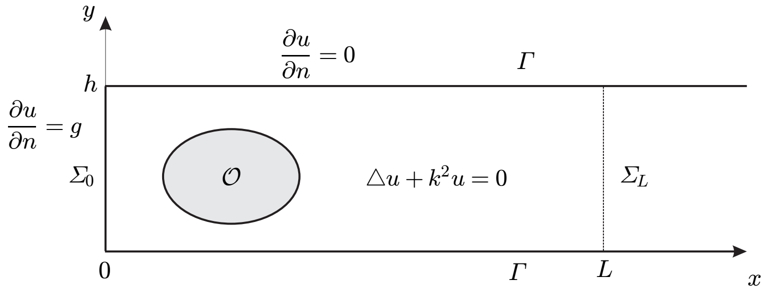
XLiFE++ Practice 1

Helmholtz problem in a waveguide

We are interested in solving Helmholtz 2D problem in a waveguide with different numerical methods. This is a locally perturbated semi-infinite waveguide (perturbation is located in $]0, L[\times[0, h])$:



The problem to solve is :

$$(1)\quad \begin{cases} \Delta u + k^2 u = 0 & \text{on } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \cup \partial\mathcal{O} \\ \dfrac{\partial u}{\partial n} = g & \text{on } \Sigma_0 \\ u(x,y) = \displaystyle\sum_{n\geq 0} \alpha_n\, e^{i\beta_n x} \varphi_n(y) & \forall x > L \quad \textit{(radiation condition)} \end{cases}$$

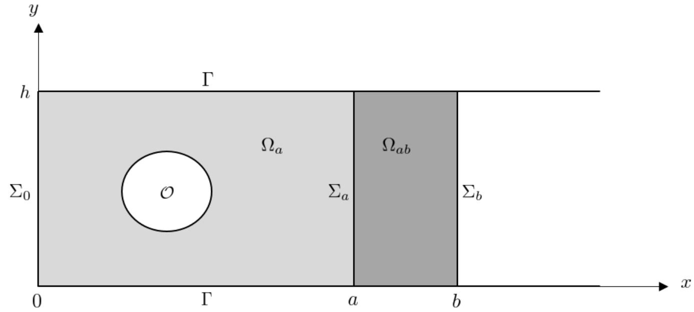
$$\text{where } \begin{cases} \beta_n = \sqrt{k^2 - \left(\dfrac{n\pi}{h}\right)^2} \text{ with } \mathcal{I}m\, \beta_n \geq 0 \text{ and } \mathcal{R}e\, \beta_n \geq 0 \\ \varphi_n(y) = a_n \cos(\dfrac{n\pi y}{h}) \text{ with } a_n = \sqrt{\dfrac{2}{h}} \text{ if } n > 0 \text{ and } a_0 = \sqrt{\dfrac{1}{h}}. \end{cases}$$

Outside the perturbated area, the solution can be written as a series (Σ_a being any section of the waveguide with $a > L$):

$$(2)\quad u(x,y) = \sum_{n\geq 0} \left(\int_{\Sigma_a} u\varphi_n \right) e^{i\beta_n(x-a)} \varphi_n(y) \quad \forall x \geq a,\, \forall y \in]0, h[.$$

The purpose of this training consists in coding the following methods (coupled with a finite elements approximation): low frequency approximation, PML method, and DtN boundary condition.

In the following, the following geometrical configuration will be used:



Exercise 1 - Low frequency approximation

When $k < \frac{\pi}{h}$ there is only one propagative mode ($n = 0$, $\beta_0 = k$). Every other mode are exponential decaying, so they can be neglected. Far from the perturbation, solution looks like $a\, e^{i\,k\,x}$ and satisfies:

$$\frac{\partial u}{\partial n} \approx i\,k\,u \text{ on } \Sigma_a \ (a > L).$$

In this context, the following problem is solved:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_a = \Omega \cap]0,a[\times]0,h[\\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \cup \partial\mathcal{O} \\ \dfrac{\partial u}{\partial n} = g & \text{on } \Sigma_0 \\ \dfrac{\partial u}{\partial n} = i\,k\,u & \text{on } \Sigma_a \ (a > L) \end{cases}$$

Its weak form is:

(3)
$$\left| \begin{array}{l} \text{Find } u \in H^1(\Omega_a) \text{ such that } \forall v \in H^1(\Omega_a) \\ \displaystyle\int_{\Omega_a} \nabla u \nabla \bar{v} - k^2 \int_{\Omega_a} u\,\bar{v} - ik \int_{\Sigma_a} u\,\bar{v} = \int_{\Sigma_0} g\,\bar{v} \end{array} \right.$$

a - 2D program without perturbation
Using Lagrange elements to approximate u, write a program using XLiFE++ to solve 3 when no perturbation.

b - Error computation
With exact solutions when no perturbation, compute the L^2 error u.

c - 2D program with perturbation
Using Lagrange elements to approximate u, write a program using XLiFE++ to solve 3 when there is a circular perturbation.

Exercise 2 - PML approximation

The Perfectly Match Layer method (PML hereafter) consists in considering a layer with the material properties so that every mode is becomes evanescent . As a result, we obtain an absorbing layer. PML are built so that there is no reflexion on interfaces of layers.

We proceed by rewriting Helmholtz operator in PML layer as follows:

$$\frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial}{\partial x}\left(\alpha \frac{\partial u}{\partial x}\right) + k^2 u = 0$$

where $\alpha = \alpha(x)$. A sufficient condition so that the layer is absorbing is:

$$\mathcal{R}e\,\alpha > 0 \text{ and } \mathcal{I}m\,\alpha < 0$$

Introducing function:

$$\tilde{\alpha}(x) = \begin{cases} 1 & \text{if } x < a \\ \alpha(x) & \text{if } a < x < b \end{cases}$$

In this context, the problem to be solved is:

$$\begin{cases} \dfrac{\partial^2 u}{\partial y^2} + \tilde{\alpha}\dfrac{\partial}{\partial x}\left(\tilde{\alpha}\dfrac{\partial u}{\partial x}\right) + k^2 u = 0 & \text{in } \Omega_b \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \cup \partial\mathcal{O} \cup \Sigma_b \\ \dfrac{\partial u}{\partial n} = g & \text{on } \Sigma_0 \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Sigma_b. \end{cases}$$

Its weak form is:

$$(4) \quad \left| \begin{array}{l} \text{Find } u \in H^1(\Omega) \text{ such that } \forall v \in H^1(\Omega) \\ \displaystyle\int_{\Omega_a} \nabla u \nabla \bar{v} - k^2 \int_{\Omega_a} u\,\bar{v} + \frac{1}{\alpha}\int_{\Omega_b} \frac{\partial u}{\partial y}\frac{\partial \bar{v}}{\partial y} + \alpha \int_{\Omega_b} \frac{\partial u}{\partial x}\frac{\partial \bar{v}}{\partial x} - \frac{k^2}{\alpha}\int_{\Omega_b} u\,\bar{v} = \int_{\Sigma_0} g\,\bar{v} \end{array} \right.$$

a - 2D program without perturbation
Using Lagrange elements to approximate u, write a program using XLiFE++ to solve 4 when no perturbation.

b - Error computation
With exact solutions when no perturbation, compute the L^2 error u.

c - 2D program with perturbation
Using Lagrange elements to approximate u, write a program using XLiFE++ to solve 4 when there is a circular perturbation. You could try to change mesh step and coefficient and size of the PML.

Exercise 3 - DtN approximation

Series representation of the outer solution (2) allows to build a Dirichlet-to-Neuman operator that matches the values and the derivative values of the inner solution with those of the outer solution:

$$\frac{\partial u}{\partial n} = Tu \stackrel{def}{=} \sum_{n \geq 0} i\,\beta_n \left(\int_{\Sigma_a} u\varphi_n\right) \varphi_n(y) \text{ on } \Sigma_a \ (a > L).$$

Numerically, the sum is truncated to rank N.

$$T_n\, u = \sum_{0 \leq n \leq N} i\,\beta_n \left(\int_{\Sigma_a} u\varphi_n\right) \varphi_n(y) \text{ on } \Sigma_a \ (a > L).$$

So the problem to be solved is

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_a \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \cup \partial\mathcal{O} \\ \dfrac{\partial u}{\partial n} = g & \text{on } \Sigma_0 \\ \dfrac{\partial u}{\partial n} = T_N u & \text{on } \Sigma_a \ (a > L). \end{cases}$$

and its weak form is:

$$(5)\quad \left| \begin{array}{l} \text{Find } u \in H^1(\Omega_a) \text{ such that } \forall v \in H^1(\Omega_a) \\ \displaystyle\int_{\Omega_a} \nabla u \nabla \bar{v} - k^2 \int_{\Omega_a} u\,\bar{v} - \int_{\Sigma_a}\int_{\Sigma_a} u(x) \left(\sum_{0\le n\le N} i\beta_n\,\varphi_n(x)\,\varphi_n(y) \right) \bar{v}(y)\,dx\,dy = \int_{\Sigma_0} g\,\bar{v} \end{array} \right.$$

You can see a double integral with a kernel defined as a sum. This is what we call a `TensorKernel` in XLiFE++.

a - 2D program without perturbation

Using Lagrange elements to approximate u, write a program using XLiFE++ to solve 5 when no perturbation.

b - Error computation

With exact solutions when no perturbation, compute the L^2 error u.

c - 2D program with perturbation

Using Lagrange elements to approximate u, write a program using XLiFE++ to solve 5 when there is a circular perturbation. You could try to change mesh step and number of modes.

You can define another DtN operator on another section Σ_b ($b > a$). This is the "Thick DtN" approximation.
Its weak form is:

$$(6)\quad \left| \begin{array}{l} \text{Find } u \in H^1(\Omega_a) \text{ such that } \forall v \in H^1(\Omega_a) \\ \displaystyle\int_{\Omega_a} \nabla u \nabla \bar{v} - k^2 \int_{\Omega_a} u\,\bar{v} - \int_{\Sigma_b}\int_{\Sigma_a} u(x) \left(\sum_{0\le n\le N} i\alpha_n\,\varphi_n(x)\,\varphi_n(y) \right) \bar{v}(y)\,dx\,dy = \int_{\Sigma_0} g\,\bar{v} \\ \text{where } \alpha_n = \beta_n e^{i\beta_n(b-a)} \end{array} \right.$$

Exercise 4 - Comparison of approximations

a - scattering coefficients

To compute scattering coefficients $a_n = \int_{\Sigma_a} u\,\varphi_n$, you just have to do:

```
TermMatrix C(intg(Sigma_a,u*phi));
TermVector a=C*U;
```

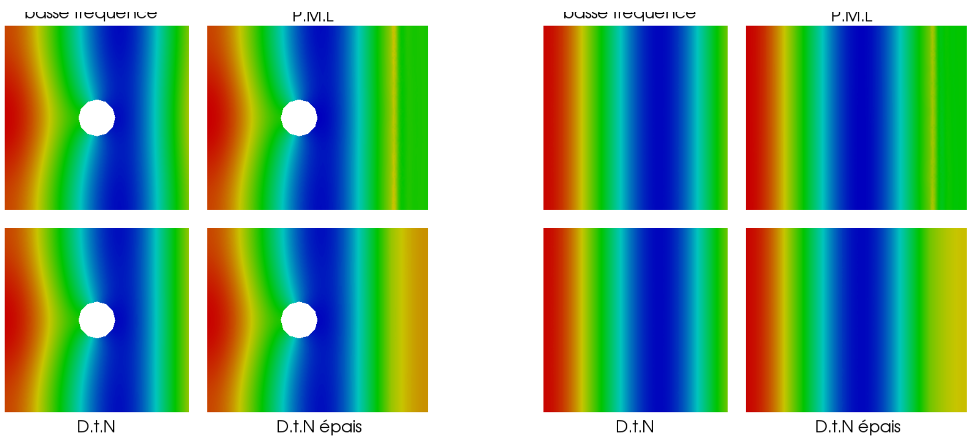


Figure 1: Real part of the field with or without perturbation