Estimating drift parameters in a fractional Ornstein Uhlenbeck process with periodic mean

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- Fractional Brownian motion and fractional Ornstein-Uhlenbeck processes
- Estimation of drift parameters

Motivation

Empirical evidence in data:

- often mean-reverting property
- specific correlation structure, e.g. long range dependence
- often saisonalities are present

Questions:

- How can we model this features?
- How can we infer involved quantities?

Fractional Brownian Motion

A fractional Brownian motion (fBm) with **Hurst parameter** $H \in (0,1)$, $B^H = \{B_t^H, t \geq 0\}$ is a zero mean Gaussian process with the covariance function

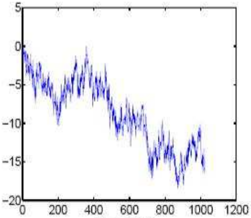
$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s,t \geq 0.$$

Properties:

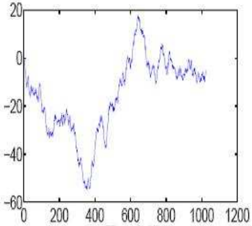
- Correlation

For $H \in (\frac{1}{2},1)$ the process possesses **long memory** and for $H \in (0,\frac{1}{2})$ the behaviour is **chaotic**.

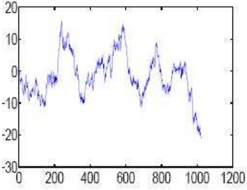
- For $H = \frac{1}{2}$, B^H coincides with the classical Brownian motion.
- Hölder continuous paths of the order $\gamma < H$.
- Gaussian increments
- Selfsimilarity: $\{a^{-H}B_{at}^H, t \geq 0\}$ and $\{B_t^H, t \geq 0\}$ have the same distribution.
- if $H \neq 0.5$ **not a semimartingale**.



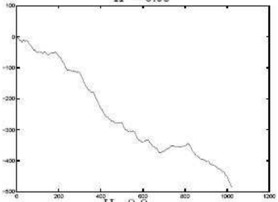
$H = 0.25$



$H = 0.65$



$H = 0.5$



H=0.9

Implications of this properties

- fractional Brownian motion is **non-Markovian**: usual martingale approaches do not work,
- increments are not independent, we cannot use classical limit theorems for independent random variables,
- Itô integration does not work, we need a different type of integration, the easiest is a **pathswise Riemann-Stieltjes integral**. Other possibility is a **divergence integral** which allows for a generalization of the Itô formula.

Ornstein-Uhlenbeck Process

A classical Ornstein-Uhlenbeck process is given by the stochastic differential equation

$$dX_t = -\lambda X_t dt + dW_t$$

where $\lambda > 0$ and W denotes a Brownian motion. It possesses the solution

$$X_t = X_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dW_s$$

and is mean-reverting.
Popular generalizations are to replace the Brownian motion by a Lévy process or a fractional Brownian motion.

Generalized fractional Ornstein-Uhlenbeck processes

We consider the stochastic process (X_t) given by the stochastic differential equation

$$dX_t = (L(t) - \alpha X_t)dt + \sigma dB_t^H,$$

with initial condition $X_0 = \xi_0$, where ξ_0 is a square integrable random variable independent of the fractional Brownian motion $(B_t^H)_{t\in\mathbb{R}}$ with a period drift function $L(t) = \sum_{i=1}^p \mu_i \varphi_i(t)$, where the functions $\varphi_i(t); i = 1, ..., p$ are bounded and periodic with the same period ν and the real numbers $\mu_i; i = 1, ..., p$ are unknown parameters as well as $\alpha > 0$. We assume that σ, $H \in (1/2, 3/4)$ and p are known. Without loss of generality we assume that the functions $\varphi_i; i = 1, ..., p$ are orthonormal in $L^2([0, \nu], \nu^{-1}\ell)$ and that $\varphi_i; i = 1, ..., p$ are bounded by a constant $C > 0$. We observe the process continuously up to time $T = n\nu$ and let $n \to \infty$.

Related work

- Dehling, Franke and Kott (2010): Estimation in periodic Ornstein-Uhlenbeck processes
- Kleptsyna and Le Breton (2002): MLE for a fractional Ornstein-Uhlenbeck process based on associated semimartingales
- Hu and Nualart (2010): Least-squares estimator for a fractional Ornstein-Uhlenbeck process.

Some analytic background

For a fixed time interval $[0,T]$ the space $\mathcal{H}$ is defined as the closure of the set of real valued step functions on $[0,T]$ with respect to the scalar product $< 1_{[0,t]}, 1_{[0,s]} >_{\mathcal{H}} = E(B_t^H B_s^H)$.
The mapping $1_{[0,t]} \to B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B^H.
Noting that

$$E(B_t^H B_s^H) = H(2H-1)\int_0^t \int_0^s |u-v|^{2H-2} du dv$$

we obtain the useful isometry properties

$$E((\int_0^t \phi(s) dB_s^H)^2) = H(2H-1)\int_0^t \int_0^t \phi(u)\phi(v)|u-v|^{2H-2} du dv$$

$$E(\int_0^t \phi(s) dB_s^H \int_0^t \int_0^s \phi(u) dB_u^H dB_s^H) = 0.$$

Divergence integral

We have to interpret the integrals $\int_0^t u_s dB_s^H$ as **divergence integral**, i.e.

$$\int_0^t u_s dB_s^H = \delta(u1_{[0,s]})$$

or

$$\int_0^t u_s dB_s^H = \int_0^t u_s \partial B_s^H + H(2H-1)\int_0^t \int_0^T D_r u_v |u-v|^{2H-2} du dv$$

If we used a straight forward **Riemann Stieltjes integral**, it has been shown in Hu and Nualart (2010) that already the simple case of estimating α in a non-periodic setting by $\hat{\alpha} = -\frac{\int_0^{n\nu} X_t dX_t}{\int_0^{n\nu} X_t^2 dt}$ would not lead to a consistent estimator. Namely in the framework of Riemann Stieltjes integrals $\hat{\alpha}$ simplifies to $-\frac{X_{n\nu}^2}{2\int_0^{n\nu} X_t^2 dt}$, which tends to zero as $n \to \infty$.

Some preliminary facts on the model

$(X_t)_{t\geq 0}$ given by

$$X_t = e^{-\alpha t}\left(\xi_0 + \int_0^t e^{\alpha s} L(s)ds + \sigma \int_0^t e^{\alpha s} dB_s^H\right);\ \ t \geq 0$$

is the unique almost surely continuous solution of equation

$$dX_t = (L(t) - \alpha X_t)dt + \sigma dB_t^H$$

with initial condition $X_0 = \xi_0$. In the following we need a stationary solution.

$(\tilde{X}_t)_{t\geq 0}$ given by

$$\tilde{X}_t := e^{-\alpha t}\left(\int_{-\infty}^t e^{\alpha s} L(s)ds + \sigma \int_{-\infty}^t e^{\alpha s} dB_s^H\right)$$

is an almost surely continuous stationary solution of the equation above. Note that for large t the difference between the two representations tends to zero.

Construction of a stationary and ergodic sequence

For the limit theorems implying consistency and asymptotic normality we need a stationary and ergodic sequence.
Assume that L is periodic with period $\nu = 1$, then the sequence of $C[0,1]$-valued random variables

$$W_k(s) := \tilde{X}_{k-1+s}, 0 \leq s \leq 1, k \in \mathbb{N}$$

is **stationary and ergodic**.

Proof.

Since L is periodic, the function

$$\tilde{h}(t) := e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} L(s) ds$$

is also periodic on $\mathbb{R}$. We have for any $t \in [0, 1]$ that

$$W_k(t) = \tilde{h}(t) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dB^H_{s+k-1} + \sigma \sum_{j=-\infty}^{0} e^{-\alpha(t+1-j)} \int_0^1 e^{\alpha s} dB^H_{s+j+k-2}.$$

Thus, we have the almost sure representation

$$W_k(\cdot) = \tilde{h}(\cdot) + F_0(Y_k) + \sum_{j=-\infty}^{0} e^{\alpha(j-1)} F(Y_{j+k-1})$$

with the functionals

$$F_0 : C[0,1] \to C[0,1]; \omega \mapsto \left(t \mapsto \sigma e^{-\alpha t} \int_0^t e^{\alpha s} d\omega(s) \right),$$

$$F : C[0,1] \to C[0,1]; \omega \mapsto \sigma e^{-\alpha t} \int_0^1 e^{\alpha s} d\omega(s)$$

and the $C[0,1]$-valued random variable $Y_l := \left[s \mapsto B^H_{s+l-1} - B^H_{l-1}; 0 \le s \le 1 \right]$. Since (Y_l) is defined via the increments of fractional Brownian motion, they form a **sequence of Gaussian random variables which is stationary and ergodic**. This implies that the sequence of $(W_k)_{k \in \mathbb{N}}$ is stationary and ergodic.

Motivation of the estimator

We start with the more general problem of a $p+1$-dimensional parameter vector $\theta = (\theta_1, ..., \theta_{p+1})$ in the stochastic differential equation

$$dX_t = \theta f(t, X_t)dt + \sigma dB_t^H,$$

where $f(t,x) = (f_1(t,x), ..., f_{p+1}(t,x))^t$ with suitable real valued functions $f_i(t,x); 1 \leq i \leq p$. A discretization of the above equation on the time interval $[0, T]$ yields for $\Delta t := T/N$ and $i = 1, ..., N$

$$X_{(i+1)\Delta t} - X_{i\Delta t} = \sum_{j=1}^{p+1} f_j(i\Delta t, X_{i\Delta t})\theta_j \Delta t + \sigma \left(B_{(i+1)\Delta t}^H - B_{i\Delta t}^H \right).$$

Now we can use a least-squares approach and minimize

$$\mathcal{G} : (\theta_1, ..., \theta_{p+1}) \mapsto \sum_{i=1}^{N} \left(X_{(i+1)\Delta t} - X_{i\Delta t} - \sum_{j=1}^{p+1} f_j(i\Delta t, X_{i\Delta t})\theta_j \Delta t \right)^2.$$

Least-squares estimator for general setting

As in Franke and Kott (2013) in a Lévy setting a least-squares estimator may be deduces which motivates the continuous time estimator $\hat{\theta}_T = Q_T^{-1} P_T$ with

$$Q_T = \begin{pmatrix} \int_0^T f_1(t, X_t) f_1(t, X_t) dt & \dots & \int_0^T f_1(t, X_t) f_{p+1}(t, X_t) dt \\ \vdots & & \vdots \\ \int_0^T f_{p+1}(t, X_t) f_1(t, X_t) dt & \dots & \int_0^T f_{p+1}(t, X_t) f_{p+1}(t, X_t) dt \end{pmatrix}$$

and

$$P_T := \left(\int_0^T f_1(t, X_t) dX_t, ..., \int_0^T f_p(t, X_t) dX_t \right)^t .$$

Least-squares estimator for fractional OU-process

In the special case of the fractional Ornstein Uhlenbeck process we have $\theta = (\mu_1, ..., \mu_p, \alpha)$ and $f(t,x) := (\varphi_1, ..., \varphi_p, -x)^t$. This yields for $T = n\nu$ the estimator

$$\hat{\theta}_n := Q_n^{-1} P_n$$

with

$$P_n := \left(\int_0^{n\nu} \varphi_1(t) dX_t, ..., \int_0^{n\nu} \varphi_p(t) dX_t, - \int_0^{n\nu} X_t dX_t \right)^t$$

and

$$Q_n := \begin{pmatrix} G_n & -a_n \\ -a_n^t & b_n \end{pmatrix},$$

where

$$G_n := \begin{pmatrix} \int_0^{n\nu} \varphi_1(t)\varphi_1(t)dt & \dots & \int_0^{n\nu} \varphi_1(t)\varphi_p(t)dt \\ \vdots & & \vdots \\ \int_0^{n\nu} \varphi_p(t)\varphi_1(t)dt & \dots & \int_0^{n\nu} \varphi_p(t)\varphi_p(t)dt \end{pmatrix} = n\nu I_p,$$

$$a_n^t := \left(\int_0^{n\nu} \varphi_1(t)X_t dt, \dots, \int_0^{n\nu} \varphi_p(t)X_t dt \right)$$

and

$$b_n := \int_0^{n\nu} X_t^2 dt.$$

Representation of the estimator

For $\nu = 1$ we have $\hat{\theta}_n = \theta + \sigma Q_n^{-1} R_n$ with

$$R_n := \left(\int_0^n \varphi_1(t) dB_t^H, ..., \int_0^n \varphi_p(t) dB_t^H, - \int_0^n X_t dB_t^H \right)^t$$

and

$$Q_n^{-1} = \frac{1}{n} \begin{pmatrix} I_p + \gamma_n \Lambda_n \Lambda_n^t & \gamma_n \Lambda_n \\ \gamma_n \Lambda_n^t & \gamma_n \end{pmatrix}.$$

with

$$\Lambda_n = (\Lambda_{n,1}, ..., \Lambda_{n,p})^t := \left(\frac{1}{n} \int_0^n \varphi_1(t) X_t dt, ..., \frac{1}{n} \int_0^n \varphi_p(t) X_t dt \right)^t$$

and

$$\gamma_n := \left(\frac{1}{n} \int_0^n X_t^2 dt - \sum_{i=1}^p \Lambda_{n,i}^2 \right)^{-1}.$$

Consistency of the estimator

First we can establish by the isometry property of fractional integrals and properties of multiple Wiener integrals that for $H \in (1/2, 3/4)$ the sequence $n^{-H} R_n$ is bounded in L^2.
Secondly we may show:
As $n \to \infty$ we obtain that nQ_n^{-1} converges almost surely to

$$C := \begin{pmatrix} I_p + \gamma \Lambda \Lambda^t & \gamma \Lambda \\ \gamma \Lambda^t & \gamma \end{pmatrix},$$

where

$$\Lambda = (\Lambda_1, ..., \Lambda_p)^t := \left(\int_0^1 \varphi_1(t) \tilde{h}(t) dt, ..., \int_0^1 \varphi_p(t) \tilde{h}(t) dt \right)^t$$

and

$$\gamma := \left(\int_0^t \tilde{h}^2(t) dt + \sigma^2 \alpha^{-2H} H \Gamma(2H) - \sum_{i=1}^{p} \Lambda_i^2 \right)^{-1},$$

with $\tilde{h}(t) := e^{-\alpha t} \sum_{i=1}^p \mu_i \int_{-\infty}^t e^{\alpha s} \varphi_i(s) ds$. Both together imply weak consistency for $H \in (1/2, 3/4)$.

Asymptotic normality

For $H \in (1/2, 3/4)$ we obtain for least-squares estimator $\hat{\theta}_n$

$$n^{1-H}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2 C \Sigma_0 C)$$

with

$$\Sigma_0 := \begin{pmatrix} \bar{G} & -\bar{a} \\ -\bar{a}^t & \bar{b} \end{pmatrix},$$

where

$$\bar{G} := \begin{pmatrix} \alpha_H \int_0^1 \int_0^1 \varphi_1(s)\varphi_1(t)|t-s|^{2H-2} dsdt & \dots & \alpha_H \int_0^1 \int_0^1 \varphi_1(s)\varphi_p(t)|t-s|^{2H-2} dsdt \\ \vdots & & \vdots \\ \alpha_H \int_0^1 \int_0^1 \varphi_p(s)\varphi_1(t)|t-s|^{2H-2} dsdt & \dots & \alpha_H \int_0^1 \int_0^1 \varphi_p(s)\varphi_p(t)|t-s|^{2H-2} dsdt \end{pmatrix},$$

$$\bar{a}^t := \left(\alpha_H \int_0^1 \int_0^1 \varphi_1(s)\tilde{h}(t)|t-s|^{2H-2} dsdt, \dots, \alpha_H \int_0^1 \int_0^1 \varphi_p(s)\tilde{h}(t)|t-s|^{2H-2} dsdt\right),$$

$$\bar{b} := \alpha_H \int_0^1 \int_0^1 \tilde{h}(s)\tilde{h}(t)|t-s|^{2H-2} dsdt,$$

$$\alpha_H = H(2H-1),$$

$$\tilde{h}(t) := e^{-\alpha t} \sum_{i=1}^{p} \mu_i \int_{-\infty}^{t} e^{\alpha s} \varphi_i(s) ds$$

Proof.

By the representation

$$\hat{\theta}_n - \theta = \sigma Q_n^{-1} R_n$$

and the almost sure convergence of $nQ_n^{-1} \to C$ it is sufficient to prove that as $n \to \infty$

$$\left(n^{-H} \int_0^n \varphi_1(t) dB_t^H, \ldots, n^{-H} \int_0^n \varphi_p(t) dB_t^H, -n^{-H} \int_0^n X_t dB_t^H \right)^t$$

$$\xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_0).$$

We may replace X_t by $\tilde{X}_t$, since $n^{-H} \int_0^n (X_t - \tilde{X}_t) dB_t^H \xrightarrow{P} 0$ as $n \to \infty$.

Now using the representation $\tilde{X}_t = \tilde{Z}_t + \tilde{h}(t)$ we may deduce that $\tilde{Z}_t = \sigma e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} dB_s^H$ does not contribute to the covariance matrix. Namely the contributions to the off-diagonal elements in $\bar{a}$ and the mixed term of $\bar{b}$ are zero by the isometry formula for multiple Wiener integrals of different order. Furthermore, $(n^{-H} \int_0^n \tilde{Z}_t dB_t^H) \to 0$ as $n \to \infty$, since we know by Hu and Nualart (2010) that $\frac{1}{n}(\int_0^n \tilde{Z}_t dB_t^H)$ is convergent and $2H > 1$ for $1/2 < H < 3/4$.

Hence it is sufficient to show that for the 1-periodic functions φ_i $(1 \le i \le p)$ and $\tilde{h}$ as $n \to \infty$

$$\left(n^{-H} \int_0^n \varphi_1(t) dB_t^H, \ldots, n^{-H} \int_0^n \varphi_p(t) dB_t^H, -n^{-H} \int_0^n \tilde{h}(t) dB_t^H \right)^t$$

$$\xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_0).$$

Auxiliary limit theorem

Let f_k $(1 \le k \le m)$ be periodic real valued functions with period 1, then for $H > 1/2$ and $n \to \infty$

$$\left(n^{-H} \int_0^n f_1(t) dB_t^H, \ldots, n^{-H} \int_0^n f_m(t) dB_t^H \right)^t$$

$$\xrightarrow{\mathcal{D}} \mathcal{N} \left(0, H(2H-1) \left(\int_0^1 \int_0^1 f_i(t) f_j(s) |t-s|^{2H-2} ds dt \right)_{1 \le i,j \le m} \right).$$

Proof.

Since f_k is periodic with period 1, we may write for $1 \leq k \leq m$

$$n^{-H} \int_0^n f_k(t) dB_t^H = n^{-H} \sum_{i=1}^n \int_{i-1}^i f_k(t) dB_t^H = n^{-H} \sum_{i=1}^n Y_i^k$$

with

$$Y_i^k \sim \mathcal{N}\left(0, H(2H-1) \int_0^1 \int_0^1 f_k(t) f_k(s) |t-s|^{2H-2} ds dt\right)$$

and

$$\begin{aligned} Cov(Y_i^k, Y_j^l) &= \rho_H(|i-j|) H(2H-1) \int_0^1 \int_0^1 f_k(t) f_l(s) |t-s|^{2H-2} ds dt \\ &\sim n^{2H-2} H^2 (2H-1)^2 \int_0^1 \int_0^1 f_k(t) f_l(s) |t-s|^{2H-2} ds dt \end{aligned}$$

for $1 \leq i, j \leq n$ and $1 \leq k, l \leq m$, since

$$\rho_H(n) = \frac{1}{2}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}).$$

Discussion I

The rate of convergence n^{1-H} is **slower** than in the Brownian case. Furthermore, it is also slower than the rate $n^{1/2}$ for the mean reverting parameter in a fractional Ornstein Uhlenbeck setting with $L = 0$. This is due to the special structure of our drift coefficient, which in our setting also dominates the component of α leading to a slower rate even for α and a different entry in the covariance matrix.
Note that if $\mu_i = 0$ for $i = 1, \cdots, p$ our asymptotic variance is degenerate which corresponds to the case in Hu and Nualart (2010) with the faster rate of convergence.

Discussion II

Unless in the Brownian case $\Sigma_0 \neq C^{-1}$.
This is due to the **isometry formula for fractional Brownian motion** with $H > 1/2$, which is not simply derived from the scalar product in L^2, but from the scalar product in a larger Hilbert space $\mathcal{H}$.
Namely for a fixed time interval $[0, T]$ the space $\mathcal{H}$ is defined as the closure of the set of real valued step functions on $[0, T]$ with respect to the scalar product $< 1_{[0,t]}, 1_{[0,s]} >_{\mathcal{H}} = (B_t^H B_s^H)$.

Conclusion

For the model

$$dX_t = (\sum_{i=1}^{p} \mu_i \phi_i(t) - \alpha X_t)dt + \sigma dB_t^H$$

we constructed a least-squares estimator, which is for $H \in (1/2, 3/4)$

- consistent as $T \to \infty$
- asymptotically normal with rate T^{1-H}