Interaction processes for union of facets, a limit behaviour

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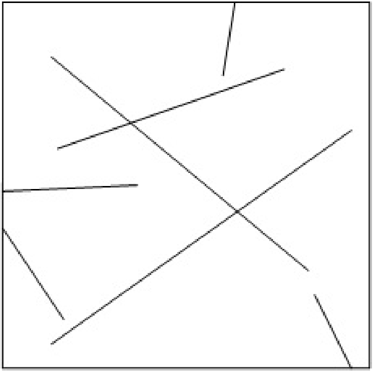
Geometrical facets in $\mathbb{R}^d$

- We study facets - compact subsets of hyperplanes in $\mathbb{R}^d$
- We are interested in the asymptotic distribution of their interaction statistics



- Each facet is determined by three parameters
  - centre $z \in [0, b]^d$, $b \in \mathbb{R}$,
  - size $r \in (r_0, r_1)$, $0 < r_0 < r_1 \leq \infty$,
  - normal vector to the hyperplane containing facet $\phi \in \mathbb{S}^{d-1}$,

  and it is defined as $\alpha(z, r, \phi) \subset \mathbb{R}^d$:

$$\alpha(z, r, \phi) = \{x, \langle x - z, \phi \rangle = 0, \|x - z\|_\infty \leq r\}$$

- $Y = [0, b]^d \times (r_0, r_1) \times \mathbb{S}^{d-1}$ is a space of facets and **N** is space of all finite counting measures on Y, $y \in \mathbf{N} : y \subset Y$
- Let $y \subset Y$ be some finite set of facets, we define statistic in a form

$$G_j(y) = \sum_{(\alpha_1, \ldots, \alpha_j) \in y^j_{\neq}} \mathbb{H}^{d-j}(\cap_{i=1}^{j} \alpha_i), \; j \in \{1, \ldots, d\},$$

  where $\mathbb{H}^{d-j}$ is Hausdorff measure of order $d - j$.

Statistics of facets in $\mathbb{R}^3$

$\mathbf{H}^2(y) = A(y)$ area of a facet,

$\mathbf{H}^1(y_1 \cap y_2) = L(y_1 \cap y_2)$ length of intersection of two facets,

$\mathbf{H}^0(y_1 \cap y_2 \cap y_3) = \mathbf{1}(y_1 \cap y_2 \cap y_3 \neq \emptyset)$.

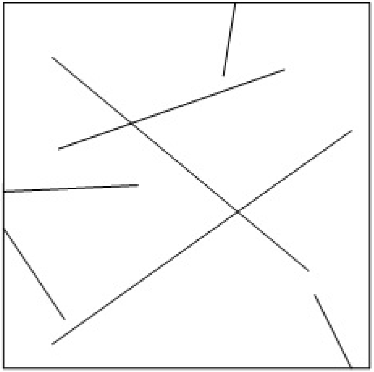
$G_1(\mathbf{x}) = \sum_{y \in \mathbf{x}} A(y)$,

$G_2(\mathbf{x}) = \frac{1}{2} \sum_{(y_1, y_2) \in \mathbf{x}^2_{\neq}} L(y_1 \cap y_2)$,

$G_3(\mathbf{x}) = \frac{1}{6} \sum_{(y_1, y_2, y_3) \in \mathbf{x}^3_{\neq}} \mathbf{1}(y_1 \cap y_2 \cap y_3 \neq \emptyset)$

Increasing number of facets

- We want to explore the distribution of the statistics, which can be quite complicated
- Therefore we will be interested in asymptotic distribution of statistics with increasing ”intensity”of facets distributions



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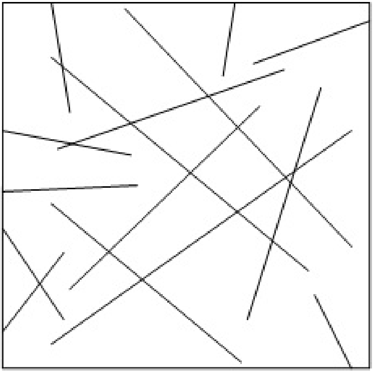

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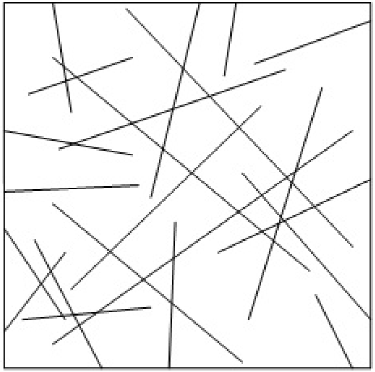

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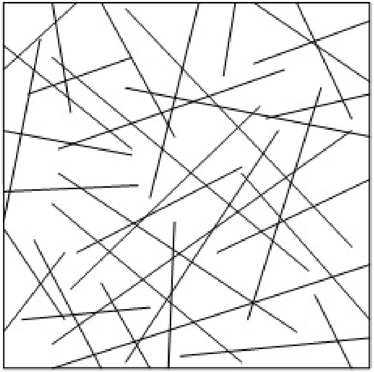


Poisson process of facets

- Consider Poisson process $\eta_a : (\Omega, \mathcal{A}, P) \to (\mathbf{N}, \mathcal{N})$ of facets with intensity measure

$$\lambda_a(\mathrm{d}(z, r, \phi)) = a\lambda(\mathrm{d}(z, r, \phi)) = a\chi(z)\mathrm{d}z Q(\mathrm{d}r) V(\mathrm{d}\phi),$$

  where
  - a is intensity parameter, which will tend to infinity,
  - χ is intensity function of the facet centres,
  - Q is probability distribution of the facet sizes,
  - V is probability distribution of the facet orientations.

- Then number of facets in set $D = A \times B \times C$, $A \subset [0, b]^d, B \subset (r_0, r_1), C \subset \mathbb{S}^{d-1}$ has Poisson distributon with intensity $\lambda(D) = a \int_A \chi(z) \mathrm{d}z Q(B) V(C)$, i.e.

$$P\left(\sum_{\alpha \in \eta_a} I(\alpha \in D) = k\right) = \frac{\lambda(D)^k}{k!} e^{-\lambda(D)}$$

- For sets $D_1, D_2 \subset Y, D_1 \cap D_2 = \emptyset$, the number of facets in D_1 and D_2 are independent random variables.

Poisson process of facets

- Solution to this problem is already known (follows from Last et al., 2014)

$$\frac{G_j(\eta_a) - \mathbb{E}G_j(\eta_a)}{a^{j-\frac{1}{2}}} \to N(0, \theta_j),\ j \in \{1, \dots, d\}$$

where

$\theta_j = \frac{1}{(j-1)!} \int_Y \left(\int_{Y^{j-1}} \mathbb{H}^{d-j}(\cap_{i=1}^{j-1} \alpha_i \cap \beta) \lambda(\mathrm{d}(\alpha_1, \dots, \alpha_{j-1})) \right)^2 \lambda(\mathrm{d}\beta)$

- Drawback of Poisson process is that it does not take interaction among the facets into the distribution. To generalize this model we can take this into account and consider process with density with respect to current Poisson process (Gibbs process).

Process with density

- We consider process $\mu_a^{(l)}, l = 2, \ldots, d$ with density

$$p(x) = c_a \exp(\nu G_l(x)), x \in \mathbf{N},$$

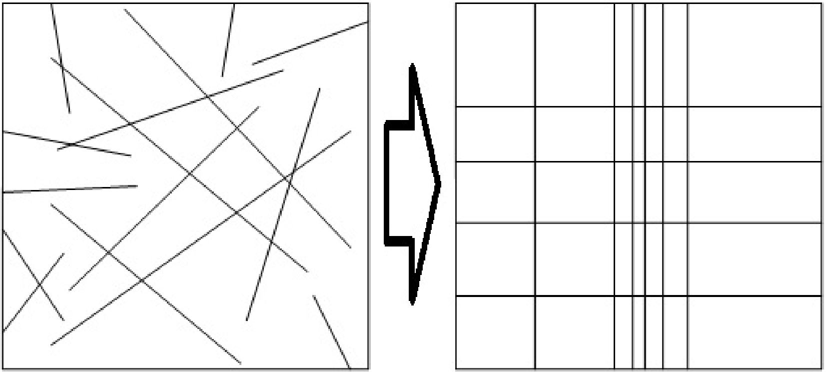
with respect to the Poisson process η_a, where

  - $\nu < 0$ to assure $p \in L_1(P_{\eta_1}) \cap L_2(P_{\eta_1})$,
  - c_a is selected to fullfill $\int_N p(x) dP_{\eta_a} = 1$.

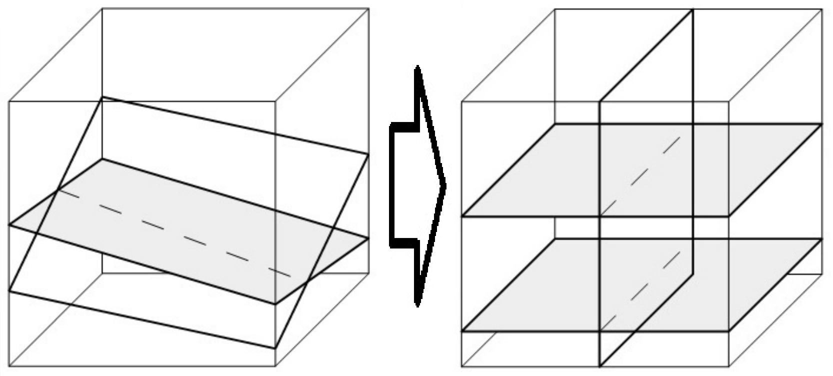
Special assumptions

- To simplify these calculations we consider 2 limitations to intensity measure of the reference Poisson process:
  - size of the fasets is fixed to $2b$
  - orientation distribution is uniform distribution on discrete set of orientations - d elementary vectors in $\mathbb{R}^d$
- Two facets with different orientation have always a non-empty intersection.
- For facets $\alpha_1, \dots, \alpha_j$ with different orientations it holds $b^{d-j} \leq \mathbb{H}^{d-j}(\cap_{i=1}^{j} \alpha_i) \leq (2b)^{d-j}$.
- This allows us to calculate probabilities of such lower and upper bound using basic combinatorics, which is significant in the calculation process.

Special assumptions in $\mathbb{R}^2$



Special assumptions in $\mathbb{R}^3$



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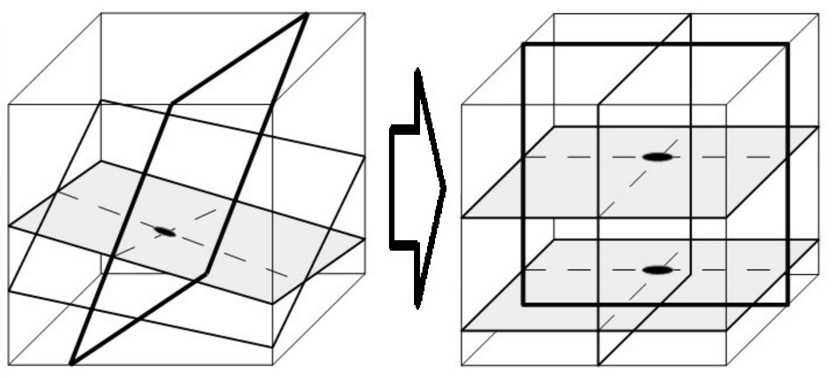

Theorem 1

Denote $\tilde{G}_j(\mu_a^{(c)}) = \frac{G_j(\mu_a^{(c)}) - \mathbb{E}G_j(\mu_a^{(c)})}{a^{j-\frac{1}{2}}}$, $1 \leq j \leq d$, $2 \leq c \leq d$, *then*

$$(\tilde{G}_1(\mu_a^{(c)}), \ldots, \tilde{G}_d(\mu_a^{(c)})) \xrightarrow{\mathcal{D}} \mathbf{Z},\ c = 2, \ldots, d, \tag{1}$$

as a *tends to infinity, where* $\mathbf{Z} \sim N(0, \Sigma)$, $\Sigma = \{\theta_{ij}\}_{i,j=1}^{d}$,

$$\theta_{kl} = \frac{(c-1)}{d^{k+l-1}} \binom{c-2}{k-1} \binom{c-2}{l-1} I_{kl},$$

$$I_{kl} = \int_{([0,b]^d)^{k+l-1}} \mathbb{H}^{d-k}(\cap_{i=1}^{k}(s_i, 2b, e_i))$$

$$\times\, \mathbb{H}^{d-l}(\cap_{i=2}^{l}(s_{i+k-1}, 2b, e_i) \cap (s_1, 2b, e_1))\times$$

$$\times\, \chi(s_1)\mathrm{d}s_1 \ldots \chi(s_{k+l-1})\mathrm{d}s_{k+l-1},$$

Interpretation of results

- Asymptotic variance of distribution of the statistics of the reference Poisson process is

$$\theta'_{jj} = \frac{d}{d^{2l-1}} \binom{d-1}{j-1}^2 I_{jj}.$$

- We can see some orientations missing asymptotically in the process with density.
- In the process $\mu_a^{(l)}$, which has volume of intersections among l-tuples of facets in the density function, there are missing $d - l + 1$ orientations, therefore we can choose facets only from $l - 1$ possible orientations, there are no intersections among l-tuples of facets.
- In examined cases the term in exponential of density always vanishes.

Sketch of the Proof

- To show that asymptotic distribution is normal we use method of moments, i.e. we show that limit of all moments are equal to moments of normal distribution.

$$\mathbb{E}\left(\frac{G_j(\mu_a^{(l)}) - \mathbb{E}G_j(\mu_a^{(l)})}{a^{j-\frac{1}{2}}}\right)^m \to 0, \qquad m \text{ odd}$$

$$\to (m-1)!!\sigma^m, \qquad m \text{ even}$$

- We have moment formulas for the process, which we use.
- In the case of first moment

$$\mathbb{E}G_j(\mu_a^{(l)}) = a^j \int_{Y^j} \mathbb{H}^{d-j}(\cap_{i=1}^j \alpha_i)\rho_j(\alpha_1,\dots,\alpha_j,\mu_a^{(l)})\lambda(\mathrm{d}(\alpha_1,\dots,\alpha_j)),$$

where

$$\rho_j(\alpha_1,\dots,\alpha_j,\mu_a^{(l)}) = \frac{\mathbb{E}\exp(\nu G_l(\eta_a \cup \{\alpha_1,\dots,\alpha_j\}))}{\mathbb{E}\exp(\nu G_l(\eta_a))}$$

is product density (intensity for $j=1$) of the Gibbs faset process.

- and in general case

$$\mathbb{E}(G_j(\mu_a^{(l)}))^m = \sum_{\sigma\in\Pi_{j,\dots,j}} a^{|\sigma|} \int_{Y^{|\sigma|}} \left(\left(\mathbb{H}^{d-j}\right)^{\otimes m}\right)_\sigma (\alpha_1,\dots,\alpha_{|\sigma|})\times$$

$$\times\rho_{|\sigma|}(\alpha_1,\dots,\alpha_{|\sigma|},\mu_a^{(l)})\lambda(\mathrm{d}(\alpha_1,\dots,\alpha_{|\sigma|}))$$

Sketch of the Proof

- First we calculate $\lim_{a\to\infty} \rho$.
- Secondly we replace the ρ with it's limit value in integrand.
- Then we substitute λ for the special intensity function.
- In the last step we show, that are moments of normalized random variable are asymptotically equal to moments of normal distribution.

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