Estimating geometric anisotropy in spatial point patterns

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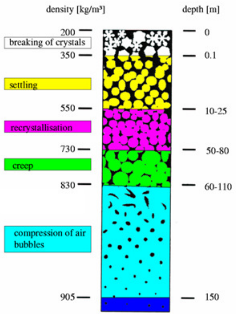
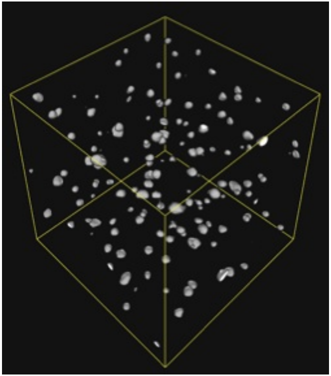
Motivating problem

- Polar ice has information on the climate of the past
- To be able to interprete ice core records, one has to know how old the ice is
- There are theories connecting the dynamics of glaciers to the age of ice
- Question: How can we estimate the deformation in polar ice?
- Method: Polar ice is compacted snow. If we go deep enough, the air pores are isolated in the ice.
  → Study the anisotropy (deformation) of these air inclusions in the ice samples.



Two data sets from Antarctica

- Talos: compressed in z direction, stretched in xy
- EDML: compression and lateral flow

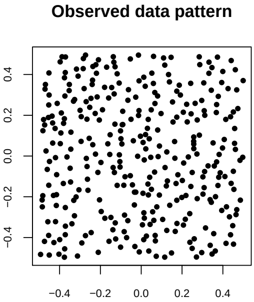


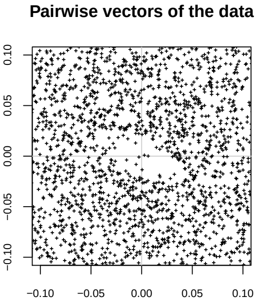
Examples of earlier work on anisotropic point patterns

- Directional versions of summary statistics Ripleys K function and the pair correlation (e.g. Ohser and Stoyan, 1981; Penttinen and Stoyan, 1989; Stoyan and Beneš, 1991; Redenbach et al., 2009)
- Spectral analysis (Mugglestone and Renshaw, 1996)
- Wavelets (e.g. Rosenberg, 2004; Mateu and Nicolis, 2012)
- Geometric anisotropy (Guan et al., 2006; Møller and Toftaker, 2014)
- Cylindrical K function and anisotropic model (Møller et al., 2016)

Is the point pattern below (left) isotropic?

Can be visualized by plotting the pairwise difference vectors $x_i - x_j$ for each point pair, a Fry plot (right).





The pairwise difference vectors not rotationally invariant
$\rightarrow$ pattern anisotropic

Set-up

- The underlying point process X is (regular,) stationary and isotropic
- We observe a point pattern $y = \{y_i : y_i \in W, i = 1, ...n\}$ in the window $W \subset \mathbb{R}^d$, $d = 2, 3$ which is a realization of the transformed process

$$Y := TX = \{Tx : x \in X\},$$

where $T : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear mapping
- Since X is isotropic, we can decompose the mapping into two matrices,

$$T = RC$$

where R is a rotation matrix and C is a diagonal scaling matrix that compresses and stretches the dimensions.
- Note that Y is stationary

Estimating the transformation in two steps

- First, we estimate the rotation R by fitting ellipsoids to the contours of directed cumulant of the difference vectors
- Second, we estimate the scaling C by numerically finding the most likely compression by transforming the back-rotated data

Estimating rotation

- We analyse the Fry points, i.e. the pairwise difference vectors

$$F(\mathbf{y}) := \{f_{ij} = y_i - y_j : \ y_i, y_j \in \mathbf{y}\}$$

  in the data, which are also transformed by T

- Note that the sector K function can be defined as

$$K(u, r) := \frac{1}{\lambda}\mathbb{E}N_{\mathbf{y}}(u, r),$$

  where

$$N_{\mathbf{y}}(u, r) := \sum_{f_{ij} \in F(\mathbf{y})} 1(f_{ij} \in S_\epsilon(u, r))$$

  counts the Fry points in a sector $S_\epsilon(u, r)$ centered in the origin and having direction u with central angle $\epsilon > 0$ up to distance r.

Estimating rotation (continues)

- Let $r_l(u)$ be the smallest distance having l points in the sector $S_\epsilon(u)$, and U the set of chosen directions
- The set of pseudo-Fry points

$$G_l := \{r_l(u) : u \in U\},$$

is a collection of the lth nearest Fry points in different directions.
- For each l, we assume that the observed pseudo-Fry points $g_i \in G_l$ follow the measurement error model

$$g_i = e_i + \varepsilon_i, \quad i = 1, ..., n(G_l),$$

where e_i are the ellipsoid points fulfilling the origin centred quadratic equation

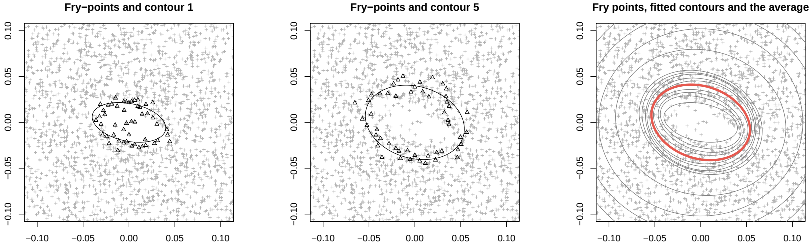
$$e_i^T A_l e_i = m_l$$

with a scale parameter m_l, ε_i's are independent Gaussian measurement errors, and $n(G_l)$ is the number of directions u.

Estimating rotation (continues)

- The model can be fitted by the penalized and adjusted ordinary least squares to obtain $\hat{A}_l$ following Kukush et al. (2004).
- The connection between the original scaling C and the scaling C_l depends on the transformation T but also on the unknown true process.
  $\rightarrow$ We fix the scaling parameter and estimate only the rotation R_l for each chosen l.
- Finally, the rotation is estimated as the "mean" of the estimated R_l's by a Monte Carlo approach.

Estimating the ellipsoid based on 54 directions



Estimating scaling

- Back-rotate $\tilde{\mathbf{y}} = \hat{R}^{-1}\mathbf{y}$ so that we get approximately $C\mathbf{x}$, where $\mathbf{x}$ is the original isotropic pattern.
- We assume volume preservation, i.e. $|T| = 1$.
- We follow Redenbach et al. (2009) who considered the estimation of C under the assumption spheroidal transformation $C = diag(\gamma^{-\frac{1}{2}}, \gamma^{-\frac{1}{2}}, \gamma)$, with $0 < \gamma < 1$.
- To estimate γ, a grid of compression parameters $\{\gamma_1, ..., \gamma_m\}$ is chosen. Then, the data are back-transformed by $C(\gamma_i)^{-1}$ and the final estimate for γ is the γ_i that minimizes

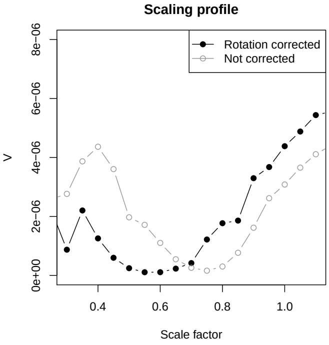
$$\int_a^b \sum_{i<j} |\hat{K}(r, u_i) - \hat{K}(r, u_j)| dr$$

where $\hat{K}(r, u)$ is an estimator for the directed K function, and directions $u_1, u_2, ..., u_d$ are along the coordinate axes in $\mathbb{R}^d$.

Simulation study in 2D

Strauss process with different interaction parameters (0.01, 0.1, 0.3); compression (0.5, 0.7, 0.9); rotation 20 degrees; approximately 300 points/realization

- rotation angle well estimated when there is strong regularity/compression in the data, variation increases when the structure effects weaken
- compression well estimated
- correcting for rotation beneficial when strong regularity/compression present, more beneficial if the rotation angle is close to 45 degrees



Two ice data sets

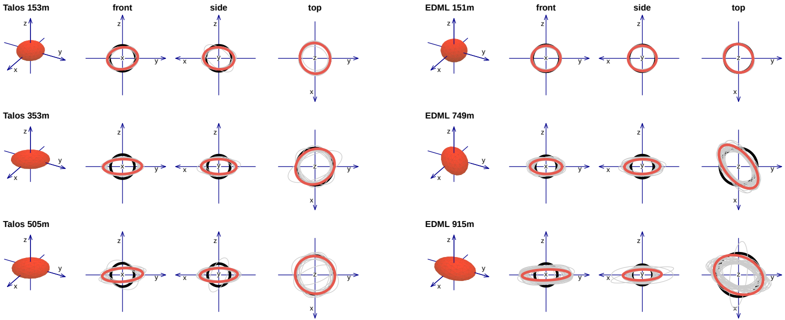
Talos Dome, Antarctica (Talos)

- ice from depths 153m, 353m, and 505m
- volume preserving spherical compression, where the ice is stretched isotropically in the xy plane to compensate for the vertical compression

Dronning Maud Land, East Antarctica (EDML)

- ice from depths 151m, 749m, and 915m
- volume preserving transformation, where the compression is accompanied by a lateral flow

Ice data



Talos 153m
front
side
top
EDML 151m
front
side
top
Talos 353m
EDML 749m
Talos 505m
EDML 915m

Conclusions

- We have a new non-parametric method for detecting, testing and characterizing anisotropy in spatial point patterns.
- The method was developed with regular patterns in mind, but it can be applied to clustered patterns as well.
- The choice of the direction grid is crucial: Increasing the number of directions increases the number of contour sample points for the ellipsoid estimation, but also increases the error model variance due to narrower sectors catching Fry points less frequently.

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