Stein's method for Gibbs point processes

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joint work with Dominic Schuhmacher

Nantes, April 7 2016

Theorem (Stein's Lemma)

Let $Z \sim \mathcal{N}(0,1)$. Then

$$\mathbf{E}f'(Z) - \mathbf{E}Zf(Z) = 0$$

for all functions such that the above expectations exist. Conversely, every random variable satisfying this equation for a large enough class of functions f is necessarily the standard normal distribution.

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- The operator $(\mathcal{A}f)(x) = f'(x) - xf(x)$ characterises the normal distribution.

- The *Stein equation*

$$f(x) - \mathbf{E}f(Z) = \mathcal{A}h_f(x)$$

is solved by the *Stein solution*

$$h_f(x) = e^{\frac{x^2}{2}} \int_{-\infty}^{x} (f(y) - \mathbf{E}f(Z))e^{-\frac{y^2}{2}}\, dy.$$

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- Assume we want to bound $\sup_{f \in \mathcal{F}} |\mathbf{E}f(X) - \mathbf{E}f(Z)|$. Using the Stein equation we can bound $\sup_{f \in \mathcal{F}} |\mathbf{E}\mathcal{A}h_f(X)|$ instead.
- The *Stein factors* $\|h'_f\|$ and $\|h''_f\|$ play a crucial role in bounding $\mathbf{E}\mathcal{A}h_f(X)$.

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- Every $\xi \in \mathfrak{N}$ can be written as $\xi = \sum_{i=1}^{n} \delta_{x_i}$ for some $x_1, \ldots, x_n \in \mathcal{X}$, and where δ_x denotes the Dirac measure at point x.

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(b) *The numbers of points in any two disjoint sets are independent:*
$\Xi(A)$, $\Xi(B)$ independent for any $A, B \in \mathcal{B}$ with $A \cap B = \emptyset$.

- We write $\Xi \sim \text{PoP}(\lambda)$.

- A function $u\colon \mathfrak{N} \to \mathbb{R}_+$ is called **hereditary** if $u(\xi) = 0$ implies $u(\eta) = 0$ for all point configurations $\xi, \eta \in \mathfrak{N}$ with $\xi \subset \eta$.

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- A Gibbs process is completely described by its **conditional intensity** $\lambda(\cdot \,|\, \cdot)$, where

$$\lambda(x \,|\, \xi) = \frac{u(\xi + \delta_x)}{u(\xi)} \quad \text{for all } \xi \in \mathfrak{N},\ x \in \mathcal{X} \text{ with } \xi(\{x\}) = 0.$$

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- We write $\mathrm{Gibbs}(\lambda)$ for the distribution of this Gibbs process.

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- A Gibbs process is a *pairwise interaction process* (PIP) if the conditional intensity is of the form $\lambda(x \,|\, \xi) = \beta(x) \prod_{y \in \xi} \varphi(x, y)$, for a $\beta \colon \mathcal{X} \to \mathbb{R}_+$ and a symmetric *interaction function* $\varphi \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, e.g. for the *Strauss process*

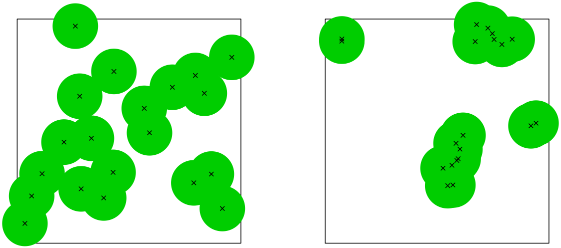
$$\varphi(x, y) = \begin{cases} \gamma & \text{if} \quad d_0(x, y) \le r; \\ 1 & \text{otherwise,} \end{cases}$$

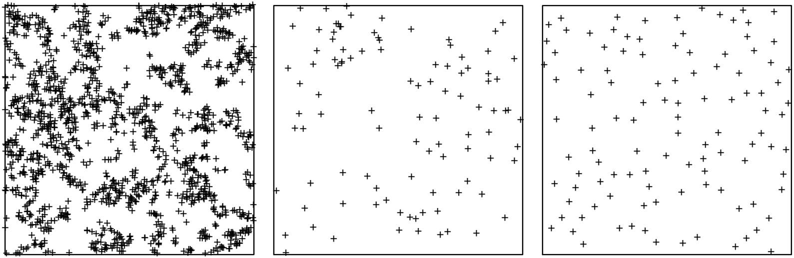
for a $r > 0$ and a $\gamma \in [0, 1]$.

- The *area-interaction process (AIP)* has the conditional intensity

$$\lambda(x \mid \xi) = \beta\gamma^{-\alpha(U_r(\xi+\delta_x)\setminus U_r(\xi))},$$

where $\beta, \gamma, r > 0$ and $U_r(\xi) = \bigcup_{x\in\xi} \mathbb{B}_r(x)$ denotes the green area.





Left: Aip with $\gamma = 100$, Middle: Aip with $\gamma = 0.01$, Right: Strauss process with $\gamma = 0$

- Suppose that we have birth rates and death rates

$$b(\cdot \,|\, \cdot) \colon \mathcal{X} \times \mathfrak{N} \to \mathbb{R}_+ \quad \text{with } \bar{b}(\xi) := \int b(x \,|\, \xi)\, \alpha(dx) < \infty;$$
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- Let $\bar{a}(\xi) = \bar{b}(\xi) + \bar{d}(\xi)$.
- A $\mathrm{SBD}^{(\xi_0)}(b, d)$-process is a pure-jump Markov process on $\mathfrak{N}$ that starts in $\xi_0 \in \mathfrak{N}$ and holds each state ξ for an $\mathrm{Exp}(\bar{a}(\xi))$-distributed time, after which
  - (a) with probability $\bar{b}(\xi)/\bar{a}(\xi)$ a point is added, positioned according to the density $b(\cdot \mid \xi)\big/\bar{b}(\xi)$, or
  - (b) with probability $d(x \mid \xi)/\bar{a}(\xi)$ the point at x is deleted.

- In what follows always $b(\cdot\,|\,\cdot) = \lambda(\cdot\,|\,\cdot)$, $d \equiv 1$ ("unit per-capita death rate") and $\lambda(\cdot\,|\,\cdot)$ is *locally stable*, i.e. there exists a constant c^* such that

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- Let $Z = \big(Z(t)\big)_{t \geq 0} \sim \mathrm{SBD}(\lambda, 1)$. Then
- Z is non-explosive;
- Z has Gibbs(λ) as its unique stationary distribution.
- Z has the infinitesimal generator

$$\mathcal{A}h(\xi) = \int_{\mathcal{X}} \big[h(\xi+\delta_x) - h(\xi)\big]\,\lambda(x\,|\,\xi)\,\alpha(dx) + \int_{\mathcal{X}} \big[h(\xi-\delta_x) - h(\xi)\big]\,\xi(dx)$$

for certain functions $h\colon \mathfrak{N} \to \mathbb{R}$.

- Our goal is to define a coupling of two Z and $\tilde{Z}$ SBD$(\lambda, 1)$'s which follow the same dynamics but are started in different configurations.
- Given that at time t the processes are in states $Z(t) = \xi$ and $\tilde{Z} = \tilde{\xi}$, propose a birth with rate $\max(\lambda(\cdot\,|\,\xi), \lambda(\cdot\,|\,\tilde{\xi}))$. The first process Z accepts it with probability $\lambda(\cdot\,|\,\xi)/\max(\lambda(\cdot\,|\,\xi), \lambda(\cdot\,|\,\tilde{\xi}))$ and $\tilde{Z}$ accepts it with probability $\lambda(\cdot\,|\,\xi)/\max(\lambda(\cdot\,|\,\tilde{\xi}), \lambda(\cdot\,|\,\tilde{\xi}))$.

Coupling

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- The deaths are coupled in the obvious manner, each point dies with rate one independently of the others but of course the common points of Z and $\tilde{Z}$ die together.

Expected coupling time

- Let

$$\varepsilon = \sup_{\|\xi-\eta\|=1} \int_{\mathcal{X}} |\lambda(x \mid \xi) - \lambda(x \mid \eta)| \, \alpha(dx) \quad \text{and} \quad c = c^* \alpha(\mathcal{X}).$$

- Consider the *coupling time* $\tau = \inf\{t \geq 0 \colon Z_t^{\xi} = \tilde{Z}_t^{\xi+\delta_x}\}$.

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Example

For a Strauss processes in $\mathbb{R}^d$ with parameters β, γ, R we get $\varepsilon = \beta(1-\gamma)\alpha_d R^d$, where α_d is the volume of the unit ball in $\mathbb{R}^d$.

Our goal:
Find upper bound for the total variation distance

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- **Barbour and Brown (1992).** Poisson process approximation.

Set-up for general probability metrics

- Let $\mathrm{H} \sim \mathrm{Gibbs}(\lambda)$. Suppose we want to bound

$$d\big(\mathscr{L}(\Xi), \mathrm{Gibbs}(\lambda)\big) = \sup_{f \in \mathcal{F}} \big| \mathbb{E} f(\Xi) - \mathbb{E} f(\mathrm{H}) \big|,$$

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  where $\mathcal{F}$ is some class of functions.
- For the total variation metric

$$\mathcal{F} = \mathcal{F}_{TV} = \{1_C\,;\; C \in \mathcal{N}\}.$$

Setting up the Stein equation

- For every $f \in \mathcal{F}$ find $h = h_f : \mathfrak{N} \to \mathbb{R}$ such that

$$f(\xi) - \mathbb{E}f(\mathrm{H}) = \mathscr{A}h_f(\xi) \quad \text{for all } \xi \in \mathfrak{N}, \qquad \text{(Stein equation)}$$

where $\mathscr{A}$ is the generator of a Markov process with stationary distribution Gibbs(λ) (generator approach).

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- Natural choice: the $\mathrm{SBD}^{(\xi)}(\lambda, 1)$-process $Z^{(\xi)} := \big(Z_t^{(\xi)}\big)_{t \geq 0}$ from earlier.

- It can be shown that for bounded f the function $h = h_f : \mathfrak{N} \to \mathbb{R}$,

$$h(\xi) := -\int_0^\infty \left[\mathbb{E}f(Z_t^{(\xi)}) - \mathbb{E}f(\mathrm{H})\right] dt,$$

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- To bound $\sup_{f \in \mathcal{F}_{TV}} |\mathbf{E}\mathcal{A}h_f(\Xi)|$ it turns out that the key ingredient is the *Stein factor*

$$c_1(\lambda) = \sup_{f \in \mathcal{F}_{TV}, x \in \mathcal{X}, \xi \in \mathcal{N}} |h_f(\xi + \delta_x) - h_f(\xi)|.$$

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$$c_1(\lambda) = \sup_{f \in \mathcal{F}_{TV}, x \in \mathcal{X}, \xi \in \mathcal{N}} |h_f(\xi + \delta_x) - h_f(\xi)|.$$

- Note that

$$h_f(\xi + \delta_x) - h_f(\xi) = -\int_0^\infty \left[\mathbb{E}f(Z_t^{(\xi)}) - \mathbb{E}f(Z_t^{(\xi+\delta_x)})\right] dt.$$

- Thus $c_1(\lambda)$ can be bounded by the expected coupling time of two SBDP's starting in configurations differing by one point.

Upper bound

Theorem (Schuhmacher and S, 2012)

For any two Gibbs point processes

Ξ with conditional intensity $\nu(\cdot \mid \cdot)$,

H with locally stable conditional intensity $\lambda(\cdot \mid \cdot)$,

there exists a finite constant $c_1(\lambda)$ such that

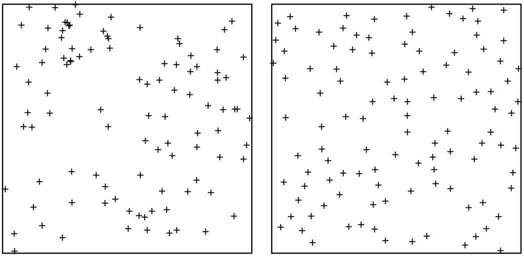
$$d_{\mathrm{TV}}\big(\mathscr{L}(\Xi), \mathscr{L}(\mathrm{H})\big) \leq c_1(\lambda) \int_{\mathcal{X}} \mathbb{E}\big|\nu(x \mid \Xi) - \lambda(x \mid \Xi)\big| \, \alpha(dx).$$

The Stein factor $c_1(\lambda)$ is bounded by the expected coupling time from earlier.

- Suppose that $\mathcal{X} \subset \mathbb{R}^d$, and $\Xi \sim$PIP(β, φ_1) and H $\sim$PIP(β, φ_2) are stationary and inhibitory, i.e. β is constant and $\varphi_i(x, y) = \varphi_i(x - y) \leq 1$ for all $x, y \in \mathcal{X}$. Then

$$d_{TV}(\mathscr{L}(\Xi), \mathscr{L}(\mathrm{H})) \leq c_1(\lambda)\,\beta\,\mathbb{E}|\Xi| \int_{\mathbb{R}^d} |\varphi_1(x) - \varphi_2(x)|\,dx.$$

Convergence of the Area interaction process



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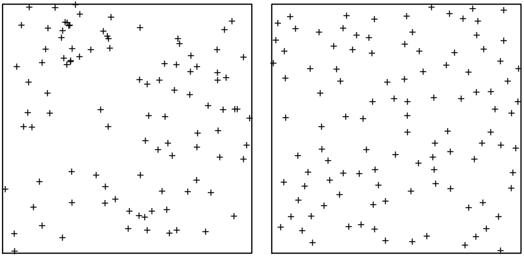

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- Suppose that $\mathcal{X} \subset \mathbb{R}^d$, and $\Xi \sim \mathrm{AIP}\big(\beta\gamma^{\alpha_d (R/2)^d}, \gamma; R/2\big)$ and $\mathrm{H} \sim \mathrm{Strauss}(\beta, 0; R)$, where α_d is the volume of the unit ball in $\mathbb{R}^d$. Then

$$d_{TV}(\mathscr{L}(\Xi), \mathscr{L}(\mathrm{H})) \le c_1(\lambda)\, 2d\alpha_d R^{d-1}\, \beta\, \mathbb{E}|\Xi| \big(\log \gamma^{-\alpha_d}\big)^{-1/d}.$$

Non locally stable Gibbs processes

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Non locally stable Gibbs processes

- Although most of the Gibbs processes considered in spatial statistics are locally stable, there exist some notable exceptions (e.g. the *Lennard - Jones process*).
- For pairwise interaction processes satisfying some mild assumptions, the previous Theorem can be generalised to

$$d_{\mathrm{TV}}\big(\mathscr{L}(\Xi), \mathscr{L}(\mathrm{H})\big) \leq C_1 \|\varphi_1 - \varphi_2\|_{L^1} + C_2(C_1),$$

where the constant C_2 can be chosen arbitrarily small causing a larger C_1.

- Assume that $\mathcal{F} = \{f\}$ consists of only one function, namely $f(\xi) = \prod_{x \in \xi} g(x)$. Then

$$\Psi_{\Xi}(g) = \mathbf{E}\Big(\prod_{x \in \Xi} g(x) \Big)$$

is called probability generating functional.

The probability generating functional

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  is called probability generating functional.
- Assume that our our Gibbs processes live on a subset of $\mathbb{R}^d$ and that the H is a homogeneous Poisson process with intensity ν. Then from the Stein equation we get

$$\Psi_\Xi(g) - \exp\Big(-\nu\int_{\mathbb{R}^d} 1 - g(x)\,dx\Big) = \mathbf{E}\big(\mathcal{A}h_f(\Xi)\big).$$

Theorem

Let Ξ be a stationary and locally stable Gibbs point process on $\mathbb{R}^d$ with intensity λ and local stability constant c^*. Then

$$1 - \lambda G \leq \Psi_\Xi(g) \leq 1 - \frac{\lambda}{c^*}\left(1 - e^{-c^* G}\right),$$

where $G = \int_{\mathbb{R}^d} 1 - g(x)\,dx$.

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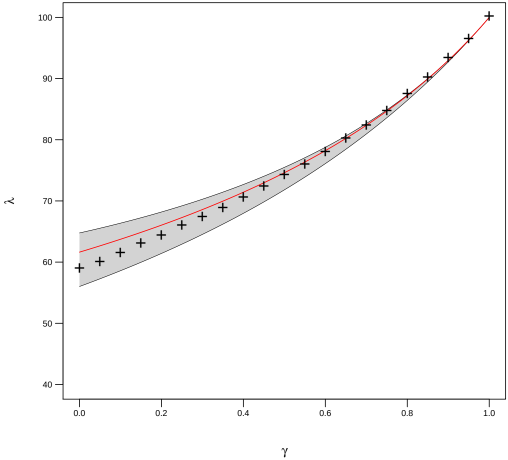
Theorem

Let $\Xi \sim \mathrm{PIP}(\beta, \varphi)$ *be inhibitory, with finite interaction range, i.e.* $1 - \varphi$ *has bounded support, and with intensity* λ*. Then*

$$\frac{\beta}{1 + \beta G} \leq \lambda \leq \frac{\beta}{2 - e^{-\beta G}},$$

where $G = \int_{\mathbb{R}^d} 1 - \varphi(x)\, dx$*.*



The graphic shows the intensities of Strauss processes in $\mathbb{R}^2$ with $\beta = 100$ and $r = 0.05$. The crosses are simulated values, the red line is the PS-approximation and the grey area corresponds to our bounds.

D. Schuhmacher and K. Stucki.
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