Two nonparametric strategies for estimating the jump rate of a piecewise-deterministic Markov process

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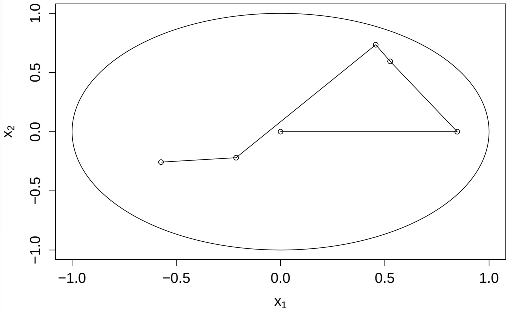
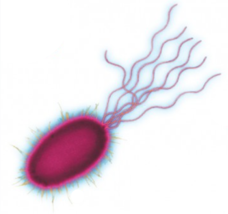
Outline

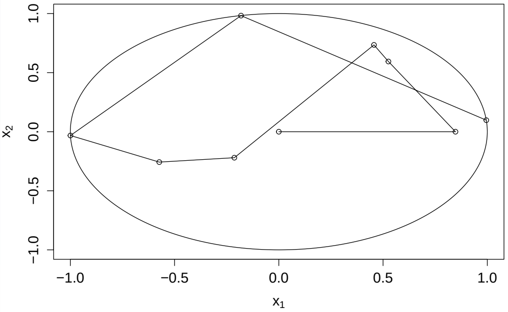
1. **Problem formulation**
   - Motivation: bacterial motility
   - Piecewise-deterministic Markov process
2. Strategies for estimating the jump rate
3. Optimal estimation of the jump rate
4. Numerical illustration

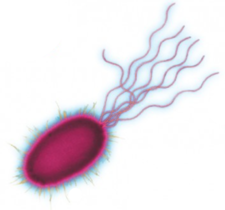
Motivation: bacterial motility





Motivation: bacterial motility





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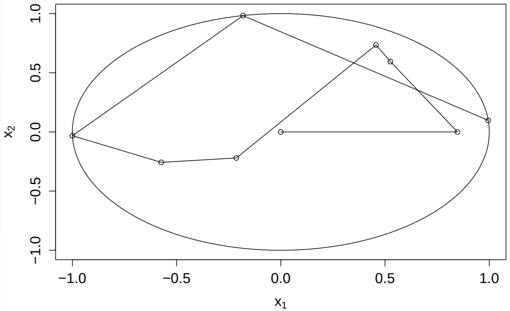

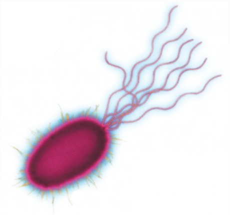

Is this a "uniform" random walk or is there an attractive chemical signal?

Piecewise-deterministic Markov process

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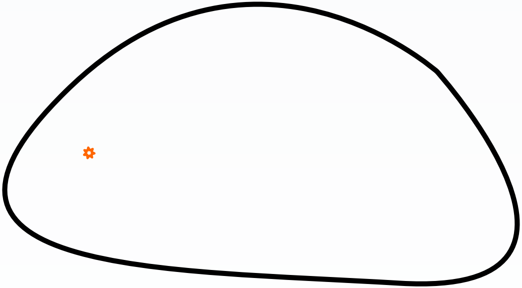
At time T_1 the process "jumps" according to $\mathcal{Q}$,

$$\mathbb{E}\left[\varphi(X_{T_1}) \,|\, \Phi(x, T_1)\right] = \int \varphi(u) \mathcal{Q}(\Phi(x, T_1), \mathrm{d}u).$$

And so on...

Piecewise-deterministic Markov process



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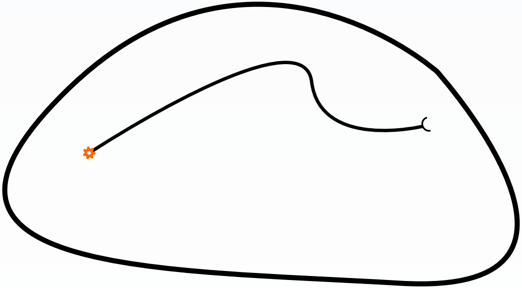

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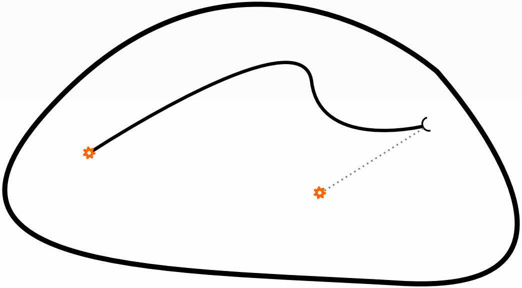

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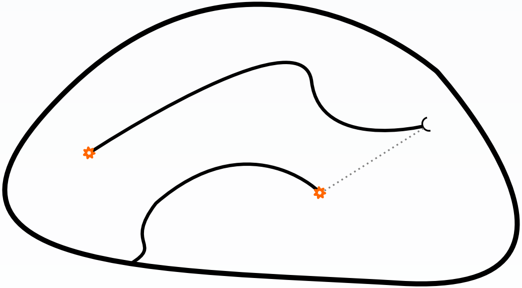

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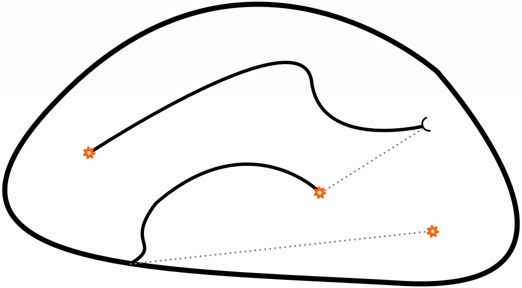

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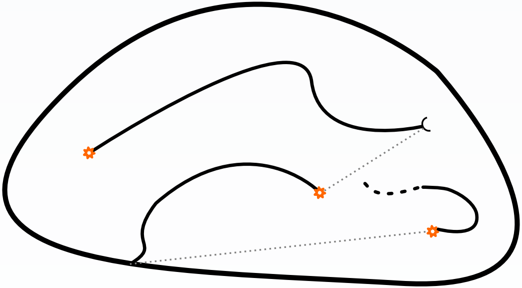

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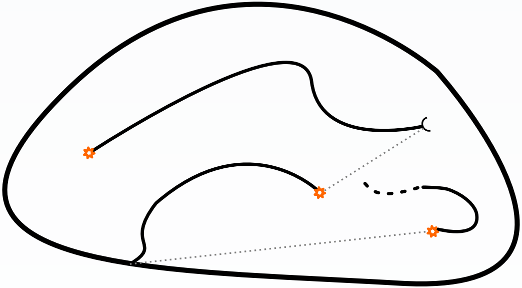

Observation of $Z_n = X_{T_n}$ and $S_{n+1} = T_{n+1} - T_n$ within a long time interval

Aim: nonparametric estimation of the jump rate λ

Outline

1. Problem formulation
2. **Strategies for estimating the jump rate**
   - Multiplicative intensity model
   - Ratio density function over survival function
3. Optimal estimation of the jump rate
4. Numerical illustration

Multiplicative intensity model

Characterization of the jump rate λ:

$$\mathbb{P}(S_{n+1} > t \,|\, Z_n = x) = \exp\left(-\int_0^t \lambda\left(\Phi(x,s)\right) \mathrm{d}s\right) \mathbb{1}_{\{\Phi(x,t)\in E\}}.$$

Conditionnaly on $Z_n = x$, we observe the (right-censored) time S_{n+1} distributed according to the non homogeneous rate $\lambda \circ \Phi(x,t)$.

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Nelson-Aalen strategy

$$M(t) = N(t) - \int_0^t \underset{\substack{\uparrow \\ \text{jump rate of interest}}}{\alpha(s)} Y(s) \mathrm{d}s$$

Multiplicative intensity model

Conditionally on Z_n,

$$t \mapsto \mathbb{1}_{\{S_{n+1} \leq t\}} - \int_0^t \lambda(\Phi(Z_n, s))\mathbb{1}_{\{S_{n+1} \geq s\}} \mathrm{d}s$$

is a continuous-time martingale.

But the sum over n is generally not a martingale

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Solution:

→ Estimation for the double-marked renewal process (Z_n, Z_{n+1}, S_{n+1})

→ Back to the initial process:
 – discretization of the state space
 – estimation of 2 other functionals

Ratio density function over survival function

Conditionally on $Z_n = x$, the distribution of S_{n+1} admits a density $f(x, \cdot)$ on the interval

$$(0, \inf\{t > 0 : \Phi(x,t) \in \partial E\}),$$

and

$$f(x,t) = \lambda(\Phi(x,t)) \exp\left(-\int_0^t \lambda(\Phi(x,s))\mathrm{d}s\right)$$

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As a consequence,

$$\lambda \circ \Phi(x,t) = \frac{f(x,t)}{G(x,t)}$$

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As a consequence,

$$\lambda \circ \Phi(x,t) = \frac{f(x,t)}{G(x,t)} = \frac{\nu_\infty(x) f(x,t)}{\nu_\infty(x) G(x,t)}$$

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3. **Optimal estimation of the jump rate**
   - Class of estimators indexed by the flow
   - How to choose among this class?
4. Numerical illustration

Class of estimators indexed by the flow

$$\widehat{\mathcal{F}}^n(x,t) = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{v_i^d w_i}\mathbb{K}_d\left(\frac{Z_i - x}{v_i}\right)\mathbb{K}_1\left(\frac{S_{i+1}-t}{w_i}\right)$$

$$\widehat{\mathcal{G}}^n(x,t) = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{v_i^d}\mathbb{K}_d\left(\frac{Z_i - x}{v_i}\right)\mathbb{1}_{\{S_{i+1}>t\}}$$

Estimation of the composed function $\lambda \circ \Phi$

$$\widehat{\lambda \circ \Phi}^n(x,t) = \frac{\widehat{\mathcal{F}}^n(x,t)}{\widehat{\mathcal{G}}^n(x,t)}$$

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Estimation of the composed function $\lambda\circ\Phi$

$$\widehat{\lambda\circ\Phi}^n(x,t) = \frac{\widehat{\mathcal{F}}^n(x,t)}{\widehat{\mathcal{G}}^n(x,t)}$$

Pointwise convergence of $\widehat{\lambda\circ\Phi}^n(x,t)$ $\qquad x\in E$ and $\Phi(x,t)\in E$

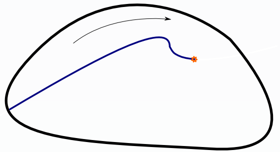
$$\widehat{\lambda\circ\Phi}^n(x,t) \xrightarrow{a.s.} \lambda\circ\Phi(x,t)$$

$$n^{\frac{1-\alpha d-\beta}{2}}\left(\widehat{\lambda\circ\Phi}^n(x,t) - \lambda\circ\Phi(x,t)\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\tau_1^2\,\tau_d^2\,\lambda\circ\Phi(x,t)}{(1+\alpha d+\beta)\nu_\infty(x)G(x,t)}\right)$$

Class of estimators indexed by the flow

Estimation of $\lambda(x)$ for some $x \in E$

$$\mathcal{C}_x = \{\Phi(x, -t) \ : \ t \geq 0\} \cap E$$



For any $\xi \in \mathcal{C}_x$, there exists a unique time $t = \tau_x(\xi)$ such that $\Phi(\xi, \tau_x(\xi)) = x$

In particular, $\lambda \circ \Phi(\xi, \tau_x(\xi)) = \lambda(x)$

Thus, $\widehat{\lambda \circ \Phi}^n(\xi, \tau_x(\xi))$ estimates $\lambda(x)$, for any $\xi \in \mathcal{C}_x$

How to choose among this class?

We propose to choose among this class the estimate with the minimal asymptotic variance in the central limit theorem:

$$\frac{\tau_1^2\,\tau_d^2\,\lambda\circ\Phi(\xi,\tau_x(\xi))}{(1+\alpha d+\beta)\nu_\infty(\xi)G(\xi,t)} \propto \left(\nu_\infty(\xi)G(\xi,\tau_x(\xi))\right)^{-1} = \kappa_x(\xi)^{-1}$$

$$\widehat{\lambda}^n(x) = \widehat{\lambda\circ\Phi}^n(\xi^\star,\tau_x(\xi^\star)) \qquad \text{for} \quad \xi^\star = \arg\max_{\xi\in\mathcal{C}_x} \kappa_x(\xi)$$

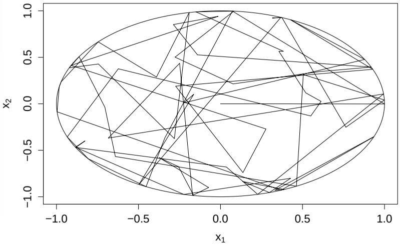
$$\widehat{\widehat{\lambda}}^n(x) = \widehat{\lambda\circ\Phi}^n(\xi^\star,\tau_x(\xi^\star)) \qquad \text{for} \quad \xi^\star = \arg\max_{\xi\in\mathcal{C}_x} \widehat{\mathcal{G}}^n(\xi,\tau_x(\xi))$$

Outline

Back to bacterial motility

- State space: $E = \{x \in \mathbb{R}^2 \, : \, |x| < 1\} \times (0, 2\pi)$
- $\Phi((x_1, x_2, \theta)\, ,\, t) = (x_1 + t\cos\theta, x_2 + t\sin\theta, \theta)$
- $\mathcal{Q}((x_1, x_2, \theta), \cdot) = \delta_{\{x_1, x_2\}} \mathcal{U}_{(0,2\pi)}$
- $\lambda(x_1, x_2, \theta) = \lambda(x_1, x_2)$

The rate λ is not direction-dependent. But is it really location-dependent?



Estimation of the jump rate

Estimation of λ at $x = (x_1, x_2, \theta) \in E$

- The curve $\mathcal{C}_x$ is given by

$$\mathcal{C}_x = \{(x_1 - t\cos\theta, x_2 - t\sin\theta, \theta) \,:\, t \geq 0\} \cap E$$

- Compute the optimal point $\xi^\star \in \mathcal{C}_x$ that maximizes $\widehat{\mathcal{G}}^n(\xi, \tau_x(\xi))$
- $\widehat{\lambda}^n_{\xi^\star}(x)$ is a good estimate of $\lambda(x)$

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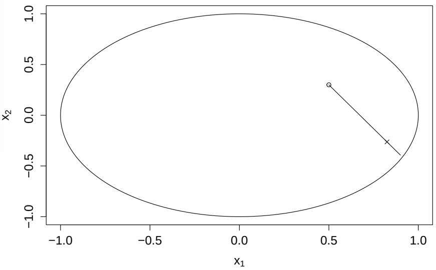
- $\widehat{\lambda}^n_{\xi^\star}(x)$ is a good estimate of $\lambda(x) = \lambda(x_1, x_2, \not\theta)$
- By only changing θ, one obtains another good estimate of $\lambda(x_1, x_2)$

Estimation of the jump rate

Estimation at $x = (0.5, 0.3)$



θ	$\widehat{\lambda}^{n}_{\xi^\star}(x)$
$2\pi/3$	1.31

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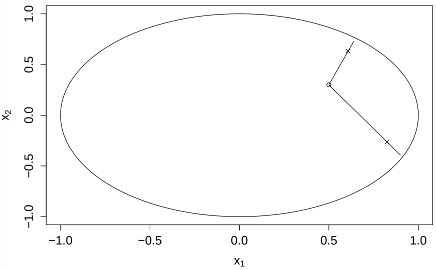

θ	$\widehat{\lambda}^n_{\xi^\star}(x)$
$2\pi/3$	1.31
$7\pi/5$	0.80

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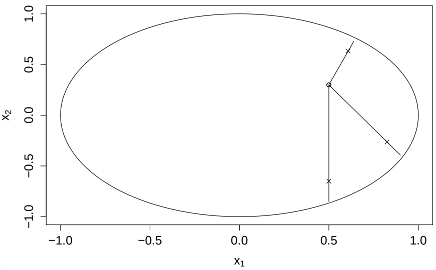

θ	$\widehat{\lambda}^{n}_{\xi^\star}(x)$
$2\pi/3$	1.31
$7\pi/5$	0.80
$5\pi/2$	1.41

Estimation of the jump rate

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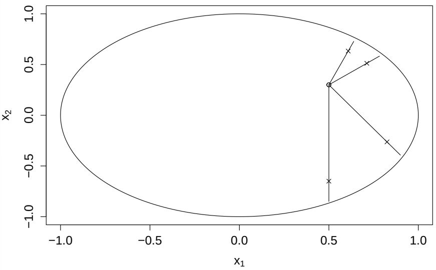

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$\pi/2$	1.13

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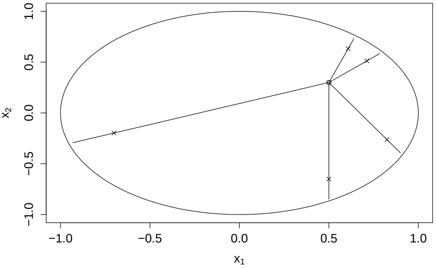
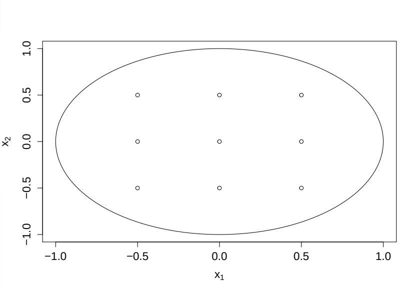


θ	$\widehat{\lambda}^n_{\xi^\star}(x)$
$2\pi/3$	1.31
$7\pi/5$	0.80
$5\pi/2$	1.41
$\pi/2$	1.13
$\pi/8$	0.91

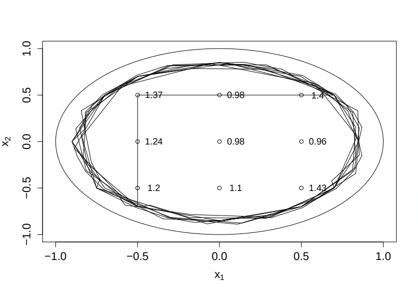
Estimation of the jump rate

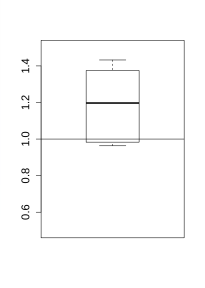
Estimation at 9 target points



Estimation of the jump rate

Estimation at 9 target points from $n = 20\,000$ observed jumps





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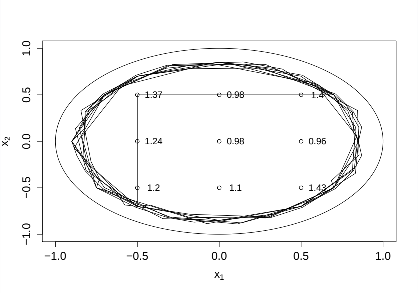
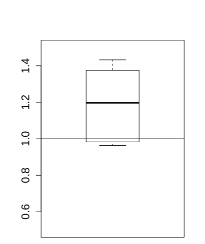
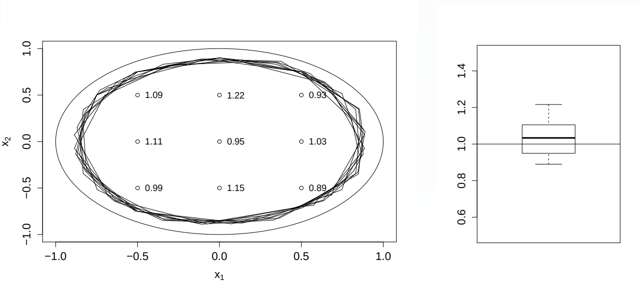




Remark: $\kappa_x(\xi) = \nu_\infty(\xi) G(\xi, \tau_x(\xi)) = \nu_\infty(\xi) \exp(-|x - \xi|)$

Estimation of the jump rate

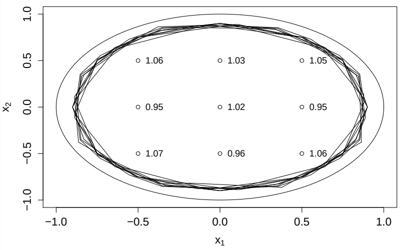
Estimation at 9 target points from $n = \ 50\,000$ observed jumps

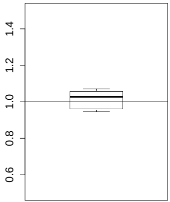


Remark: $\kappa_x(\xi) = \nu_\infty(\xi) G(\xi, \tau_x(\xi)) = \nu_\infty(\xi) \exp(-|x - \xi|)$

Estimation of the jump rate

Estimation at 9 target points from $n = 100\,000$ observed jumps





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