))H_\beta(\overline{\mathbf{Y}}'(t))H_{\alpha'}(\mathbf{Y}(s))H_{\beta'}(\overline{\mathbf{Y}}'(s)]d\lambda_m(t)d\lambda_m(s).(*)$$

So by using the diagram formula for fourth random Gaussian vectors we get that

$$\mathbb{E}[H_\alpha(\mathbf{Y}(t))H_\beta(\overline{\mathbf{Y}}'(t))H_{\alpha'}(\mathbf{Y}(s))H_{\beta'}(\overline{\mathbf{Y}}'(s))]$$

$$=: \mathcal{H}_d^q(< s, t >).$$

Where $\mathcal{H}_d^q(< s, t >)$ is a certain polynomial function of covariances between the processes and their derivatives on the sphere.

Now we can compute the limit of the variance in each term of the chaos expansion. This is

$$\frac{d^{m/4}}{2}\int_{S^m}\int_{S^m}\mathcal{H}_d^q(< s,t >)\lambda_m(s)\lambda_m(t).$$

$$= 2\frac{\lambda_m^2(S^m)}{w_{m-1}}\int_0^{\sqrt{d}\pi/2} d^{(m-1)/2}\sin^{m-1}(\frac{t}{\sqrt{d}})\mathcal{H}_d^q(\cos^d(\frac{t}{\sqrt{d}}))dt.$$

where we have applied the invariance with respect the orthogonal group and that the domain of integration reduces, by symmetry, to the interval $[0,\sqrt{d}\pi/2]$.
The convergence now follows by using for the covariances $\rho_{..}$ the same methods used to establish the asymptotic for the variance.

J-M Azaïs and M. Wschebor. Level sets and extrema of random processes and fields. John Wiley & Sons Inc. (2009).

F. Dalmao. Asymptotic variance and CLT for the number of zeros of Kostlan Shub Smale random Polynomials. C.R. Acad. Sci. Paris Ser. I 353 (2015) 1141-1145.

A. Edelman and E. Kostlan. How many zeros of a random polynomial are real? Bull. Amer. Math. Soc. 32 (1995), no. 1, 1-37.

A. Estrade and J.R. León. A central limit theorem for the Euler characteristic of a Gaussian excursion set. To appear in The Annals of Probability.

E. Kostlan. On the distribution of roots of random polynomials. From Topology to Computation: Proceedings of the Smalefest, 419-431, Springer (1993).

G. Peccati and M. Taqqu. Wiener chaos: moments, cumulants and diagrams. A survey with computer implementation. Bocconi & Springer Series, 1 (2011).

M. Shub and S. Smale. Complexity of Bézout's theorem. II. Volumes and Computational algebraic geometry 267-285, Progr. Math., 109, Birkhäuser (1993).