# Variational Aspects of Boundary Integral Equations


Martin Costabel

IRMAR, Université de Rennes 1


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# Outline

1. What sort of “variational” methods, and why?
   - Positivity, Strong Ellipticity
   - Mapping Properties: From Lax-Milgram via Gårding back to Fredholm
   - Main Motivation: Stability of Projection Methods
2. A quick recipe: Construction of boundary integral equations
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3. Some history: 180 years of boundary integral equations
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   - C. Neumann
   - Poincaré

# Starting point: Basic Hilbert space theory, Lax-Milgram lemma

$X$ real Hilbert space, dual space $X'$

$a : X \times X \to \mathbb{R}$ bilinear form that is

1. bounded: $|a(x,y)| \leq M\|x\|\|y\| \quad \forall x, y \in X$
2. positive definite: $a(x,x) \geq \alpha\|x\|^2 \quad \forall x \in X$

Then $a(x,y) = \langle Ax, y\rangle$, where $A : X \to X'$ is an isomorphism with bounded inverse $\|A^{-1}\| \leq \frac{1}{\alpha}$.

In other words, the problem:

Given $f \in X'$, find $u \in X$ such that $\boxed{a(u,v) = \langle f, v\rangle \quad \forall v \in X}$

has a unique solution, and one has the estimate $\|u\| \leq \frac{1}{\alpha}\|f\|$.

If $a$ is symmetric, then $u$ is the unique minimizer of the functional

$$J : v \mapsto a(v,v) - 2\langle f, v\rangle$$

(hence “variational”)

Suppose $A = A_0 + B$, where $A_0 : X \to X'$ is an isomorphism and $B : X \to X'$ is compact, then $A$ is a Fredholm operator of index zero, that is: the kernel $\ker A$ and cokernel $X'/AX$ have the same finite dimension.
In particular, if $A$ is injective, it is also surjective, hence an isomorphism.
("Uniqueness implies existence")
If $A_0$ is positive definite, then the bilinear form $a$ satisfies a Gårding inequality with a compact bilinear form $b$

$$a(x,x) \geq \alpha \|x\|^2 - b(x,x)\,.$$

$a$ and $A$ are sometimes called "strongly elliptic" or "coercive".

Projection methods or Petrov-Galerkin methods: Equation

$$(Eq(A)) \qquad Au = f, \qquad A : X \to Y, \quad f \in Y \text{ given}, \quad u \in X \text{ unknown}$$

Subspaces $X_N \subset X$, $T_N \subset Y'$, $\dim X_N = \dim T_N = N$. Equations

$$(Eq_N(A)) \qquad \langle Au_N, t\rangle = \langle f, t\rangle \quad \forall t \in T_N, \qquad u_N \in X_N \text{ unknown}$$

Special case: Galerkin methods (conforming, no DG!): $Y = X'$, $T_N = X_N$.
"Projection": The mapping $u \mapsto u_N$ is a projection, namely

$$\text{if } u \in X_N, \text{ then } u_N = u\,.$$

**Basic fact 1, Lax-Milgram – Céa**

If $Y = X'$ Hilbert spaces, $T_N = X_N$, $A : X \to Y$ positive definite, then for given $f \in Y$ there exists a unique $u_N$, and one has the quasi-optimality estimate with $C = \frac{M}{\alpha}$

$$\|u - u_N\| \le C \inf\{\|u - v\| \mid v \in X_N\}$$

# Stability of projection methods

## Basic fact 2, Compact perturbation

Assume that $A = A_0 + B$, $A, A_0 : X \to Y$ isomorphisms, $B : X \to Y$ compact.
If the method $(X_N, T_N)_{N\in\mathbb{N}}$ is stable for $A_0$, that is
(with $u_{0,N}$ defined by $(\mathrm{Eq}_N(A_0))$)

$$\forall f \quad \exists \text{ unique } u, u_{0,N}, \text{ and } \|u_{0,N}\| \le C\,\|u\|, \qquad C \text{ independent of } f, N$$

then the same method is asymptotically stable for $A$, that is

$$\exists N_0\, \forall N \ge N_0 : \quad \exists \text{ unique } u_N \in X_N \text{ and } \|u_N\| \le C\,\|u\|$$

Céa's quasi-optimality is true for $N \ge N_0$.

Technical assumption needed (dual approximation property):
$X, Y$ reflexive Banach spaces and there exist projections $P_N : Y' \to T_N$
that converge strongly to the identity in $Y'$.

## Executive version

For a strongly elliptic invertible operator, every reasonable Galerkin method is asymptotically stable and convergent.

# A quick recipe: Construction of boundary integral equations

## Ingredients

(for solving a constant-coefficient elliptic boundary value problem with BIEs)

- 1 fundamental solution
- 1 Green formula
- 1 set of jump relations

## Directions

1. Write Green formula in distributional form
2. Apply fundamental solution by convolution to obtain integral representation
3. Apply jump relations to obtain Calderón projector
4. Stir in boundary conditions to get boundary integral equations

Aim:
Boundary integral formulation for Dirichlet boundary value problem in bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \quad \text{(works best with } f = 0\text{)} \\ u = g & \text{on } \Gamma = \partial\Omega \end{cases}$$

Fundamental solution

$$G(x) = \frac{e^{-|x|}}{4\pi|x|} \qquad \Rightarrow \qquad LG = \delta$$

2nd Green formula for a bounded domain $D$, unit outer normal $\boldsymbol{n}$, $u, v \in C^2(\overline{D})$

$$\int_D (uLv - vLu)\, dx = -\int_{\partial D} (u\partial_n v - v\partial_n u)\, ds$$

Will be applied to $D = \Omega$ and $D = B_R \setminus \Omega$, $B_R$ large ball.
Jump relations: See later

# Preparation 1: 2nd Green formula

Write Green formula in distributional form:
$u$ piecewise smooth with compact support (smooth on $\overline{\Omega}$ and on $\overline{\complement\Omega}$), jumps on $\Gamma$

$$[\gamma u] = \gamma^+ u - \gamma^- u; \quad [\partial_n u] = \partial_n^+ u - \partial_n^- u$$

$$u^+ = u\big|_{\complement\Omega},\ u^- = u\big|_{\Omega},\ \gamma^{\pm} u = u^{\pm}\big|_{\Gamma},\ \partial_n^{\pm} u = \partial_n u^{\pm}$$

Green formula ($f = Lu\big|_{\mathbb{R}^3\setminus\Gamma}$):

$$\boxed{Lu = f - [\partial_n u]\,\delta_\Gamma - [\gamma u]\,\partial_n\delta_\Gamma}$$

The distributions $\delta_\Gamma$ and $\partial_n\delta_\Gamma$ with support on $\Gamma$ are defined by

$$\langle \delta_\Gamma, \phi\rangle = \int_\Gamma \phi\, ds; \qquad \langle \partial_n\delta_\Gamma, \phi\rangle = -\int_\Gamma \partial_n\phi\, ds$$

$\psi \mapsto \psi\delta_\Gamma$: adjoint $\gamma'$ to trace operator $\gamma$, “single layer”
$v \mapsto -v\partial_n\delta_\Gamma$: adjoint $(\partial_n)'$ to trace operator $\partial_n$, “double layer”

Proof of Green formula in distributional form:
$u$ piecewise smooth with compact support, test function $\phi \in C_0^\infty(\mathbb{R}^3)$,
$\Omega^- = \Omega$, $\Omega^+ = \complement\Omega$.
Write Green formula for $D = \Omega^\pm$ and add:

$$\int_{\Omega^\pm} (uL\phi - f\phi)dx = \int_\Gamma (\pm u\partial_n\phi \mp \partial_n u\phi)ds = \mp\langle \gamma^\pm u\partial_n\delta_\Gamma + \partial_n^\pm u\delta_\Gamma, \phi\rangle$$

$\Longrightarrow$

$$\langle Lu, \phi\rangle = \int_{\mathbb{R}^3} uL\phi\, dx = \int_{\mathbb{R}^3} f\phi\, dx - \langle [\gamma u]\partial_n\delta_\Gamma + [\partial_n u]\delta_\Gamma, \phi\rangle$$

This is equivalent to

$$Lu = f - [\partial_n u]\,\delta_\Gamma - [\gamma u]\,\partial_n\delta_\Gamma$$

The condition of compact support can be dropped.

Recall Green's formula in distributional form

$$Lu = f - [\partial_n u]\delta_\Gamma - [\gamma u]\partial_n \delta_\Gamma$$

If $u$ has compact support, then it can be represented as a convolution with the fundamental solution ("free space Green function"): $u = G * Lu$.
This gives the integral representation formula with 3 potentials

$$\boxed{u = \mathscr{N} f - \mathscr{S}[\partial_n u] + \mathscr{D}[\gamma u]}$$

Volume (or Newton) potential: $\mathscr{N} f(x) = \int_{\mathbb{R}^3} G(x-y) f(y) dy$

Single layer potential: $\mathscr{S}\psi(x) = \int_\Gamma G(x-y)\psi(y) ds(y)$

Double layer potential: $\mathscr{D}v(x) = \int_\Gamma \partial_{n(y)} G(x-y) v(y) ds(y)$

It is possible to read the jump relations from this formula:

$$[\gamma\mathscr{S}\psi] = 0 = [\partial_n \mathscr{D}v]\,; \quad [\partial_n \mathscr{S}\psi] = -\psi\,; \quad [\gamma\mathscr{D}v] = v$$

Recall the jump relations:

$$[\gamma\mathscr{S}\psi] = 0 = [\partial_n\mathscr{D}v]\,; \quad [\partial_n\mathscr{S}\psi] = -\psi\,; \quad [\gamma\mathscr{D}v] = v$$

Definition of the 4 classical boundary integral operators ($x \in \Gamma$):
Single layer potential

$$V\psi = \gamma^-\mathscr{S}\psi = \gamma^+\mathscr{S}\psi; \quad V\psi(x) = \int_\Gamma G(x-y)\psi(y)ds(y)$$

Double layer potential

$$Kv = \tfrac{1}{2}(\gamma^+ + \gamma^-)\mathscr{D}v; \quad Kv(x) = \int_\Gamma \partial_{n(y)}G(x-y)v(y)ds(y)$$

Normal derivative of single layer potential

$$K'\psi = \tfrac{1}{2}(\partial_n^+ + \partial_n^-)\mathscr{S}\psi; \quad K'\psi(x) = \int_\Gamma \partial_{n(x)}G(x-y)\psi(y)ds(y)$$

Normal derivative of double layer potential

$$Wv = -\partial_n^-\mathscr{D}v = -\partial_n^+\mathscr{D}v; \quad Wv(x) = -\int_\Gamma \partial_{n(x)}\partial_{n(y)}G(x-y)v(y)ds(y)$$

From the jumps and the mean traces we obtain one-sided jump relations:

$$\gamma^+\mathscr{S} = \gamma^-\mathscr{S} = V; \quad \partial_n^+\mathscr{S} = -\frac{1}{2} + K'; \quad \partial_n^-\mathscr{S} = \frac{1}{2} + K'$$

$$\gamma^+\mathscr{D} = \frac{1}{2} + K; \quad \gamma^-\mathscr{D} = -\frac{1}{2} + K; \quad \partial_n^+\mathscr{D} = \partial_n^-\mathscr{D} = -W$$

Applied to the piecewise smooth function $u = \mathscr{D}v - \mathscr{S}\psi$ in $\complement\Omega$, $u = 0$ in $\Omega$ this means

$$\begin{pmatrix}\gamma^+ u\\ \partial_n^+ u\end{pmatrix} = \begin{pmatrix}\frac{1}{2}+K & -V\\ -W & \frac{1}{2}-K'\end{pmatrix}\begin{pmatrix}v\\ \psi\end{pmatrix} =: \mathscr{C}^+\begin{pmatrix}v\\ \psi\end{pmatrix}$$

Because of $\begin{pmatrix}\gamma^+ u\\ \partial_n^+ u\end{pmatrix} = \begin{pmatrix}[\gamma u]\\ [\partial_n u]\end{pmatrix}$, on can choose $\begin{pmatrix}v\\ \psi\end{pmatrix} = \begin{pmatrix}\gamma^+ u\\ \partial_n^+ u\end{pmatrix}$, hence

$$\left(\mathscr{C}^+\right)^2 = \mathscr{C}^+$$

$\mathscr{C}^+$ is the Calderón projector for the exterior domain $\complement\Omega$.

Exterior Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } \complement\Omega \\ \gamma^+ u = g & \end{cases}$$

Available: 2 integral relations $\begin{pmatrix} \gamma^+ u \\ \partial_n^+ u \end{pmatrix} = \mathscr{C}^+ \begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+K & -V \\ -W & \frac{1}{2}-K' \end{pmatrix} \begin{pmatrix} v \\ \psi \end{pmatrix}$.

Direct method: Choice $v = g = \gamma^+ u$, unknown $\psi = \partial_n^+ u$.

1. 1st integral relation $\boxed{V\psi = (-\frac{1}{2} + K)g}$

2. 2nd integral relation $\boxed{\frac{1}{2}\psi + K'\psi = -Wv}$

Indirect method 1: Single layer representation $u = \mathscr{S}\psi$, $v = 0$

3. 1st integral relation $\boxed{V\psi = g}$

Indirect method 2: Double layer potential representation $u = \mathscr{D}v$, $\psi = 0$

4. 1st integral relation $\boxed{\frac{1}{2}v + Kv = g}$

Further possibilities: Infinitely many linear combinations...

# Examples of compact BIOs

$K$ integral operator with kernel $k$:

$$K : u \mapsto Ku, \quad Ku(x) = \int_D k(x,y)u(y)dy$$

1. Weakly singular kernels: $|k(x,y)| \le C|x-y|^{-n+\alpha}$, $\alpha > 0$,
$n$: dimension of $D$ (bounded) $\implies K$ compact in $L^p(D)$.
If $k$ is continuous (smooth) for $x \neq y$, then $K$ compact in $C(\overline{D})$, Sobolev spaces...
2. Double layer potential on smooth ($C^{1+\alpha}$) surface $D$ in $\mathbb{R}^3$:

$$4\pi k(x,y) = \partial_{n(y)} \frac{1}{|x-y|} = \frac{\boldsymbol{n}(y)\cdot(x-y)}{|x-y|^3}$$

Proof: By Taylor, $x - y = \boldsymbol{t}_x(y)|x-y| + O(|x-y|^{1+\alpha}) \Rightarrow$
$|k(x,y)| = O(|x-y|^{-2+\alpha})$: weakly singular ($n = 2$)
3. Difference in wave number: $G_k(x) = \frac{e^{ik|x|}}{4\pi|x|}$
$\implies$ convolution with $G_{k_1} - G_{k_2} : H^s_{\mathrm{comp}}(\mathbb{R}^3) \to H^{s+4}_{\mathrm{loc}}(\mathbb{R}^3)\ \forall s \in \mathbb{R}$
By Rellich, the following are compact:
$V_k - V_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$, $K_k - K_0 : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$, etc.,
even for Lipschitz $\Gamma$

# Historical time frame: 180 years of boundary integral equations

| | |
|---|---|
| 1828 | Green: "An Essay on the Application of mathematical Analysis to the theories of Electricity and Magnetism" ▸ Green's Mill ▸ Green himself |
| •1838–1840 | C.-F. Gauss: 2 papers and 1 book on Magnetism, Potential Theory<br>Single layer potential, 1st kind integral equation, computations |
| •1870–1877 | C. Neumann: Double layer potential, 2nd kind integral equation |
| •1896 | H. Poincaré: "La méthode de Neumann et le problème de Dirichlet" |
| 1900 | Fredholm: "Sur une nouvelle méthode pour la résolution du problème de Dirichlet." |
| | ................. |
| 1956–1957 | Calderón – Zygmund: "On singular integrals" |
| 1959–1964 | Agmon – Douglis – Nirenberg: "Estimates near the boundary. . . " |
| 1965– | many authors: Pseudodifferential Operators |
| •1973 | J.-C. Nedelec – J. Planchard: "Une méthode variationnelle d'éléments finis pour la résolution numérique d'un problème extérieur dans $\mathbb{R}^3$" |
| 1976– | Wendland – Hsiao et. al.: Analysis of Boundary Element Methods |
| •2001 | O. Steinbach – W. Wendland: " On C. Neumann's method for second-order elliptic systems in domains with non-smooth boundaries " |
| ••2007 | M. Costabel: " Some historical remarks on the positivity of boundary integral operators" |

# Variational Story 1: Gauss' Missing Theorem

Gauss 1839: First kind integral equation for the gravity potential in $\complement\Omega$

$$V\phi(x) \equiv \int_\Gamma \frac{\phi(y)\,ds(y)}{4\pi|x-y|} = f(x), \quad x \in \Gamma$$

Variational approach: Minimize $\frac{1}{2}\langle \phi, V\phi\rangle - \langle f, \phi\rangle$.
Needed: Bilinear form $\langle \phi, V\psi\rangle$ is positiv definite.

- 2 principal methods: With or without looking at the integral operator.

**[Gauss 1839] Looking at the kernel, obvious estimate**

$$\int \frac{\phi(x)\phi(y)}{|x-y|} ds(y) ds(x) \geq \frac{\|\phi\|^2_{L^1(\Gamma)}}{\mathrm{diam}(\Gamma)} \quad \text{if } \phi \geq 0$$

Gauss himself deplored that he needed the positivity of $\phi$ ($\rightarrow$ variational inequality) and wished that one could prove positivity of the bilinear form without this assumption, but found that this is "not evident".
Mystery: He had (almost) all the ingredients in his paper:

Jump relations and Green's formula.

# Story 1: Gauss' Missing Theorem, proved (without looking at the kernel)

Jump relation for the single layer potential (for the Laplace operator, $L = -\Delta$)

$$u(x) = \mathscr{S}\phi(x) = \int_\Gamma \frac{\phi(y)\,ds(y)}{4\pi|x-y|} \text{ in } \mathbb{R}^3 \qquad \Longrightarrow \qquad \phi = -[\partial_n u] = \partial_n^- u - \partial_n^+ u$$

Green's (1st) formula

$$\Delta u = 0 \quad \Longrightarrow \quad \int_{\Omega^\pm} |\nabla u|^2\,dx = \mp \int_\Gamma u \partial_n^\pm u\,ds$$

Adding up $\quad (V\phi = \gamma u,\, \phi = -[\partial_n u])$ :

$$\boxed{\langle \phi, V\phi \rangle = \int_{\mathbb{R}^3} |\nabla \mathscr{S}\phi|^2\,dx}$$

This is $> 0$ if $\phi \neq 0$.

**[Nedelec-Planchard 1973]**

$\|\phi\|_V^2 = \langle \phi, V\phi \rangle$ defines a norm on $H^{-\frac{1}{2}}(\Gamma)$, equivalent to the Sobolev norm.

**Similarly,**

$\|\cdot\|_{V^{-1}}$ and $\|\cdot\|_W$ define equivalent norms on $H^{\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$

Trace lemma: $H^{\frac{1}{2}}(\Gamma)$ is the space of traces of $H^1(\Omega)$ (and of $H^1(\complement\Omega)$ !)

$$\|g\|_{H^{\frac{1}{2}}(\Gamma)} = \inf_{u \in H^1(\Omega), \gamma u = g} \|u\|_{H^1(\Omega)}$$

$$\simeq \inf_{u \in H^1(\complement\Omega), \gamma u = g} \|u\|_{H^1(\Omega)} \simeq \inf_{u \in H^1(\mathbb{R}^3), \gamma u = g} \|u\|_{H^1(\Omega)}$$

$H^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H^{\frac{1}{2}}(\Gamma)$ with $L^2(\Gamma)$ as pivot

$$H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)$$

This definition is sufficient to prove the Nedelec-Planchard Theorem
Proof here for $L = -\Delta + 1$:
Green's formula gives for $u = \mathscr{S}\phi$

$$\langle \phi, V\phi \rangle = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx = \|u\|^2_{H^1(\mathbb{R}^3)}$$

Thus we have to show that

$$\|u\|^2_{H^1(\mathbb{R}^3)} \simeq \|\phi\|^2_{H^{-\frac{1}{2}}(\Gamma)}$$

1st Green formula for $\Omega$ Lipschitz

$$\int_\Omega (\nabla u \cdot \nabla v + \Delta u\, v)dx = \int_\Gamma \partial_n^- u\, \gamma v\, ds$$

With $Lu = 0$, this can be written as

$$\big(u, v\big)_{H^1(\Omega)} = \langle \partial_n^- u, \gamma v\rangle$$

Taking the inf over $v \in H^1(\Omega)$, $\gamma v = g$, in $|\langle \partial_n^- u, g\rangle| \le \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$, we find by definition of the norm in $H^{\frac{1}{2}}(\Gamma)$ and of the dual norm

$$|\langle \partial_n^- u, g\rangle| \le \|u\|_{H^1(\Omega)} \|g\|_{H^{\frac{1}{2}}(\Gamma)} \Longrightarrow \|\partial_n^- u\|_{H^{-\frac{1}{2}}(\Gamma)} \le \|u\|_{H^1(\Omega)}$$

Conversely,

$$\|u\|^2_{H^1(\Omega)} = \langle \partial_n^- u, \gamma v\rangle \le \|\partial_n^- u\|_{H^{-\frac{1}{2}}(\Gamma)} \|\gamma u\|_{H^{\frac{1}{2}}(\Gamma)} \le \|\partial_n^- u\|_{H^{-\frac{1}{2}}(\Gamma)} \|u\|_{H^1(\Omega)}$$

Altogether we have shown the identity

$$\|\partial_n^- u\|_{H^{-\frac{1}{2}}(\Gamma)} = \|u\|_{H^1(\Omega)}$$

For $\complement\Omega$, the same holds for an equivalent norm.

$$\mathscr{D}v(x) = \frac{1}{4\pi}\int_\Gamma v(y)\partial_{n(y)}|x-y|^{-1}ds(y)\,, \quad Kv = \mathscr{D}v|_\Gamma$$

Jump relations for the double layer potential $u = \mathscr{D}v$

$$[\partial_n u]_\Gamma = 0\,; \quad [\gamma u]_\Gamma = v\,; \quad \gamma^{\pm}u = (\pm\tfrac{1}{2} + K)v$$

2nd kind integral equation for the Dirichlet problem $\Delta u = 0$ in $\Omega$, $u = g$ on $\Gamma$

$$(\tfrac{1}{2} - K)v = -g \quad \text{or} \quad (1-N)v = -2g \quad \text{with } N = 2K$$

If one can show that $N$ is a contraction in some Banach space, one gets a unique solution by successive approximation (“Neumann series”)

$$v = -2\sum_{\ell=0}^{\infty} N^\ell g\,.$$

First approach (looking at the kernel)

$$d\theta_x(y) = -\frac{n(y)\cdot(y-x)}{2\pi|x-y|^3}\,ds(y)$$

is for $x \in \Gamma$ a measure (solid angle) of total mass 1 on $\Gamma$,
positive if $\Omega$ is convex.

**[C. Neumann 1877] Using hard analysis**

If $\Omega$ is convex, but not the intersection of 2 convex cones, then $N = 2K$ is a contraction on $L^\infty(\Gamma)/\mathbb{R}$ in a norm equivalent to the $L^\infty$ norm.

Second approach (without integral operators)

**[Poincaré 1896] An energy equilibrium inequality**

There exists a constant $\mu > 0$ depending on $\Omega$ such that

1. If $u$ is a double layer potential, then

$$\frac{1}{\mu}\int_{\complement\Omega}|\nabla u|^2 \le \int_{\Omega}|\nabla u|^2 \le \mu\int_{\complement\Omega}|\nabla u|^2$$

2. If $u$ is a single layer potential, then

$$\int_{\Omega}|\nabla u|^2 \le \mu\int_{\complement\Omega}|\nabla u|^2 \quad \text{and if } \int_{\Gamma} u = 0 \text{ then } \int_{\complement\Omega}|\nabla u|^2 \le \mu\int_{\Omega}|\nabla u|^2$$

Poincaré: Proved for simply connected smooth domains.
Korn, Steklov. . . : For Lyapunov domains.
Nowadays easy exercise for Lipschitz domains (if $\complement\Omega$ is connected). ▸ easy exercise

**[Stekloff 1900]**

Nous appellerons ce théorème *théorème fondamental*. . . .
Nous verrons dans ce qui va suivre, que *la solution de tous les problèmes fondamentaux de la Physique mathématique se ramène à la démonstration complète du théorème fondamental*.

$$u = \mathscr{S}\phi \quad \Longrightarrow \quad \int_{\Omega^\pm} |\nabla u|^2 = \mp \int_\Gamma u \partial_n^\pm u = \int_\Gamma V\phi (\tfrac{1}{2} \mp K')\phi$$

**[Co 2007] Corollary of the "Théorème fondamental"**

The operators $A = \frac{1}{2} - K'$ and $B = \frac{1}{2} + K'$ are bounded selfadjoint operators on the space $H^{-\frac{1}{2}}(\Gamma)$ with norm $\|\cdot\|_V$ satisfying $A + B = 1$.

1. $A$ is positive definite, hence $B$ is a contraction, with norm
$$\|B\| \le \frac{\mu}{1+\mu}.$$
2. On the subspace $H_0^{-\frac{1}{2}} = \{\phi \mid \langle \phi, 1 \rangle = 0\}$, $B$ is positive definite, hence both $A$ and $N' = A - B$ are contractions, and the Neumann series converges in the norm $\|\cdot\|_V$.

Proof of 1: Poincaré $\Rightarrow \langle V\phi, B\phi \rangle \le \mu \langle V\phi, A\phi \rangle \Rightarrow$
$\|\phi\|_V^2 = \langle V\phi, \phi \rangle = \langle V\phi, (A+B)\phi \rangle \le (1+\mu)\langle V\phi, A\phi \rangle \Rightarrow A$ pos. def. and
$\langle V\phi, B\phi \rangle = \langle V\phi, \phi \rangle - \langle V\phi, A\phi \rangle \le (1 - \frac{1}{1+\mu})\|\phi\|_V^2 = \frac{\mu}{1+\mu}\|\phi\|_V^2$

Same results for $\frac{1}{2} \pm K$ in the space $H^{\frac{1}{2}}(\Gamma)$ with norm defined by the quadratic form of $V^{-1}$.

Thank you for your attention!

# APPENDICES

of the rectangular co-ordinates, and tending to increase them. Then $\varrho$ representing the density of the electricity on an element $d\sigma$ of the surface, and $r$ the distance between $d\sigma$ and $p$, any other point of the surface, the equation for determining $\varrho$ which would be employed in the ordinary method, when the problem is reduced to its simplest form, is known to be

$$(a) \qquad \text{cons} = a = \int \frac{\varrho\, d\sigma}{r} - \int (X\,dx + Y\,dy + Z\,dz);$$

the first integral relative to $d\sigma$ extending over the whole surface $A$, and the second representing the function whose complete differential is $X\,dx + Y\,dy + Z\,dz$, $x$, $y$ and $z$ being the co-ordinates of $p$.

This equation is supposed to subsist, whatever may be the position of $p$, provided it is situate upon $A$. But we have no general theory of equations of this description,

.........

It only remains therefore to find a function $V'$ which satisfies the partial differential equation, becomes equal to $\overline{V'}$ when $p$ is upon the surface $A$, vanishes when $p$ is at an infinite distance from $A$, and is besides such, that none of its differential co-efficients shall be infinite, when the point $p$ is exterior to $A$.

All those to whom the practice of analysis is familiar, will readily perceive that the problem just mentioned, is far less difficult than the direct resolution of the equation $(a)$, and therefore the solution of the question originally proposed has been rendered much easier by what has preceded.

back

For simplicity, we show it for $L = -\Delta + 1$.
Let $u = \mathscr{S}\phi$ be a single layer potential, $\phi = -[\partial_n u]$.
We have already seen that $\|u\|^2_{H^1(\Omega)} = \|\partial_n^- u\|^2_{H^{-\frac{1}{2}}(\Gamma)}$, hence

$$\|u\|^2_{H^1(\Omega)} = \langle \partial_n^- u, \gamma^- u\rangle \le \|\partial_n^- u\|_{H^{-\frac{1}{2}}(\Gamma)} \|\gamma^- u\|_{H^{\frac{1}{2}}(\Gamma)} \le \|\partial_n^- u\|_{H^{-\frac{1}{2}}(\Gamma)} \|u\|_{H^1(\Omega)}$$

Equality $\|u\|_{H^1(\Omega)} = \|\gamma^- u\|_{H^{\frac{1}{2}}(\Gamma)}$ follows.
Similarly for the exterior domain: $\|u\|_{H^1(\complement\Omega)} \simeq \|\gamma^+ u\|_{H^{\frac{1}{2}}(\Gamma)}$.
Since $\gamma^- u = \gamma^+ u$, it follows that

$$\|u\|_{H^1(\Omega)} \simeq \|u\|_{H^1(\complement\Omega)} . \qquad Q.E.D.$$

The proof for a double layer potential is similar.
Note: An essential point here is that we know that the spaces of interior and exterior traces are the same:

$$\gamma^- H^1(\Omega) = H^{\frac{1}{2}}(\Gamma) = \gamma^+ H^1(\complement\Omega) .$$

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# George Green’s Windmill in Nottingham

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