Palm distributions for log Gaussian Cox processes

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Content:

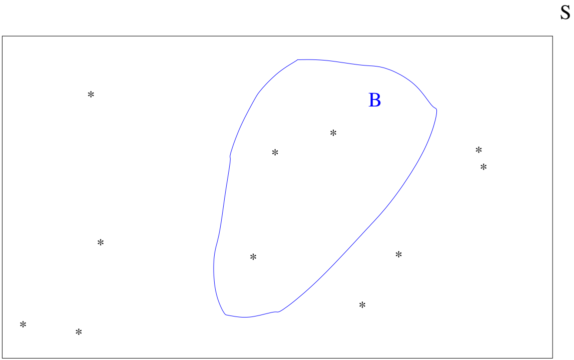
- elementary introduction to Palm distributions for spatial point processes
- Palm distributions for log Gaussian Cox processes

Spatial point process

A spatial point process $\mathbf{X}$ on $\mathbb{R}^d$ is a locally finite random subset of $\mathbb{R}^d$: if $S \subset \mathbb{R}^d$ is bounded then $\mathbf{X} \cap S$ is finite.

Conditioning in spatial point processes I

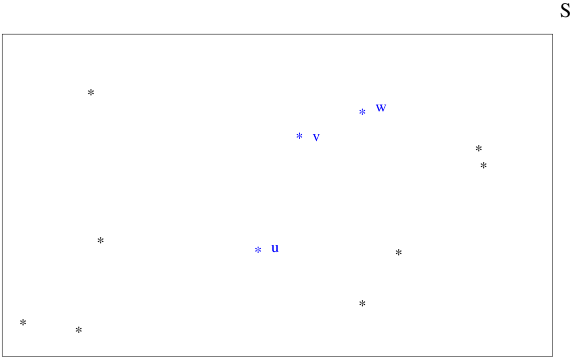
We may consider conditional distribution of $\mathbf{X} \cap (S \setminus B)$ given $\mathbf{X} \cap B = \mathbf{z}$



Note here $\mathbf{z}$ is the information regarding the presence of three points in B *as well as* the absence of further points in B.

Conditioning in spatial point processes II

We may consider conditional distribution of $\mathbf{X}$ given $\mathbf{z} \subseteq \mathbf{X}$



Note here we only condition on the information that the points in $\mathbf{z} = \{u, v, w\}$ belong to $\mathbf{X}$

In this talk we focus on this type of conditional distribution - a Palm distribution.

Palm distribution

Palm distributions named after Swedish statistician and electrical engineer Conrad ‘Conny’ Palm due to his seminal dissertation from 1943.

Extensive literature in probability theory: e.g. Ryll-Nardzewski, Jagers, Last , Kallenberg, Thorisson, point process monograph by Daley and Vere-Jones.

Modern Palm theory based on disintegration of Campbell measure.

Disintegration

Intensity measure:

$$\mu(A) = \mathbb{E} \sum_{u \in \mathbf{X}} 1[u \in A], \quad A \subseteq \mathbb{R}^d$$

Reduced Campbell measure (F set of locally finite point configurations)

$$C^!(A \times F) = \mathbb{E} \sum_{u \in \mathbf{X}} 1[u \in A, \mathbf{X} \setminus \{u\} \in F]$$

Disintegration

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Note $\mu(A) = 0 \Rightarrow C^!(A \times F) = 0$ for all F.

Hence, by disintegration, under regularity conditions,

$$C^!(A \times F) = \int_A P^!_u(F)\mu(\mathrm{d}u)$$

where for each u $P^!_u(\cdot)$ is a point process distribution called the reduced Palm distribution of $\mathbf{X}$ given u.

Palm distribution: distribution of $\mathbf{X}_u^! \cup \{u\}$ if $\mathbf{X}_u^!$ distributed as $P_u^!(\cdot)$

General definition of reduced Palm distribution quite abstract and technical.

$P_u^!(\cdot)$ represents conditional distribution of $\mathbf{X} \setminus \{u\}$ given $u \in \mathbf{X}$ but this maybe not so obvious.

In elementary probability theory, conditional distributions defined in terms of ratios of joint and marginal densities.

Similar approach possible for Palm distributions if the point process distribution specified by a density.

Natural for statisticians who like likelihood functions !

Assume distribution of $\mathbf{X}$ is specified in terms of a density f with respect to a Poisson process $\mathbf{Z}$.

This means

$$P(\mathbf{X} \in F) = \mathbb{E}\Big[1[\mathbf{Z} \in F] f(\mathbf{Z})\Big]$$

Restriction: in practice this requires that $\mathbf{X}$ is a *finite* point process.

Assume $\mathbf{X}$ confined to bounded S and $\mathbf{Z}$ Poisson process of intensity one.

For pairwise distinct $u_1, \ldots, u_m$ we can define the m'th order joint intensity as

$$\rho^{(m)}(u_1, \ldots, u_m) = \mathbb{E} f(\mathbf{Z} \cup \{u_1, \ldots, u_m\})$$

Hence with $\mathbf{x} = \{x_1, \ldots, x_n\} \subset S$,

$$f_{u_1,\ldots,u_m}(\mathbf{x}) = \frac{f(\mathbf{x} \cup \{u_1, \ldots, u_m\})}{\rho^{(m)}(u_1, \ldots, u_m)}$$

defines probability density of a point process (again with respect to $\mathbf{Z}$).

Interpretation

$\rho^{(m)}(u_1, \ldots, u_m)\mathrm{d}u_1 \cdots \mathrm{d}u_m$ 'probability' that '$u_1, \ldots, u_m \in \mathbf{X}$'.

$f(\mathbf{x})\mathrm{d}x_1 \cdots \mathrm{d}x_n$ 'probability' that '$\mathbf{X} = \mathbf{x}$' for $\mathbf{x} = \{x_1, \ldots, x_n\}$.

Thus by elementary definition of conditional probability,

$$f_{u_1,\ldots,u_m}(\mathbf{x})\mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{f(\mathbf{x} \cup \{u_1, \ldots, u_m\})\mathrm{d}x_1 \cdots \mathrm{d}u_m}{\rho^{(m)}(u_1, \ldots, u_m)\mathrm{d}u_1 \cdots \mathrm{d}u_m}$$

'conditional probability' that $\mathbf{X} \setminus \{u_1, \ldots, u_m\} = \mathbf{x}$ given $u_1, \ldots, u_m \in \mathbf{X}$.

Defining ($m = 1$)

$$P^!_u(F) = P(\mathbf{X}^!_u \in F) = \mathbb{E}\Big[1[\mathbf{Z} \in F] f_u(\mathbf{Z})\Big]$$

it is easy to check that

$$C^!(A \times F) = \int_A P^!_u(F)\mu(\mathrm{d}u)$$

So defining a reduced Palm distribution in terms of the density $f_{u_1,\ldots,u_m}(\cdot)$ complies with the usual definition (for all $m \geq 1$)

Example: Poisson process

Density of Poisson process with intensity function ρ:

$$f(\mathbf{x}) \propto \prod_{v \in \mathbf{x}} \rho(v)$$

and

$$\rho^{(m)}(u_1, \dots, u_m) = \prod_{i=1}^{m} \rho(u_i)$$

Thus

$$f_{u_1,\dots,u_m}(\mathbf{x}) \propto \frac{\prod_{v \in \mathbf{x}} \rho(v) \prod_{i=1}^{m} \rho(u_i)}{\prod_{i=1}^{m} \rho(u_i)} = \prod_{v \in \mathbf{x}} \rho(v)$$

Hence reduced Palm distribution of Poisson process is just the Poisson process itself.

Example: Cox process

Given non-negative random function $\Lambda(\cdot)$, $\mathbf{X}$ is a Poisson process with intensity function $\Lambda(\cdot) = \lambda(\cdot)$.

I.e. $\mathbf{X}$ has conditional density

$$f(\mathbf{x}|\Lambda) \propto \prod_{v \in \mathbf{x}} \Lambda(v)$$

and density

$$f(\mathbf{x}) = \mathbb{E} f(\mathbf{x}|\Lambda)$$

Moreover,

$$\rho^{(m)}(u_1, \dots, u_m) = \mathbb{E} \prod_{i=1}^{m} \Lambda(u_i)$$

so

$$f_{u_1,\dots,u_m}(\mathbf{x}) = \frac{f(\mathbf{x} \cup \{u_1, \dots, u_m\})}{\rho^{(m)}(u_1, \dots, u_m)} = \mathbb{E}\left[f(\mathbf{x}|\Lambda) \frac{\prod_{i=1}^{m} \Lambda(u_i)}{\rho^{(m)}(u_1, \dots, u_m)} \right]$$

Note

$$f_{u_1,\ldots,u_m}(\mathbf{x}) = \mathbb{E}\left[f(\mathbf{x}|\Lambda)\frac{\prod_{i=1}^m \Lambda(u_i)}{\rho^{(m)}(u_1,\ldots,u_m)}\right] = \mathbb{E}\left[f(\mathbf{x}|\Lambda_{u_1,\ldots,u_m})\right]$$

where $\Lambda_{u_1,\ldots,u_m}$ absolutely continuous wrt. Λ with density

$$\frac{\prod_{i=1}^m \Lambda(u_i)}{\rho^{(m)}(u_1,\ldots,u_m)}.$$

In other words, short proof that $\mathbf{X}^!_{u_1,\ldots,u_m}$ Cox process driven by random intensity function $\Lambda_{u_1,\ldots,u_m}$ (e.g. Daley and Vere-Jones).

Log-Gaussian Cox process

Log-Gaussian Cox process **X** has

$$\Lambda(u) = \exp[Y(u)]$$

where Y Gaussian process with mean function $m(\cdot)$ and covariance function $c(\cdot,\cdot)$, $c(u,v) = \mathbb{C}\mathrm{ov}[Y(u), Y(v)]$.

Result: reduced Palm distribution of **X** given $u_1, \ldots, u_m$ is a log-Gaussian Cox process driven by

$$\Lambda_{u_1,\ldots,u_m}(v) = \exp[Y_{u_1,\ldots,u_m}(v)]$$

where $Y_{u_1,\ldots,u_m}$ Gaussian process with 'elevated' mean function

$$m_{u_1,\ldots,u_m}(v) = m(v) + \sum_{i=1}^{m} c(u_i, v)$$

and covariance function $c(\cdot,\cdot)$.

‘Paradox’

Typically covariance function is positive $c(\cdot,\cdot) > 0$.

Thus $m_{u_1,\ldots,u_m}(v) \geq m(v)$ - and increases with m.

This implies larger intensity for reduced Palm log Gaussian Cox point process $\mathbf{X}^!_{u_1,\ldots,u_m}$ than for $\mathbf{X}$ - and intensity increases with m.

‘the more points we condition on the more remaining points we expect to see’

In fact (by thinning argument) we can obtain

$$\mathbf{X} \subseteq \mathbf{X}^!_{u_1,\ldots,u_m}$$

Crucial: m and $u_1, \ldots, u_m$ *fixed and prespecified* - not a realization of a subset of $\mathbf{X}$.

Thus event $u_1, \ldots, u_m \in \mathbf{X}$ represents 'surprise' !

For comparison consider $\mathbf{Y} \subseteq \mathbf{X}$ with conditional (discrete) distribution

$$p(\mathbf{y}|\mathbf{x}) = P(\mathbf{Y} = \mathbf{y} \,|\, \mathbf{X} = \mathbf{x})$$

Then by Bayes rule, conditional density of $\mathbf{X} \setminus \mathbf{Y}$ given $\mathbf{Y} = \mathbf{y}$:

$$f(\mathbf{x}|\mathbf{y}) \propto p(\mathbf{y}|\mathbf{x} \cup \mathbf{y}) f(\mathbf{x} \cup \mathbf{y})$$

which is *not the same* as density of reduced Palm distribution:

$$f_{\mathbf{y}}(\mathbf{x}) \propto f(\mathbf{x} \cup \mathbf{y})$$

Caution needed when interpreting Palm distribution.

Conclusion

- Defining Palm distributions in terms of densities can make the topic more digestible for a general audience
- Another nice property of log Gaussian Cox processes: reduced Palm distribution of any order is simply a new log Gaussian Cox process with a modified mean function.
- Result can be used to study summary statistics as the neighbour distribution function for log Gaussian Cox processes.