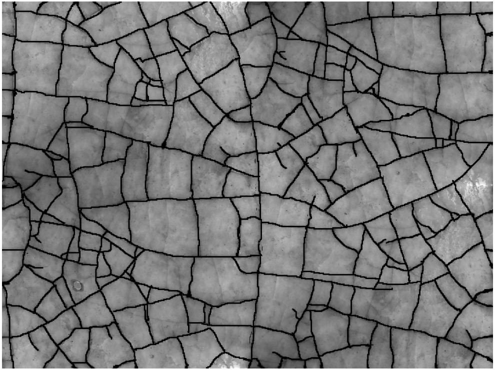
STIT tessellations – a mathematical model for structures generated by sequential division

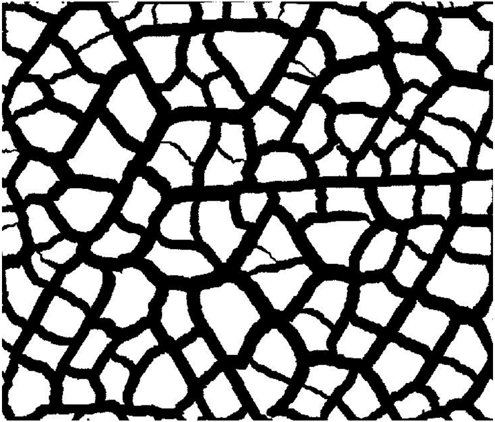
Werner Nagel, Jena

Examples of structures that can be modeled by random tessellations



Craquelée on a ceramic surface (Photo: G. Weil)

Examples of structures that can be modeled by random tessellations



Simulated cracks in soil (H.-J. Vogel)

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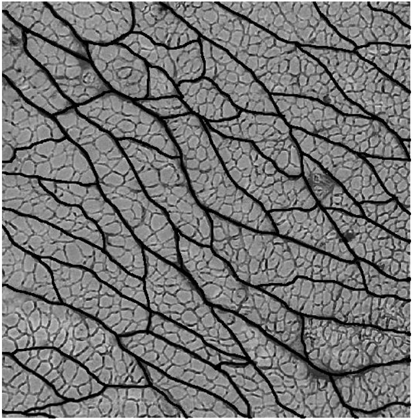

Connective tissue in a rat muscle (I. Erzen)

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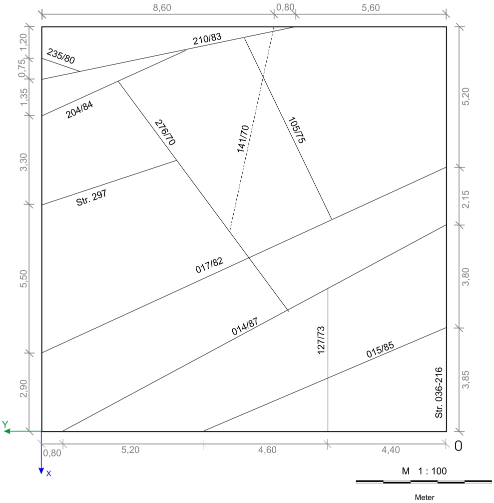
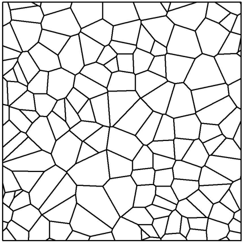


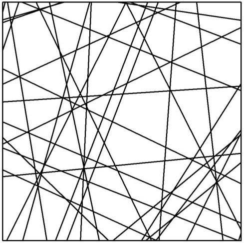
Gris Perla.jpg

Fissures in granite (D. Nikolayev, S. Siegesmund, S. Mosch, A. Hoffmann)

Mathematical models for tessellations

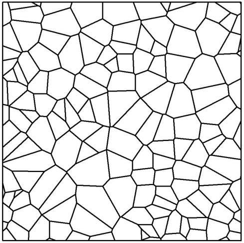


Poisson-Voronoi



Poisson line

Mathematical models for tessellations



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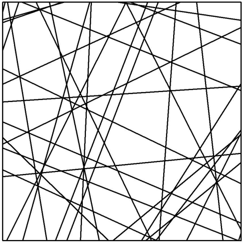

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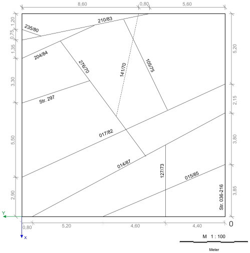

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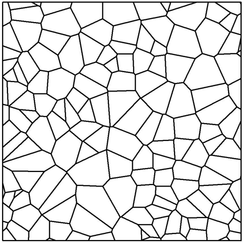

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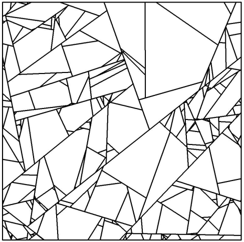

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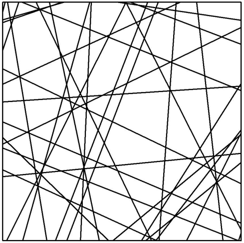

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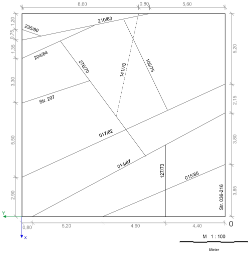

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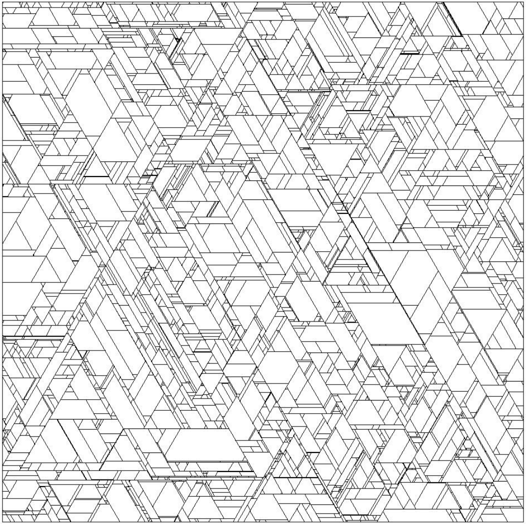
Simulations of STIT tessellations

three directions



Simulations of STIT tessellations

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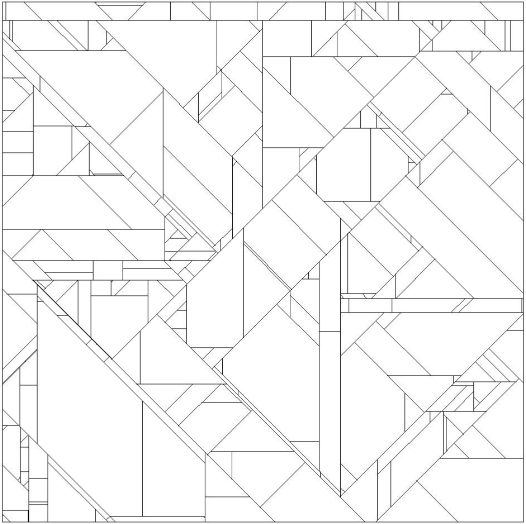

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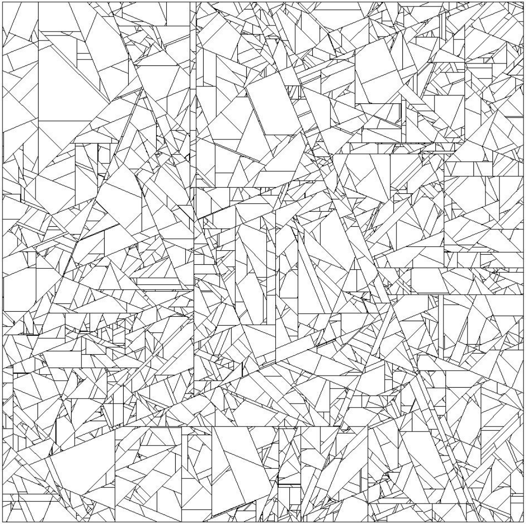
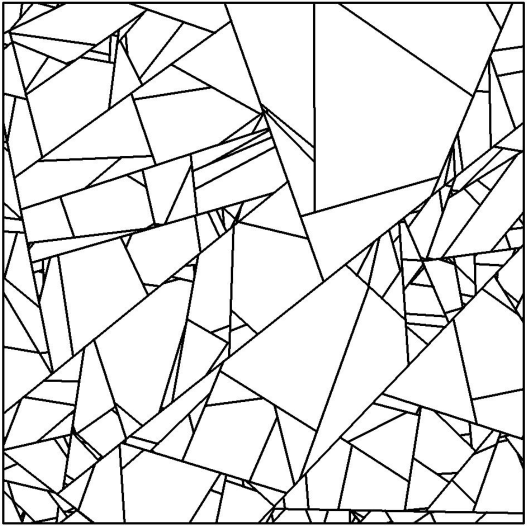


Simulations of STIT tessellations

isotropic model



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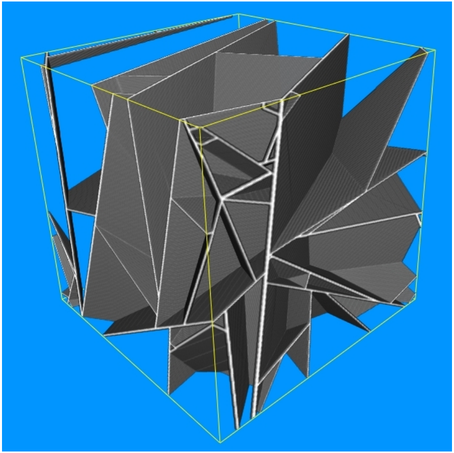
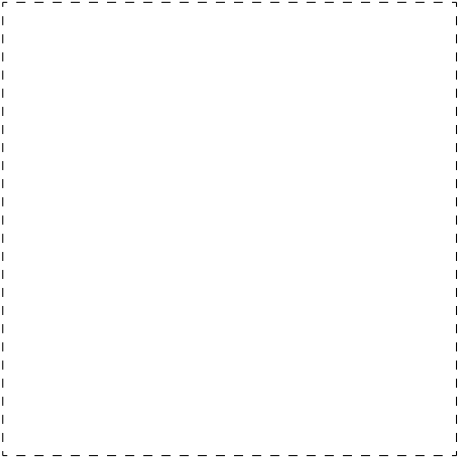


3d isotropic STIT model (Ohser/Lautensack/Sych)

Construction of STIT tessellations:

A process of cell divisions/cracks



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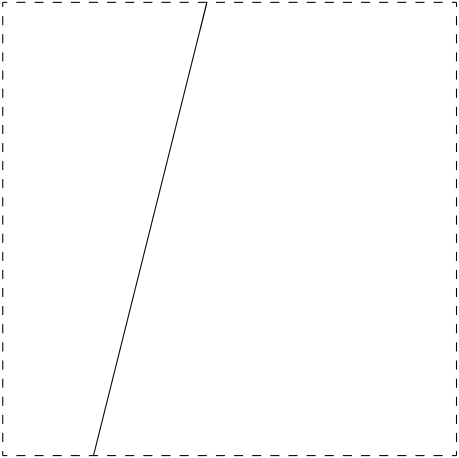

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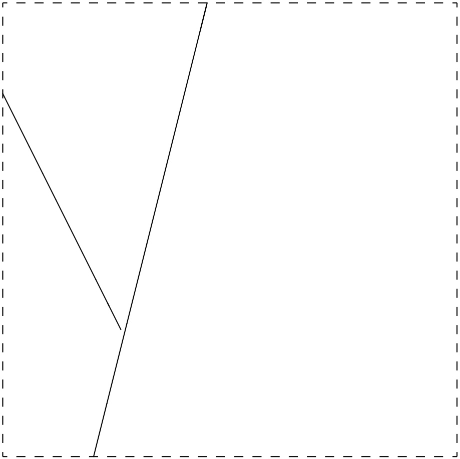

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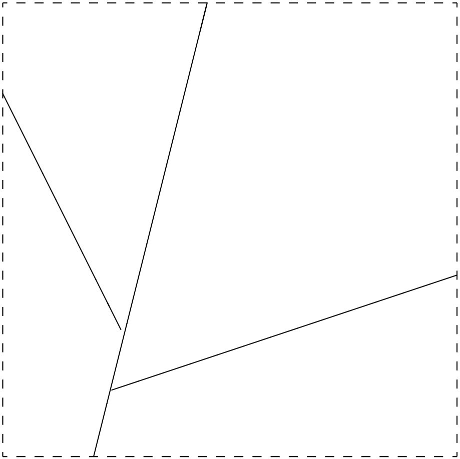

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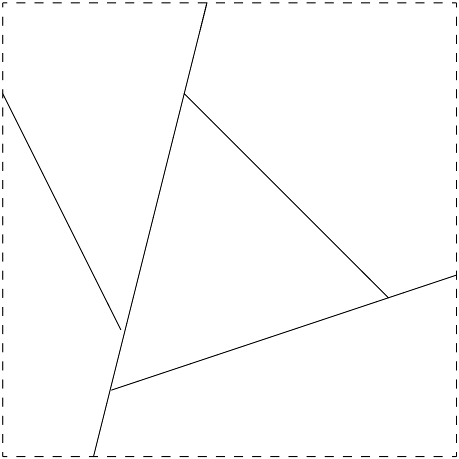

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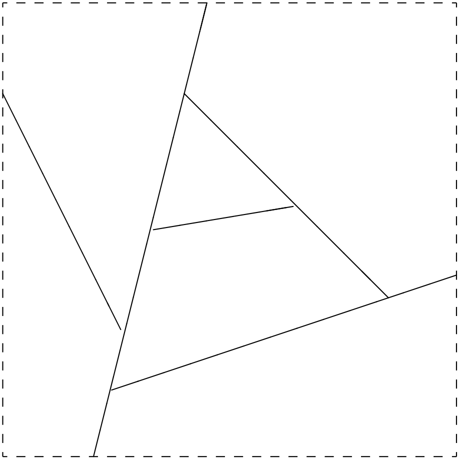

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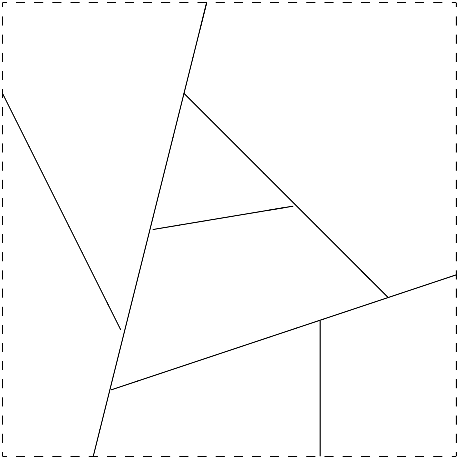

Construction of STIT tessellations

$(Y_t, t > 0)$ ... STIT process in $\mathbb{R}^d$
determined by

- Λ ... translation invariant measure on $(\mathcal{H}, \mathfrak{H})$ on the space of hyperplanes in $\mathbb{R}^d$

$$\Lambda = \text{image}\,[\ \gamma \cdot \ell \otimes \theta\ ]$$

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- $\Lambda_{[C]} = \frac{1}{\Lambda([C])} \Lambda(\cdot \cap [C])$ ... division rule

Features of STIT tessellations

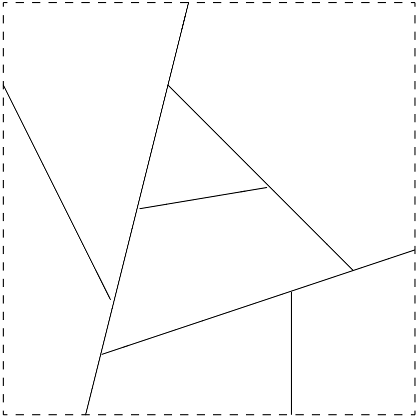
Among a variety of cell division models, (only) STIT is mathematically tractable.

- STIT 'in-between' Poisson-Voronoi and Poisson line/plane tessellations – in many aspects
- STIT is a reference model for several crack/fissure or cell division structures,
  i.e. it can be a good starting point for model fitting
- The seminal paper: Nagel/Weiss (2005). Crack STIT tessellations – characterization of the stationary random tessellations which are stable with respect to iteration. *Adv. Appl. Prob.* **37**.

A key issue: Consistency in space

For any fixed $t > 0$ and $W' \subset W$:

$[Y_t \text{ constructed in } W] \cap W' \overset{D}{=} [Y_t \text{ constructed in } W']$



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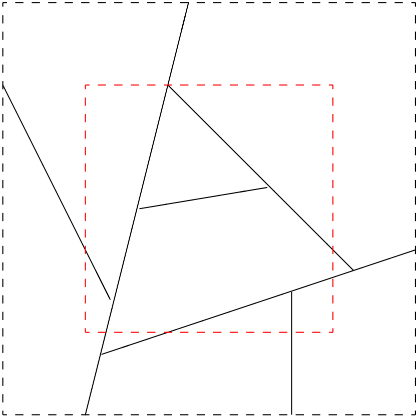

Consistency in space

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If a construction is consistent in space, then we do not have "edge effects" in bounded windows.

Moreover:
If the construction is consistent in space, then we have the
existence of a random tessellation in the whole space $\mathbb{R}^d$
such that its distribution coincides in any bounded window with the construction
(Theorem by Schneider and Weil).

Consistency in space

Theorem (N. and Weiss 2005)

The STIT tessellation process is consistent in space.

Consistency in space

Theorem (N. and Weiss 2005)

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Theorem (N. and Biehler 2015)

The STIT tessellation processes are the ONLY cell division processes which are consistent in space.

Cell Division process: Given any cell, its life time and division rule is (conditionally) independent from all the other cells.

Features of STIT tessellations

- STIT 'in-between'

  Poisson-Voronoi and Poisson line/plane tessellations –

  in many aspects

Results by

N./Weiß, and

Redenbach/Thäle

Mean values of essential parameters

Assumptions:

- $\mathbb{R}^3$, i.e. 3d case,
- homogeneous (i.e. spatially stationary) and
- isotropic (i.e. distribution invariant under rotations) tessellations.

Upper index:

P ... Poisson plane tessellation
V ... Poisson Voronoi tessellation
none ... STIT tessellation

Rescaled, such that
Mean number of cells per unit volume

$$N_3^P = N_3 = N_3^V$$

Mean values of essential parameters

Then:
Per unit volume:

	Mean
$6 \quad N_0^P = N_0 \approx 0.887 \quad N_0^V$	number of vertexes (nodes)
$4 \quad N_1^P = N_1 \approx 0.887 \quad N_1^V$	number of edges
$\frac{7}{3} \quad N_2^P = N_2 \approx 0.901 \quad N_2^V$	number of faces
$S_V^P = S_V \approx 0.853 \quad S_V^V$	total area of faces
$2 \quad L_V^P = L_V \approx 0.829 \quad L_V^V$	total length of edges

Mean values of essential parameters

Typical cell:

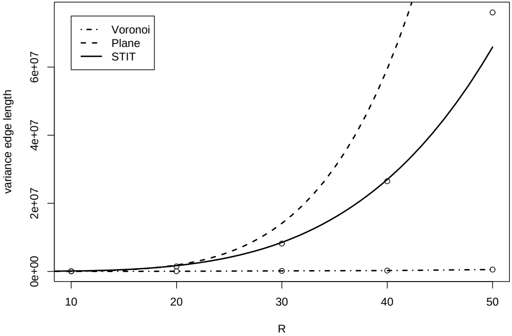
	Mean
$V_3^P = V_3 = V_3^V$	volume
$A_3^P = A_3 \approx 0.853 \; A_3^V$	surface
$B_3^P = B_3 \approx 0.829 \; B_3^V$	mean width
$\frac{3}{2} \; U_3^P = U_3 \approx 0.829 \; U_3^V$	total edge length

Mean values of essential parameters

For the typical cell:

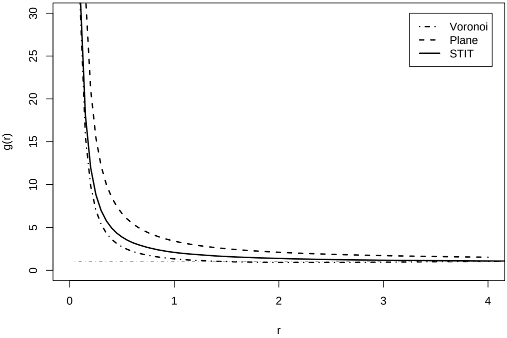
			Mean
$N_{30}^{P} = 8;$	$N_{30} = 24;$	$N_{30}^{V} \approx 27.071$	number of nodes
$N_{31}^{P} = 12;$	$N_{31} = 36;$	$N_{31}^{V} \approx 40.606$	number of edges
$N_{32}^{P} = 6;$	$N_{32} = 14;$	$N_{32}^{V} \approx 15.535$	number of faces

Second order quantities



$\mathbb{R}^3$; variance of total edge length in cube with side-length R (R./Th.)

Second order quantities



$\mathbb{R}^3$; pair correlation function of the PP of nodes (R./Th.)

Summary

- For homogeneous (spatially stationary) tessellations:
  Poisson-Voronoi tess.: always isotropic
  Poisson plane tess. and STIT: arbitrary directional distr.
- Poisson plane tess. and STIT: typical cells identically distr.
  i.e. the same shape and size distribution of the cells.
- Poisson plane tess. and STIT: different arrangement of cells
- Isotropic STIT and Poisson-Voronoi : 'topol. param.' similar
  Isotropic Poisson plane tess.: significant difference

Summary

STIT fills a gap between Poisson Voronoi and Poisson plane tessellations.

Therefore,

STIT should be taken into account, when a tessellation model is adapted to a real structure.