XLiFE++ Practice 3

BEM, FEM-BEM coupling

Exercise 1 - A Basic BEM problem.

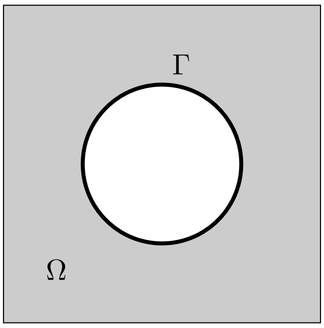

Figure 1: Domain of computation. Integral equation on Γ and evaluation of the solution in Ω.

We are interested in solving a scattering problem in 2D by boundary integral equation technique. The scattered field u solves the following Helmholtz equation:

$$(1) \qquad \left\{ \begin{array}{l} -\Delta u(x) + k^2 u(x) = 0, \text{ in } \Omega, \\ u(x) = -u_{inc}(x), \text{ on } \Gamma, \\ u(x) \text{ verifies a radiation condition at infinity.} \end{array} \right.$$

We assume that the incident wave is a plane wave:

$$u_{inc}(x) = exp(ikd \cdot x),$$

where d is the incident vector and k the wavenumber.

We use a first kind integral equation to solve the Problem (1) so we have

$$(2) \qquad u(z) = \int_\Gamma G(z,y)\phi(y)ds_y, \forall z \in \Omega,$$

where G is the 2-D Helmholtz Green kernel ($G(x,y) = -\frac{i}{4}H_0^{(1)}(k\|x-y\|)$).Then by taking the limit when $z \to x \in \Gamma$, we obtain the integral equation:

$$(3) \qquad \left| \begin{array}{l} \text{Find } \phi \in H^{-\frac{1}{2}}(\Gamma) \text{ such that} \\ -u_{\text{inc}}(x) = \displaystyle\int_\Gamma G(x,y)\phi(y)ds_y, \forall x \in \Gamma \text{ where } u_{\text{inc}} \in H^{\frac{1}{2}}(\Gamma) \end{array} \right.$$

Finally the variational formulation is:

$$(4) \qquad \left| \begin{array}{l} \text{Find } \phi \in H^{-\frac{1}{2}}(\Gamma) \text{ such that} \\ \displaystyle\int_{\Gamma\times\Gamma} \phi(y)G(x,y)\psi(x)ds_y ds_x = -\int_\Gamma u_{\text{inc}}(x)\psi(x)ds_x, \quad \forall \psi \in H^{\frac{1}{2}}(\Gamma) \end{array} \right.$$

a - Incident field

Create a function **uinc** that takes a `Point` and a `Parameters` and returns a `Complex` that corresponds to the incident field.

b - Kernel

In order to solve the integral equation, we need to construct the matrix and the right-hand side term. We will write the bilinear form and the linear form but first we create the Green Kernel of the Helmholtz 2-D equation in 4 steps:

1. Create a variable **params** of type `Parameters`.
2. Create a variable **k** of type `Real` that takes the value 3.
3. Push this variable **k** in params
4. Finally, create the kernel using: `Kernel G=Helmholtz2dKernel(params);`

c - 2D problem

By using P0 elements, write a program that solves on $\Omega = [-3, 3]^2$ with a hole defined as the unit disk.

One can adapt easily the mesh size to the wavenumber by defining a variable `Real myh = (2.*pi/(10*k))` (if one wants 10 points by wavelength) and then **_hsteps** option for the geometry.

Since the assembly of the matrix requires the evaluation of singular integrals, we have to provide an `IntegrationMethods` to handle these integrals. For 2-D problems, we use a method based on the Duffy transformation that use the Jacobian from a change of variables to regularise the integrand. We define the corresponding `IntegrationMethods`:

```
IntegrationMethods ims(Duffy, order1, distance1, rule2, order2, distance2, rule3,
    order3, ...)
```

the arguments corresponds to quadratures on the segment of *order1* for the singular integrals (self-influence and segments with a common vertex), on the segment of order *order2* for non adjacent elements with relative distance between *distance1* and *distance2*, on the segment of order *order3* for non adjacent elements with relative distance greater than *distance2*. We suggest to use:

```
IntegrationMethods ims(Duffy, 5, 0., Gauss_Legendre, 5, 1., Gauss_Legendre, 4,
    2.,Gauss_Legendre, 3);
```

d - Integral representation

In order to visualize the solution on a domain, we need to compute the representation of the solution using the integral representation formula (8). We can do this by defining a `LinearForm` **lfIR** with additional information for the quadrature (specially for points close to Γ) but also a `Space` of piecewise linear function on the domain Ω, denoted by **IRspace** and an `Unknown` on this space **IRu**. Then, we call the function `integralRepresentation` :

```
TermVector IRsol=integralRepresentation(IRu, omega, lfIR,U);
```

For the `LinearForm`, we suggest to add the arguments _GaussLegendreRule, 10 at the end in order to use a Gauss-Legendre quadrature rule of order 10 to compute the integrals arising in the integral representation.

We can save the solution (that is the scattered field) using

```
saveToFile("solIR",IRsol,_vtu);
```

e - Computing the total field

Since the scattered field u verifies $u = u_T - u_{\text{inc}}$, we have add the incident field to obtain the total field. We create a `TermVector` by interpolating the incident field at the degree of freedom of the visualisation domain **omega**:

```
TermVector Uinc(IRu,omega,uincF);
```

Then create a `TermVector` that is the sum of the scattering field **IRsol** and the incident field **Uinc** and save it in a file defined by its filename:

```
String filename="total_field_k_"+tostring(k)
```

f - 3D problem

Solve the same problem but with Γ the surface of a ball.

The 3D Helmholtz Kernel is obtained by: **Kernel** G =**Helmholtz3dKernel**(params);

The integration method will be: **IntegrationMethods** ims(Sauter_Schwab,3, 0., symmetrical_Gauss, 3);

The domain used for the integral representation will be a plane define by $\{(x,y) \in [-3,3]^2, z=0\}$minus a hole corresponding to the unit ball.

g - Extra questions

Try to solve the same problem for different k and/or using other integral formulation (with a double layer potential for example) and/or with different shapes for the obstacle: `Square`/`Cube`, `Triangle`/`Tetrahedron`, `Ellipse`/`Ellipsoid`, ...

Exercise 2 - FEM-BEM in 2-D

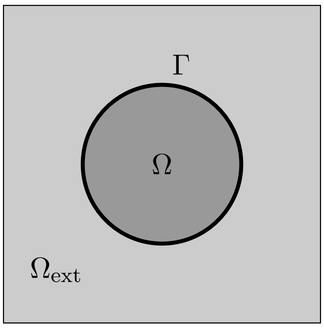

Figure 2: Domain of computation. Integral equation on Γ, Finite Element in Ω and integral representation of the BEM solution on Ω_{ext}.

We will solve the scattering by a **penetrable** disk:

$$(5) \qquad \begin{cases} -\Delta u(x) + k^2 u(x) = 0, \text{ in } \Omega_{\text{ext}}, \\ -\Delta u(x) + \frac{k^2}{c^2(x)} u(x) = 0, \text{ in } \Omega, \\ u(x) = -u_{inc}(x), \text{ on } \Gamma, \\ u(x) \text{ verifies a radiation condition at infinity.} \end{cases}$$

We assume that the incident wave is a plane wave:

$$u_{inc}(x) = exp(ik \cdot x_1),$$

where k is the wavenumber and x_1 is the first component of x.

Moreover, we introduce $\eta^2(x) = \frac{1}{c^2(x)}$.

We decompose the problem in a coupled system of two equations:

- in the FEM part, the solution solves the following equation:

$$\Delta u + k^2 \eta^2 u = 0$$

which gives the variational formulation:

$$(6) \quad \left| \begin{array}{l} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla u(x) \cdot \nabla \bar{v}(x) dx - k^2 \int_\Omega \eta^2(x) u(x) \bar{v}(x) dx - \int_\Gamma \lambda(x) \bar{v}(x) dx = 0, \forall v \in H^1(\Omega), \end{array} \right.$$

where $\lambda = \dfrac{\partial u}{\partial n}$ is the normal trace of the total field u on Γ.

- in the BEM part, we solve:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_{\text{ext}} \\ u = -u_i & \text{on } \Gamma \end{cases}$$

The scattered field verifies $u_s(x) = -S_\Gamma \lambda(x) + K_\Gamma u(x), x \in \Omega_{\text{ext}}$, with S_Γ and K_Γ being respectively the single and double layer boundary potentials:

$$S_\Gamma \phi(x) = \int_\Gamma G(x,y)\phi(y) dy, \quad K_\Gamma \phi(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n_y} \phi(y) dy, \quad \text{where } G(x,y) = \frac{e^{ik\|x-y\|}}{4\pi\|x-y\|}$$

Since $u_s = u - u_{\text{inc}}$, and by taking the limit when x goes to Γ, we obtain the integral equation:

$$\left(\frac{I}{2} - K_\Gamma\right) u(x) + S_\Gamma \lambda(x) = u_{\text{inc}}(x), x \in \Gamma.$$

The resulting variational formulation for the BEM part is then:

$$(7) \quad \left| \begin{array}{l} \text{Find } (u, \lambda) \in H^{1/2}(\Omega) \times H^{-1/2}(\Gamma) \text{ such that} \\ \displaystyle\int_\Gamma u(x)\bar{\tau}(x) dx - \int_{\Gamma\times\Gamma} u(y) \frac{\partial G(x,y)}{\partial n_y} \bar{\tau}(x) dy dx + \int_{\Gamma\times\Gamma} \lambda(y) G(x,y) \bar{\tau}(x) dy dx \\ \displaystyle = \int_\Gamma u_{\text{inc}}(x) \bar{\tau}(x) dx, \quad \forall \tau \in H^{1/2}(\Gamma) \end{array} \right.$$

By adding the variational formulations defined in (6) **and** (7) **relatives to the two linked problems, we obtain the final variational formulation.**

a - Incident field: Create a function **uinc** that takes a **Point** and a **Parameters** and returns a **Complex** that corresponds to the incident field.

b - Propagation velocity
Define a function **eta2** ($\eta^2(x) = \frac{1}{c^2(x)}$) that returns $\frac{1}{4}$ for the moment (later we will change this function).

c - 2D problem
By using P1 elements on Ω and Γ, write a program that solves this FEM-BEM formulation on $\Omega = [-3, 3]^2$ with a hole defined as the unit disk.

When doing FEM-BEM coupling, you have to disable the renumbering optimisation (usually used in FEM) by adding an additional parameter **false** at the end of the instruction to create a **Space**:

```
Space V(Omega, P1, "V", false);
```

d - Integral representation The integral representation of the BEM solution on Ω_{ext} is:

$$u_s(x) = -\int_\Gamma G(x,y)\frac{\partial u(y)}{\partial n}dy + \int_\Gamma \frac{\partial G(x,y)}{\partial n_y}u(y)dy, \tag{8}$$

with u_s the scatterer field.

In order to compute the BEM solution on the domain Ω_{ext}, we need to use the integral representation formula (8).

First, we generate a new **Mesh** with a large square (with domain_name = OmegaExt) **-** the circle used to define **omega** and load the domain: **omegaExt**. Define a **Space** of piecewise linear functions on the domain **omegaExt**, denoted by **IRspace** and an **Unknown** on this space **IRu**. This new **Space** and **Unknown** will be used to do the integral representation.
Then, we call the function **integralRepresentation** :

```
TermVector IRsol=integralRepresentation(IRu, omegaExt,
    intg(gamma,G*Sol(u0),_GaussLegendreRule, 10))+ ...
```

The Gauss-Legendre quadrature rule of order 10 added is to define a quadrature rule to compute the integrals arising in the integral representation. If there are only points really close to Γ then it can be good to increase the order of this quadrature.

e - Extra questions
Try to solve the same problem for different *eta*2 and try to use non-constant functions.

f - Building the total field everywhere
By using the routine `merge` on two `TermVector`, adapt your code to have only one *vtu* file with the FEM and the BEM solution.