A stochastic Galerkin method for the nonlinear Boltzmann equation with uncertainty

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Joint work with Shi Jin

Overview

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The Boltzmann equation (dimensionless form)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\mathrm{Kn}} \mathcal{Q}(f, f)(\mathbf{v}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \ \mathbf{v} \in \mathbb{R}^d$$

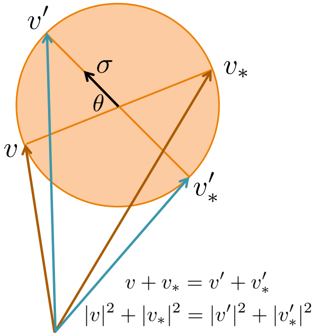
- $f(t, \mathbf{x}, \mathbf{v})$ is the **phase space distribution function** of time t, position $\mathbf{x}$, and velocity $\mathbf{v}$
- Kn is the **Knudsen number**, ratio of the mean free path and the characteristic length scale
- $\mathcal{Q}(f, f)$ is the **collision operator**, a quadratic integral operator modeling the binary interaction of particles

Collision operator

$$\mathcal{Q}(f,f)(\mathbf{v}) = \int_{\mathbb{R}^d}\int_{S^{d-1}} B(\mathbf{v}-\mathbf{v}_*,\sigma)[f(\mathbf{v}')f(\mathbf{v}_*') - f(\mathbf{v})f(\mathbf{v}_*)]\,\mathrm{d}\sigma\mathrm{d}\mathbf{v}_*$$

$(\mathbf{v},\mathbf{v}_*)$ and $(\mathbf{v}',\mathbf{v}_*')$ are the velocity pairs before and after collision:

$$\begin{cases} \mathbf{v}' = \dfrac{\mathbf{v}+\mathbf{v}_*}{2} + \dfrac{|\mathbf{v}-\mathbf{v}_*|}{2}\sigma \\ \mathbf{v}_*' = \dfrac{\mathbf{v}+\mathbf{v}_*}{2} - \dfrac{|\mathbf{v}-\mathbf{v}_*|}{2}\sigma \end{cases}$$



$$B(\mathbf{v}-\mathbf{v}_*,\sigma) = B\left(|\mathbf{v}-\mathbf{v}_*|, \frac{\sigma\cdot(\mathbf{v}-\mathbf{v}_*)}{|\mathbf{v}-\mathbf{v}_*|}\right)$$

Variable hard sphere (VHS) model

$B = b_\lambda|\mathbf{v}-\mathbf{v}_*|^\lambda,\ -d < \lambda \le 1$

$\lambda = 1$: hard sphere molecule

$\lambda = 0$: Maxwell molecule

Properties of $\mathcal{Q}$

- **conservation** of mass, momentum, and energy:

$$\int_{\mathbb{R}^d} \mathcal{Q}(f,f)\,d\mathbf{v} = \int_{\mathbb{R}^d} \mathcal{Q}(f,f)\mathbf{v}\,d\mathbf{v} = \int_{\mathbb{R}^d} \mathcal{Q}(f,f)|\mathbf{v}|^2\,d\mathbf{v} = 0$$

- **Boltzmann's H-theorem**:

$$-\int_{\mathbb{R}^d} \mathcal{Q}(f,f)\ln f\,d\mathbf{v} \geq 0$$

- **equilibrium** function:

$$\text{``}=\text{''} \Longleftrightarrow \mathcal{Q}(f,f) = 0 \Longleftrightarrow f = \mathcal{M} := \frac{\rho}{(2\pi T)^{d/2}} e^{-\frac{(\mathbf{v}-\mathbf{u})^2}{2T}}$$

with density $\rho := \int f\,d\mathbf{v}$; bulk velocity $u := \frac{1}{\rho}\int f\mathbf{v}\,d\mathbf{v}$; temperature $T := \frac{1}{d\rho}\int f|\mathbf{v}-\mathbf{u}|^2\,d\mathbf{v}$

Maxwell boundary condition

For any boundary point $\mathbf{x} \in \partial\Omega$, let $n(\mathbf{x})$ be the unit normal vector to the boundary, pointed to the gas, then the **in-flow boundary** condition is: for $(\mathbf{v} - \mathbf{u}_w) \cdot n > 0$,

$$f(t, \mathbf{x}, \mathbf{v}) = (1-\alpha) f(t, \mathbf{x}, \mathbf{v} - 2[(\mathbf{v} - \mathbf{u}_w) \cdot n]n) \\ + \frac{\alpha}{(2\pi)^{\frac{d-1}{2}} T_w^{\frac{d+1}{2}}} e^{-\frac{|\mathbf{v}-\mathbf{u}_w|^2}{2T_w}} \int_{(\mathbf{v}-\mathbf{u}_w)\cdot n<0} f(t, \mathbf{x}, \mathbf{v}) |(\mathbf{v} - \mathbf{u}_w) \cdot n| \, d\mathbf{v}$$

- $\mathbf{u}_w = \mathbf{u}_w(t, \mathbf{x})$, $T_w = T_w(t, \mathbf{x})$ are the velocity and temperature of the wall (boundary)
- $0 \le \alpha \le 1$ is the **accommodation coefficient**
  $\alpha = 1$: purely diffusive boundary
  $\alpha = 0$: purely specular reflective boundary

Overview

Boltzmann equation with uncertainty

Sources of uncertainties in the Boltzmann equation:

- **collision kernel**: empirical collision kernels are usually used in numerical simulations which contain adjustable parameters whose values can be determined by matching directly with the measured scattering data or transport data
- **boundary data**, e.g. the wall temperature is given by measurement
- **initial data**, via initial macroscopic quantities, density, temperature, etc.

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To quantify these uncertainties and investigate the behavior of the solution, we adopt the generalized polynomial chaos (gPC) based stochastic Galerkin method.

Overview

Formulation of the problem

The stochastic Boltzmann equation can be formulated as follows:

$$\begin{cases} \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \dfrac{1}{\text{Kn}} \mathcal{Q}(f, f)(t, \mathbf{x}, \mathbf{v}, \mathbf{z}), \quad t > 0,\ \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^d,\ \mathbf{z} \in I_{\mathbf{z}}, \\ f(0, \mathbf{x}, \mathbf{v}, \mathbf{z}) = f^0(\mathbf{x}, \mathbf{v}, \mathbf{z}), \quad \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^d,\ \mathbf{z} \in I_{\mathbf{z}}, \\ f(t, \mathbf{x}, \mathbf{v}, \mathbf{z}) = g(t, \mathbf{x}, \mathbf{v}, \mathbf{z}), \quad t \geq 0,\ \mathbf{x} \in \partial\Omega,\ \mathbf{v} \in \mathbb{R}^d,\ \mathbf{z} \in I_{\mathbf{z}}, \end{cases}$$

where $\mathbf{z}$ is an n-**dimensional random vector** with support $I_{\mathbf{z}}$ characterizing the random inputs of the system. For simplicity, we assume $\mathbf{z}$ is a collection of random vectors $\mathbf{z}^B$, $\mathbf{z}^b$, $\mathbf{z}^i$ with mutually independent components

- collision kernel: $B = b_\lambda(\mathbf{z}^B)|\mathbf{v} - \mathbf{v}_*|^\lambda$
- boundary data: $T_w = T_w(t, \mathbf{x}, \mathbf{z}^b)$, $\mathbf{u}_w = \mathbf{u}_w(t, \mathbf{x}, \mathbf{z}^b)$
- initial data: $\rho^0(\mathbf{x}, \mathbf{z}^i)$, $T^0(\mathbf{x}, \mathbf{z}^i)$, $\mathbf{u}^0(\mathbf{x}, \mathbf{z}^i)$

Stochastic Galerkin method

We seek a solution in the following form

$$f(t,\mathbf{x},\mathbf{v},\mathbf{z}) \approx P_K f = \sum_{|\mathbf{k}|=0}^{K} f_{\mathbf{k}}(t,\mathbf{x},\mathbf{v})\Phi_{\mathbf{k}}(\mathbf{z}),$$

$$f_{\mathbf{k}}(t,\mathbf{x},\mathbf{v}) = \int_{I_{\mathbf{z}}} f(t,\mathbf{x},\mathbf{v},\mathbf{z})\Phi_{\mathbf{k}}(\mathbf{z})\pi(\mathbf{z})\,d\mathbf{z}.$$

Here $\mathbf{k} = (k_1,\ldots,k_n)$ is a multi-index with $|\mathbf{k}| = k_1 + \cdots + k_n$. $\{\Phi_{\mathbf{k}}(\mathbf{z})\}$ are orthonormal gPC basis functions satisfying

$$\int_{I_{\mathbf{z}}} \Phi_{\mathbf{k}}(\mathbf{z})\Phi_{\mathbf{j}}(\mathbf{z})\pi(\mathbf{z})\,d\mathbf{z} = \delta_{\mathbf{kj}}, \quad 0 \le |\mathbf{k}|, |\mathbf{j}| \le K,$$

where $\pi(\mathbf{z})$ is the probability distribution function of $\mathbf{z}$. The above approximation is optimal in space $\mathbb{P}_K^n$ (the set of all n-variate polynomials of degree up to K) in the sense that

$$\|f - P_K f\|_{L^2_\pi} = \inf_{h\in\mathbb{P}_K^n} \|f - h\|_{L^2_\pi}.$$

Stochastic Galerkin method (cont'd)

Inserting the gPC expansion into the Boltzmann equation, and performing standard Galerkin projection, we get

$$\begin{cases} \dfrac{\partial f_{\mathbf{k}}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_{\mathbf{k}} = \dfrac{1}{\mathsf{Kn}} Q_{\mathbf{k}}(P_K f, P_K f)(t, \mathbf{x}, \mathbf{v}), \quad t > 0,\ \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^d, \\ f_{\mathbf{k}}(0, \mathbf{x}, \mathbf{v}) = f_{\mathbf{k}}^0(\mathbf{x}, \mathbf{v}), \quad \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^d, \\ f_{\mathbf{k}}(t, \mathbf{x}, \mathbf{v}) = g_{\mathbf{k}}(t, \mathbf{x}, \mathbf{v}), \quad t \geq 0,\ \mathbf{x} \in \partial\Omega,\ \mathbf{v} \in \mathbb{R}^d \end{cases}$$

for each $0 \leq |\mathbf{k}| \leq K$, and

$$Q_{\mathbf{k}}(P_K f, P_K f) := \int_{I_{\mathbf{z}}} \mathcal{Q}(P_K f, P_K f)(t, \mathbf{x}, \mathbf{v}, \mathbf{z}) \Phi_{\mathbf{k}}(\mathbf{z}) \pi(\mathbf{z}) \, \mathrm{d}\mathbf{z},$$

$$f_{\mathbf{k}}^0 := \int_{I_{\mathbf{z}}} f^0(\mathbf{x}, \mathbf{v}, \mathbf{z}) \Phi_{\mathbf{k}}(\mathbf{z}) \pi(\mathbf{z}) \, \mathrm{d}\mathbf{z}, \quad g_{\mathbf{k}} := \int_{I_{\mathbf{z}}} g(t, \mathbf{x}, \mathbf{v}, \mathbf{z}) \Phi_{\mathbf{k}}(\mathbf{z}) \pi(\mathbf{z}) \, \mathrm{d}\mathbf{z}.$$

$$\mathbb{E}[f] = f_{\mathbf{0}}, \quad \mathrm{Var}[f] \approx \sum_{|\mathbf{k}|=1}^{K} f_{\mathbf{k}}^2, \quad S[f] \approx \sqrt{\sum_{|\mathbf{k}|=1}^{K} f_{\mathbf{k}}^2}.$$

Overview

Treatment of boundary condition

For the Maxwell boundary condition with uncertainty in the wall temperature T_w (assume $\mathbf{u}_w = 0$ for simplicity), $g_{\mathbf{k}}$ is given by

$$g_{\mathbf{k}} = (1-\alpha) f_{\mathbf{k}}(t, \mathbf{x}, \mathbf{v} - 2(\mathbf{v}\cdot n)n) + \alpha \sum_{|\mathbf{j}|=0}^{K} D_{\mathbf{kj}}(\mathbf{x}, \mathbf{v}) \int_{\mathbf{v}\cdot n<0} f_{\mathbf{j}}(t, \mathbf{x}, \mathbf{v}) |\mathbf{v}\cdot n| \, d\mathbf{v}$$

where

$$D_{\mathbf{kj}}(\mathbf{x}, \mathbf{v}) := \int_{I_{\mathbf{z}}} \frac{e^{-\frac{\mathbf{v}^2}{2T_w(\mathbf{x},\mathbf{z})}}}{(2\pi)^{\frac{d-1}{2}} T_w^{\frac{d+1}{2}}(\mathbf{x}, \mathbf{z})} \Phi_{\mathbf{k}}(\mathbf{z}) \Phi_{\mathbf{j}}(\mathbf{z}) \pi(\mathbf{z}) \, d\mathbf{z}.$$

Overview

Treatment of collision term

For the VHS collision kernel with uncertainty in b_λ, $Q_{\mathbf{k}}$ can be further expanded as

$$Q_{\mathbf{k}} = \sum_{|\mathbf{i}|,|\mathbf{j}|=0}^{K} S_{\mathbf{kij}} \int_{\mathbb{R}^d} \int_{S^{d-1}} |\mathbf{v} - \mathbf{v}_*|^\lambda \left[f_{\mathbf{i}}(\mathbf{v}') f_{\mathbf{j}}(\mathbf{v}'_*) - f_{\mathbf{i}}(\mathbf{v}) f_{\mathbf{j}}(\mathbf{v}_*) \right] \mathrm{d}\sigma \mathrm{d}\mathbf{v}_*,$$

where

$$S_{\mathbf{kij}} := \int_{I_{\mathbf{z}}} b_\lambda(\mathbf{z}) \Phi_{\mathbf{k}}(\mathbf{z}) \Phi_{\mathbf{i}}(\mathbf{z}) \Phi_{\mathbf{j}}(\mathbf{z}) \pi(\mathbf{z}) \, \mathrm{d}\mathbf{z}.$$

Note that $Q_{\mathbf{k}}$ still has 1, $\mathbf{v}$, $|\mathbf{v}|^2$ as collision invariants.

Evaluating $Q_{\mathbf{k}}$ is definitely the most expensive part. Can we do it efficiently?

Evaluating the collision operator — first reduction

$$Q_{\mathbf{k}} = \sum_{|\mathbf{i}|,|\mathbf{j}|=0}^{K} S_{\mathbf{kij}} \int_{\mathbb{R}^d} \int_{S^{d-1}} |\mathbf{v} - \mathbf{v}_*|^{\lambda} \left[f_{\mathbf{i}}(\mathbf{v}') f_{\mathbf{j}}(\mathbf{v}'_*) - f_{\mathbf{i}}(\mathbf{v}) f_{\mathbf{j}}(\mathbf{v}_*) \right] \mathrm{d}\sigma \mathrm{d}\mathbf{v}_*$$

For each fixed $\mathbf{k}$, decompose the symmetric matrix $(S_{\mathbf{kij}})_{N_K \times N_K}$ (via SVD) as

$$S_{\mathbf{kij}} = \sum_{r=1}^{R_{\mathbf{k}}} U_{\mathbf{i}r}^{\mathbf{k}} V_{r\mathbf{j}}^{\mathbf{k}}, \quad R_{\mathbf{k}} \leq N_K = \dim(\mathbb{P}_K^n) = \binom{n+K}{n}$$

Substituting it into $Q_{\mathbf{k}}$ and rearranging terms, we get

$$Q_{\mathbf{k}} = \sum_{r=1}^{R_{\mathbf{k}}} \int_{\mathbb{R}^d} \int_{S^{d-1}} |\mathbf{v} - \mathbf{v}_*|^{\lambda} \left[g_r^{\mathbf{k}}(\mathbf{v}') h_r^{\mathbf{k}}(\mathbf{v}'_*) - g_r^{\mathbf{k}}(\mathbf{v}) h_r^{\mathbf{k}}(\mathbf{v}_*) \right] \mathrm{d}\sigma \mathrm{d}\mathbf{v}_*$$

$$g_r^{\mathbf{k}}(\mathbf{v}) := \sum_{|\mathbf{i}|=0}^{K} U_{\mathbf{i}r}^{\mathbf{k}} f_{\mathbf{i}}(\mathbf{v}), \quad h_r^{\mathbf{k}}(\mathbf{v}) := \sum_{|\mathbf{i}|=0}^{K} V_{r\mathbf{i}}^{\mathbf{k}} f_{\mathbf{i}}(\mathbf{v}).$$

Evaluating the collision operator — second reduction

Note that

$$Q_{\mathbf{k}} = \sum_{r=1}^{R_{\mathbf{k}}} \int_{\mathbb{R}^d} \int_{S^{d-1}} |\mathbf{v} - \mathbf{v}_*|^{\lambda} \left[g_r^{\mathbf{k}}(\mathbf{v}') h_r^{\mathbf{k}}(\mathbf{v}'_*) - g_r^{\mathbf{k}}(\mathbf{v}) h_r^{\mathbf{k}}(\mathbf{v}_*) \right] \mathrm{d}\sigma \mathrm{d}\mathbf{v}_*$$

$$= \sum_{r=1}^{R_{\mathbf{k}}} \mathcal{Q}(g_r^{\mathbf{k}}, h_r^{\mathbf{k}}), \quad \mathcal{Q} \text{ is the original deterministic collision operator}$$

One can apply the fast Fourier spectral method[1] in velocity space (with slight modification).

[1] Mouhot and Pareschi, 2006.

Combining everything ...

Finally, the computational cost for evaluating

$$Q_{\mathbf{k}} = \sum_{|\mathbf{i}|,|\mathbf{j}|=0}^{K} S_{\mathbf{kij}} \int_{\mathbb{R}^d} \int_{S^{d-1}} |\mathbf{v} - \mathbf{v}_*|^{\lambda} \left[f_{\mathbf{i}}(\mathbf{v}') f_{\mathbf{j}}(\mathbf{v}'_*) - f_{\mathbf{i}}(\mathbf{v}) f_{\mathbf{j}}(\mathbf{v}_*) \right] \mathrm{d}\sigma \mathrm{d}\mathbf{v}_*$$

would be

$$O(N_K^2 N_\sigma^{d-1} N_{\mathbf{v}}^{2d}) \Longrightarrow O(R_{\mathbf{k}} N_\sigma^{d-1} N_{\mathbf{v}}^{2d}) \Longrightarrow O(R_{\mathbf{k}} N_\sigma^{d-1} N_{\mathbf{v}}^{d} \log N_{\mathbf{v}})$$

$R_{\mathbf{k}} \leq N_K = \binom{n+K}{n}$ is the dimension of n-variate polynomials of degree up to K ($N_K = 120$ if $K = 7, n = 3$; $N_K = 792$ if $K = 7, n = 5$), d is the dimension of velocity space (typically $d = 2$ or 3), N_σ is the number of discrete points in each angular direction, and $N_{\mathbf{v}}$ is the number of points in each velocity direction (typically $N_\sigma \ll N_{\mathbf{v}}$).

Time/spatial discretization and others

- We use the time-splitting framework to solve the convection part and collision part separately
- MUSCL scheme with slope limiter is applied to the spatial discretization
- The random variable $\mathbf{z}$ is assumed to be uniform distribution (Legendre polynomial chaos)
- Given $f_{\mathbf{k}}$, $\rho_{\mathbf{k}}$ is obtained by direct integration; $\rho_{\mathbf{k}}^{-1}$ is obtained by solving linear system $\rho\rho^{-1} = 1$; $\mathbf{u}_k$ and $T_{\mathbf{k}}$ are then computed in terms of $\rho_{\mathbf{k}}^{-1}$

A spectral accuracy analysis

Consider

$$\frac{\partial f_{\mathbf{k}}}{\partial t} = Q_{\mathbf{k}}(P_K f, P_K f)(t, \mathbf{v}),$$

assume

$$\hat{f}(t, \mathbf{v}, \mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \hat{f}_{\mathbf{k}}(t, \mathbf{v}) \Phi_{\mathbf{k}}(\mathbf{z})$$

is the exact solution, then under some regularity conditions, one can show that

$$\|\hat{f} - P_K f\|_{L^2_{\mathbf{v}}} \le C(t) \left\{ \frac{1}{K^m} + \|\mathbf{e}(0)\|_{L^2_{\mathbf{v}}} \right\},$$

where

$$e_{\mathbf{k}} = \hat{f}_{\mathbf{k}} - f_{\mathbf{k}}, \quad |\mathbf{k}| \le K, \qquad \mathbf{e} = (e_1, \cdots, e_{N_K})^T.$$

Overview

Homogeneous BGK equation

$$\frac{\partial f}{\partial t} = B(\mathbf{z})(\mathcal{M} - f), \quad v \in \mathbb{R}$$

with collision kernel

$$B(\mathbf{z}) = 1 + s_1 z_1 + s_2 z_2, \quad s_1 = 0.2, \; s_2 = 0.1,$$

and initial condition

$$f^0(v) = v^2 e^{-v^2}.$$

This is a particularly simple example where the Maxwellian $\mathcal{M}$ neither depends on $\mathbf{z}$ nor changes in time.

Homogeneous BGK equation

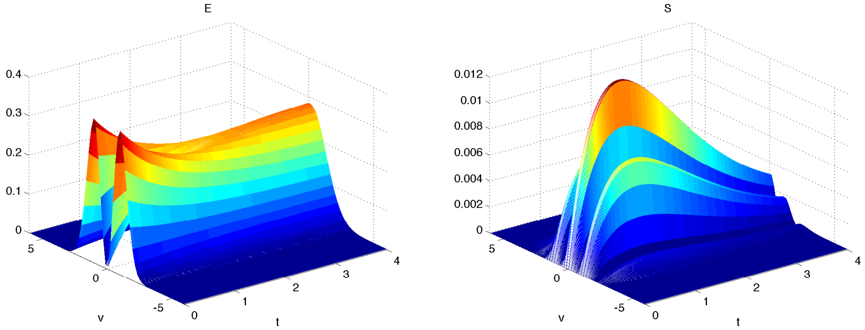

Figure: Left: $\mathbb{E}[f](t,v)$. Right: $S[f](t,v)$. $K = 7$, $N_v = 64$, $\Delta t = 0.2/32$, RK-4 for time discretization.

Homogeneous BGK equation

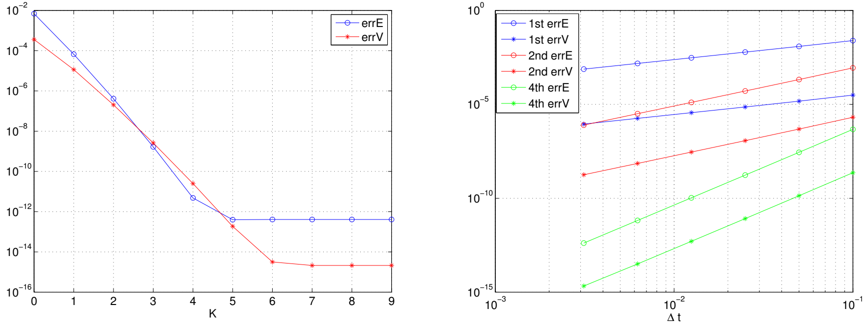

Figure: Left: spectral accuracy in K ($\Delta t = 0.2/64$, RK-4 for time discretization). Right: 1st, 2nd, and 4th order accuracy in time ($K = 9$).
$\text{errE} = \|\mathbb{E}[f] - \mathbb{E}[f^{\text{ext}}]\|_{l^1(t,v)}$, $\text{errV} = \|\text{Var}[f] - \text{Var}[f^{\text{ext}}]\|_{l^1(t,v)}$.

Boltzmann equation with random collision kernel

Assume the collision kernel

$$B(z) = 1 + sz, \quad s = 0.6,$$

the continuous initial data

$$f^0(x, \mathbf{v}) = \frac{\rho^0(x)}{4\pi T^0(x)} \left(e^{-\frac{|\mathbf{v}-\mathbf{u}^0(x)|^2}{2T^0(x)}} + e^{-\frac{|\mathbf{v}+\mathbf{u}^0(x)|^2}{2T^0(x)}} \right), \quad x \in [0, 1],$$

where

$$\rho^0(x) = \frac{2 + \sin(2\pi x)}{3}, \quad \mathbf{u}^0 = (0.2, 0), \quad T^0 = \frac{3 + \cos(2\pi x)}{4},$$

periodic boundary condition in x, and $\mathrm{Kn} = 0.1$.

Compare with stochastic collocation on a finer mesh using 20 Gauss-Legendre quadrature points.

Boltzmann equation with random collision kernel

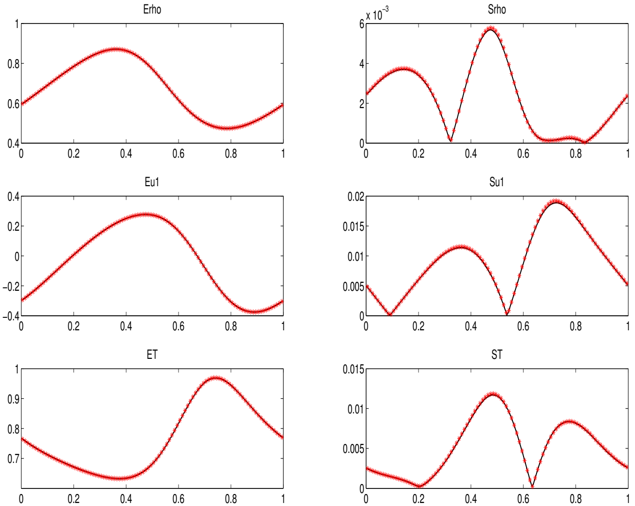

Figure: Solutions at $t = 0.2$. Solid line: collocation with $N_z = 20$, $N_{\mathbf{v}} = 64$, $N_\sigma = 8$, $N_x = 200$. Red star: Galerkin with $K = 7$, $N_{\mathbf{v}} = 32$, $N_\sigma = 4$, $N_x = 100$.

Boltzmann equation with random initial data

Consider the same initial data as before except

$$\rho^0(x, \mathbf{z}) = \frac{2 + \sin(2\pi x) + \frac{1}{2}\sin(4\pi x) z_1 + \frac{1}{3}\sin(6\pi x) z_2}{3},$$

$$T^0(x, \mathbf{z}) = \frac{3 + \cos(2\pi x) + \frac{1}{2}\cos(4\pi x) z_1 + \frac{1}{3}\cos(6\pi x) z_2}{4}.$$

These are chosen to mimic the K-L expansion. Periodic boundary condition is assumed in x.

Boltzmann equation with random initial data

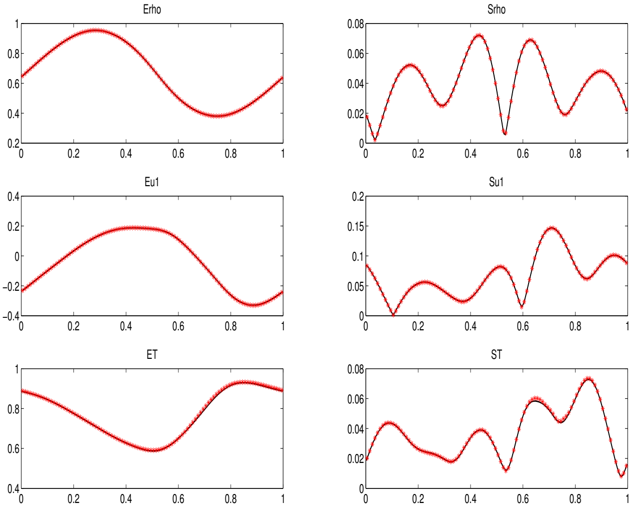

Figure: Solutions at $t = 0.1$. Solid line: collocation with $N_{\mathbf{z}} = 10$, $N_{\mathbf{v}} = 64$, $N_\sigma = 8$, $N_x = 200$. Red star: Galerkin with $K = 5$, $N_{\mathbf{v}} = 32$, $N_\sigma = 4$, $N_x = 100$.

Boltzmann equation with random initial data — shock tube problem

Consider the equilibrium initial condition with random macroscopic quantities:

$$\begin{cases} \rho_l = 1 + s_1\left(\frac{z+1}{2}\right), \quad u_l = 0, \quad T_l = 1 + s_2 z, \quad x \leq 0.5, \\ \rho_r = 0.125, \quad u_r = 0, \quad T_r = 0.25, \quad x > 0.5. \end{cases}$$

with $s_1 = 0.2$, $s_2 = 0.1$.

Boltzmann equation with random initial data

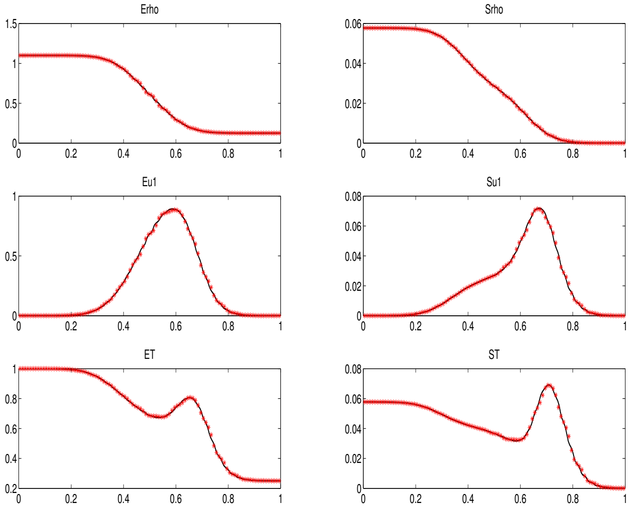

Figure: Solutions at $t = 0.1$. Solid line: collocation with $N_z = 20$, $N_{\mathbf{v}} = 64$, $N_\sigma = 8$, $N_x = 200$. Red star: Galerkin with $K = 7$, $N_{\mathbf{v}} = 32$, $N_\sigma = 4$, $N_x = 100$.

Boltzmann equation with random boundary data — sudden heating problem[2]

The gas is initially in a constant state

$$f^0(x, \mathbf{v}) = \frac{1}{2\pi T^0} e^{-\frac{\mathbf{v}^2}{2T^0}}, \quad T^0 = 1, \quad x \in [0, 1].$$

At time $t = 0$, suddenly change the wall temperature at left boundary to

$$T_w(z) = 2(T_0 + sz), \quad s = 0.2.$$

Assume purely diffusive Maxwell boundary condition at $x = 0$, and $\mathrm{Kn} = 0.1$.

[2] Aoki et al, 1991, Gamba et al, 2010, Filbet, 2012.

Boltzmann equation with random boundary data

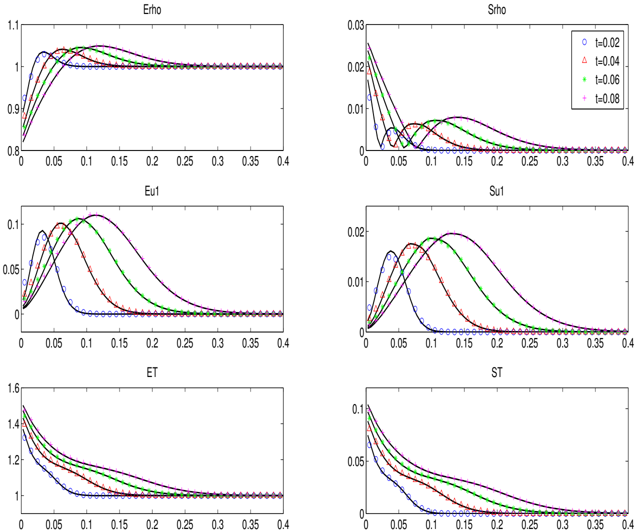

Figure: Solid line: collocation with $N_z = 20$, $N_{\mathbf{v}} = 64$, $N_\sigma = 8$, $N_x = 200$. Other legends are the Galerkin solutions at different time with $K = 7$, $N_{\mathbf{v}} = 32$, $N_\sigma = 4$, $N_x = 100$.

Overview

Conclusion and ongoing work

We introduced a gPC based stochastic Galerkin method for the nonlinear Boltzmann equation with uncertainty:

- can quantify random inputs from collision kernel, initial data, and boundary data
- a fast algorithm is constructed to accelerate the computation of collision operator

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- dimension reduction for high dimensional random input
- design asymptotic-preserving scheme in the near fluid regime (kinetic scheme for the compressible Euler equation with uncertainty)

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Thank you!