$SL_2(\mathbb{R})$

Dynamics and Harmonic Analysis

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The group $\mathrm{SL}_2(\mathbb{R})$

and some subgroups

Definition

The group $\mathrm{SL}_2(\mathbb{R})$ is the group of 2×2 real matrices of determinant 1:

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1 \right\}.$$

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The largest normal subgroup of $\mathrm{SL}_2(\mathbb{R})$ is its centre $Z = \{\pm I\}$, $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Hence the quotient group $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ has trivial centre.

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Remarquable one-parameter subgroups

$$K = \left\{ \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right] \right\}_{\theta\in\mathbb{R}}, \quad A = \left\{ \left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right] : a > 0 \right\}, \quad N = \left\{ \left[\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right] \right\}_{u\in\mathbb{R}},$$

$$A^{\perp} = \left\{ \left[\begin{smallmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{smallmatrix}\right] \right\}_{\theta\in\mathbb{R}}, \quad \overline{N} = \left\{ \left[\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right] \right\}_{v\in\mathbb{R}},$$

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Next we introduce some fundamental actions of $\mathrm{SL}_2(\mathbb{R})$ which will be further exploited later on.

Topology of $G = \mathrm{SL}_2(\mathbb{R})$

Connectness

G acts linearly on $\mathbb{R}^2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.(x, y) = (ax + by, cx + dy)$$

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The orbits are: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.

The stabiliser of the point $(1, 0)$ is the group N. Hence we have

$$\mathbb{R}^2 \setminus \{0\} = G.(1, 0) \approx G/\,\mathrm{Stab}_G[(1, 0)] = G/N.$$

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Remark (for the wine cellar)

The linear action of G on $\mathbb{R}^2$ preserves the Lebesgue measure on $\mathbb{R}^2$ and induces a linear action on the full tensor algebra of $\mathbb{R}^2$.

Topology of $G = \mathrm{SL}_2(\mathbb{R})$

Fundamental group

G (in fact $\mathrm{PSL}_2(\mathbb{R})$) acts on $\mathcal{Q}_2$, be the vector space of symmetric bilinear forms on $\mathbb{R}^2$:

$$(g, Q) \mapsto gQg^T, \qquad (g \in G, Q \in \mathcal{Q}_2)$$

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$$\mathcal{Q}_2 \leftrightarrow \mathbb{R}^3, \qquad \left[\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right] \leftrightarrow (A, B, C)$$

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Consequence

Since $\mathrm{Stab}_G[-I] = K$ and since the sheet S is contractible from the fibration $K \to G \to G/K \approx S$ we get that K injects its π_1 in G: $\pi_1(G) = \pi_1(K) \approx \mathbb{Z}$.

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Isomorphism $\mathrm{PSL}_2(\mathbb{R}) \approx \mathrm{SO}(1,2)_0$.

Topology of $G = \mathrm{SL}_2(\mathbb{R})$

Simplicity

The Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ is the tangent space $\mathfrak{sl}_2(\mathbb{R}) := T_I G$ of G at the identity I. Thus

$$\mathfrak{sl}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : \quad a, b, c \in \mathbb{R} \right\}.$$

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Generators of the one-parameters subgroups K, A, $A^\perp$, N and $\bar{N}$ are:

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h^\perp = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad n^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad n^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

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In the basis (h, n^+, n^-), the Lie algebra structure is completely determined by the commutation rules

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Consequence

$\mathrm{SL}_2(\mathbb{R})$ is a simple Lie group (no subspace of $\mathfrak{sl}_2(\mathbb{R})$ is stable for all the maps $\mathrm{ad}(X) : Y \mapsto [X, Y]$).

Linear action on $\mathbb{R}^2$

KAN decomposition

For all $(x, y) \neq (0, 0)$ $\exists!\, k_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in K$ and $\exists!\, a_r = \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix} \in A$ such that

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Since $K \cap AN = I$ we obtain

Theorem (Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$)

Every element $g \in \mathrm{SL}_2(\mathbb{R})$ can be written in a unique way as a product

$$g = kan$$

with $k \in K$, $a \in A$ and $n \in N$. In particular $\mathrm{SL}_2(\mathbb{R}) = KAN$.

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From the linear action on $\mathbb{R}^2$ we also get a number of representations of $\mathrm{SL}_2(\mathbb{R})$.

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When V is a infinite dimensional locally convex, complete, Hausdorff TVS then we require that

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If V is a Hilbert, Banach or Frechet space, this is tantamount to the continuity of the map

$$(g, v) \in G \times V \mapsto \pi(g)v \in V$$

Representations

Interlude,2

Remark

The previous hypotheses on (π, V) are sufficient to allows to define

$$\pi(f) := \int_G f(g)\pi(g)\,\mathrm{d}g \in \mathcal{L}_s(V)$$

for any function $f \in C_c(G)$. (Here $\mathrm{d}g$ is a (left)-Haar measure on G).

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Unitary representations of $\mathrm{SL}_2(\mathbb{R})$, or more generally of semi-simple (or reductive), groups split as a direct sum or direct integral of unitary irreducible representations. To some extent the goal of representation theory is the classification of all irreducible (unitary) representations of a group G and the decomposition of (remarkable) representations (e.g. the regular representation of G on $L^2(G, \mathrm{d}g)$).

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If $(V, \langle \mid \rangle)$ is a Hilbert space and G is a compact Lie group then any representation can be made unitary by defining a new product

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For a compact Lie group all irreducible representations are finite dimensional.

Example

The irreducible representations of S^1 are (one-dimensional) characters $\varepsilon_n : x \in \mathbb{R}/\mathbb{Z} \mapsto \varepsilon_n(x) := e^{2\pi inx} \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$.

Linear action on $\mathbb{R}^2$

Finite dimensional representations

The linear action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$ induces a representation on the space P_d of homogeneous polynomials $P(x, y)$ of degree d on $\mathbb{R}^2$:

$$P_d = \mathrm{span}_{\mathbb{C}}\{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}, \quad \dim P_d = d + 1$$

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$$(g.P)(x, y) = P(ax + by, cx + dy) \qquad \text{if } g^{-1} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], P \in P_d$$

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$$P_d = \mathrm{span}_{\mathbb{C}}\{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}, \quad \dim P_d = d + 1$$

$$(g.P)(x, y) = P(ax + by, cx + dy) \qquad \text{if } g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, P \in P_d$$

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None of these representations are unitarisable as $U(d)$ is compact.

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Unitary

The map

$$(\rho(g).f)(x,y) = f(g^{-1}(x,y)), \quad f \in H = L^2(\mathbb{R}^2, \mathrm{d}x\mathrm{d}y)$$

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(2) ρ commutes with the homotheties

$$(U_t f)(x,y) := (\exp(itA)f)(x,y) := e^t f(e^t x, e^t y), \quad f \in H^\pm, t \in \mathbb{R}, (x,y) \in \mathbb{R}^2.$$

It follows that ρ commutes with the spectral projectors of

$$A = \frac{1}{i}\left(\frac{\partial}{\partial r} + 1\right), \quad \left(r = \sqrt{x^2+y^2}\right),$$

In conclusion ρ leaves invariant the generalized eigenspaces of A

$$\mathcal{H}^{\pm,i\nu} = \left\{ f \in L^2_{\mathrm{loc}}(R^2) \mid f(rx,ry) = r^{-1+i\nu} f(x,y), f(-x,-y) = \pm f(x,y) \right\} \ (\nu \in \mathbb{R}).$$

Linear action on $\mathcal{Q}_2$

KAK decomposition

Every positive definite bilinear form of determinant 1 on $\mathbb{R}^2$, may be conjugated by a rotation to the diagonal bilinear form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, with $\lambda \in \mathbb{R}_+^*$. Thus

Theorem (Cartan's *KAK* decomposition for $\mathrm{SL}_2(\mathbb{R})$)

Every element $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as a product

$$g = k_1 a k_2$$

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Proof.

The map $gK \in G/K \leftrightarrow gg^T \in \mathcal{Q}_2^{(1)}$ is a bijection on the orbit of I (the positive definite matrices of determinant one). Since $gg^T = k \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} k^T$, $(k \in K)$, we get $gK = k \begin{bmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{bmatrix} K$, that is $g = k \begin{bmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{bmatrix} k'$. □

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Differently from the the Iwasawa decomposition, the Cartan *KAK* decomposition of an element is not unique.

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The normalizer of A in $\mathrm{SO}(2)$ is generated by $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

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From the Cartan's *KAK* decomposition the polar decomposition follows:

Theorem (Cartan's polar decomposition for $\mathrm{SL}_2(\mathbb{R})$)

Every element $g \in \mathrm{SL}_2(\mathbb{R})$ can be uniquely written as a product

$$g = k \exp S$$

with $k \in K$ and $S \in \mathfrak{sl}_2(\mathbb{R})$ a traceless symmetric matrix.

The action of $\mathrm{SL}_2(\mathbb{R})$ on the projective line

The linear action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$ induces an action of $\mathrm{SL}_2(\mathbb{R})$ on the projective line $\mathbb{P}^1(\mathbb{R})$. In the affine chart $x = [x : 1]$ the action is given by

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right].x = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right].[x : 1] = [(ax+b) : (cx+d)] = \frac{ax+b}{cx+d}, \quad x \in \mathbb{R}. \tag{1}$$

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As the group $\mathrm{SL}_2(\mathbb{C})$ acts linearly on $\mathbb{C}^2$ we also have a projective action of $\mathrm{SL}_2(\mathbb{C})$ on the Riemann sphere $\hat{\mathbb{C}} := \mathbb{P}^1(\mathbb{C})$.

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Restricting this action to $\mathrm{SL}_2(\mathbb{C})$ we obtain a action of $\mathrm{SL}_2(\mathbb{R})$ on $\hat{\mathbb{C}}$, which in the affine chart $z = [z : 1]$ of $\hat{\mathbb{C}}$ is also given by the formulas (1), where x is replaced by $z \in \mathbb{C}$.

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In the affine chart $z = [z : 1]$ the latter two orbits are identified are respectively, the upper half plane

$$\mathbb{H}^2 := \{z \in \mathbb{C} \mid \Im z > 0\}$$

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The subgroup

$$AN = \left\{ \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix} \mid s \in \mathbb{R}^*_+, t \in \mathbb{R} \right\},$$

acts simply transitively on $\mathbb{H}^2$ and $\mathbb{H}^2_-$ by the formulas

$$\begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix} .(\pm i) = st \pm s^2 i.$$

The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$

part 1

We shall now focus on the action of $\mathrm{PSL}_2(\mathbb{R}))$ on $\mathbb{H}^2$ given by the formulas

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).z = \frac{az+b}{cz+d}, \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}), z \in \mathbb{H}^2. \tag{2}$$

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Thus there a unique (up to rescaling) $\mathrm{SL}_2(\mathbb{R})$-invariant Riemannian metric on $\mathbb{H}^2$ given

$$\mathrm{d}s^2 = \frac{|\mathrm{d}z|^2}{(\Im z)^2} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}, \quad z = x + iy \in \mathbb{H}^2.$$

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The upper half plane $\mathbb{H}^2$ endowed with the above Riemannian metric is the *Poincaré half plane* or *Poincaré hyperbolic plane*. (The normalization of the metric is such that the curvature is constantly equal to -1).

The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$

part 2

The group $\mathrm{PSL}_2(\mathbb{R})$ is now, by definition of the metric the coonnected component of the identity of group of isometries of the Poincaré hyperbolic plane.

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$\mathrm{PSL}_2(\mathbb{R})$ acts on itself, hence on $T^1\mathbb{H}^2$, on the left and on the right. The action on the left is the differential of isometries of $\mathbb{H}^2$. The action on the right gives rise to geometrical flows.

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $T^1\mathbb{H}^2$

Geodesic flow on the Poincaré plane

Parametrize the subgroup $A \subset \mathrm{PSL}_2(\mathbb{R})$ by

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

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$$\{(e^t i, e^t i) = (\gamma(t), \gamma'(t))\}_{t\in\mathbb{R}}$$

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As isometries map geodesics to geodesics, for any $g \in \mathrm{PSL}_2(\mathbb{R})$, the path $\bar{\gamma}(t) = g.\gamma(t)$ is the unique unit speed geodesic with initial velocity $\bar{\gamma}'(0) = \mathrm{d}g_i(i) \in T^1_{g.i}\mathbb{H}^2$.

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $T^1\mathbb{H}^2$

Geodesic flow on the Poincaré plane

Parametrize the subgroup $A \subset \mathrm{PSL}_2(\mathbb{R})$ by

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Under the identification $\mathrm{PSL}_2(\mathbb{R}) \leftrightarrow T^1\mathbb{H}^2$ the path $(a_t)_{t\in\mathbb{R}}$ in $\mathrm{PSL}_2(\mathbb{R})$ is identified with the geodesic path in $T^1\mathbb{H}^2$

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Thus the path $(ga_t)_{t\in\mathbb{R}}$ is identified with the path $(\bar{\gamma}(t), \bar{\gamma}'(t))$ in $T^1\mathbb{H}^2$, the lift to $T^1\mathbb{H}^2$ of the geodesic $\bar{\gamma}$.

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$$(g, t) \in \mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R} \mapsto \Phi^A_t(g) := ga_t \in \mathrm{PSL}_2(\mathbb{R}).$$

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $T^1\mathbb{H}^2$

Horocycle flows on the Poincaré plane

Parametrize N and $\overline{N} = wNw^{-1}$ by

$$n_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \qquad \bar{n}_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}. \tag{3}$$

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Then

$$n_s a_t = a_t n_{se^{-t}}, \qquad \bar{n}_s a_t = a_t \bar{n}_{se^t} \tag{4}$$

It follows that the flows $(\Phi_t^{N,\overline{N}})_{t\in\mathbb{R}}$ on $\mathrm{PSL}_2(\mathbb{R})$ defined by

$$\Phi_t^N(g) = gn_t, \quad \Phi_t^{\overline{N}}(g) = g\bar{n}_t$$

satisfy

$$\Phi_t^A \circ \Phi_s^N = \Phi_{se^{-t}}^N \circ \Phi_t^A, \qquad \Phi_t^A \circ \Phi_s^{\overline{N}} = \Phi_{se^t}^{\overline{N}} \circ \Phi_t^A. \tag{5}$$

i.e. the map Φ_t^A contracts the flow (Φ_s^N) by a factor e^{-t} by expands the flow $(\Phi_s^{\bar{N}})$ by a factor e^t.
Equivalent formulation:

$$\mathrm{d}\Phi_t^A(n^+) = e^{-t}n^+, \quad \mathrm{d}\Phi_t^A(n^-) = e^t n^-, \quad \text{or} \quad [-h, n^\pm] = \mp n^\pm.$$

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $T^1\mathbb{H}^2$

Rotational flow and parallel transport on the Poincaré plane

The group $\bar{K}$ operates by multiplication on the right on $\mathrm{PSL}_2(\mathbb{R}) \approx T^1\mathbb{H}^2$ leaving each fiber $T^1_z\mathbb{H}^2$ stable.

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In particular the flow generated by $h^\perp = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] = k_{-\pi/2} h k_{\pi/2}$ is therefore the parallel transport along the geodesic making a right angle $\pi/2$ with the initial unit vector.

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It follows that on $\bar{K}$ invariant function the operator

$$-\frac{1}{4}(h^2 + (h^\perp)^2)$$

is the Laplace-Beltrami operator for the Poincaré metric on $\mathbb{H}^2$

Hyperbolic surfaces

and their tangent unit bundle

The group $\mathrm{SL}_2(\mathbb{R})$ has many discrete subgroups Γ. Then the quotient $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ is a smooth manifold and the quotient $\Gamma\backslash\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$ is a Riemann surface, with at most some conical points. The celebrated example is the group $\mathrm{SL}_2(\mathbb{R})$

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We get a unitary representation π of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}),\mu)$ by letting

$$(\pi(g)f)(\Gamma x) = f(\Gamma xg), \quad f \in L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}),\mu).$$

which decomposes as a direct integral or sum of irreducible represenations of $\mathrm{SL}_2(\mathbb{R})$.

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One of our goals will be to shed some light of the relation of the geometry of the surface $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ and the decomposition into irreducible representations.

Lie algebras/Lie groups generalities

The adjoint representation

For any matrix group G (i.e. closed subgroup of $\mathrm{GL}(\mathbb{R}^n)$) the Lie algebra $\mathfrak{g}_0$ of G is the tangent space $\mathfrak{g}_0 = T_I G$ of G at the identity I which we may also identify with the vector space of left invariant vector fields on G. The braket $[X, Y]$ is the commutator of the corresponding vector fields.

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G acts on itself by conjugation fixing the identity, and form this we get an action of G by automorphims of $\mathfrak{g}_0$, which for a linear group is given by

$$\mathrm{Ad}(g)X = gXg^{-1}, \qquad g \in G, X \in \mathfrak{g}$$

The map $g \mapsto \mathrm{Ad}(g)$ is a homomorphism of G in $\mathrm{Aut}(\mathfrak{g}_0)$ called the *adjoint representation*.

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Then a the Lie bracket operation arises as the infinitesimal version of this action:

$$\mathrm{ad}(Y)X := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathrm{Ad}(e^{\epsilon Y})X = [Y, X].$$

Lie algebras/Lie groups generalities

The Cartan-Killing form

For any Lie algebra $\mathfrak{g}_0$, the bilinear form

$$B(X, Y) = \mathrm{trace}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)), \qquad (X, Y \in \mathfrak{g}_0),$$

is symmetric and invariant under the adjoint action

$$B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X, Y), \qquad (X, Y \in \mathfrak{g}_0, g_0 \in G).$$

For $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$, using the commutation relations

$$[\kappa, h] = 2h^{\perp}, \quad [\kappa, h^{\perp}] = -2h, \quad [h, h^{\perp}] = -2\kappa$$

we get

$$B = 8\left[(\kappa^*)^2 - (h^*)^2 - ((h^{\perp})^*)^2\right]$$

which is a non degenerate bilinear form of signature $(1, 2)$. (Thus $\mathfrak{sl}_2(\mathbb{R})$ is the 3-dimensional lorentzian space-time).

Lie algebras

Enveloping algebra

Elements of $\mathfrak{g}_0$ corresponds to left invariant vector fields on G, i.e. first order differential operators on G. Composing such operators and considering their linear span over $\mathbb{C}$ we obtain an $\mathbb{C}$-algebra of left invariant vector differential operators on G.

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Casimir element and the center of $\mathfrak{U}(\mathfrak{g})$

From the invariance of the Cartan-Killing form we have that element

$$\Omega = -\frac{1}{4}(h^2 + (h^\perp)^2 - \kappa^2) \in \mathfrak{U}(\mathfrak{sl}_2(\mathbb{R}))$$

commutes with all $X \in \mathfrak{sl}_2(\mathbb{R})$ and therefore with all $\mathfrak{U}(\mathfrak{sl}_2(\mathbb{R}))$.

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In fact Ω generates the center $Z(\mathfrak{sl}_2(\mathbb{R}))$ of $\mathfrak{U}(\mathfrak{sl}_2(\mathbb{R}))$.

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Theorem (Dixmier-Schur Lemma)

If V is an irreducible $\mathcal{U}(\mathfrak{g})$-module of countable dimension then every $Z \in Z(\mathfrak{g})$ acts as a scalar on V.

Smooth vectors

Geodesic flow on the Poincaré plane

Definition

Let (π, V) be a representation of a Lie group G dans un TVS V. A vector $v \in V$ is of class C^r ($r \in \mathbb{N} \cup \{\infty, \omega\}$) if the map

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On note V^∞, (V^ω), the subspace of C^∞, (C^ω), vectors. Let (π, V) be a representation of a Lie group G.

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If V is a "good" space, then V^∞ is dense in V.

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Theorem (Harish-Chandra, Nelson)

If V is a Banach space, then V^ω is dense in V.

K-finite vectors

Here the appropriate setting is G a reductive linear group (a closed subgroup of $GL(n, \mathbb{R})$ [of $GL(n, \mathbb{C})$] stable under $g \mapsto (g^{-1})^T$, [under $g \mapsto (g^{-1})^\dagger$]). The $K = G \cap O(n)$ [$K = G \cap O(n)$].

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If V is an Hilbert space and (π, V) a rep of G then K-finite vectors are dense in V.

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The statement of the example is still valid in "good" spaces V.

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The statement of the example is still valid in "good" spaces V.

On note $V_{(K)}$ the subspace of V of K-finite vectors.

Isotypical components

Definition

Fix an irreducible representation (τ, W) of K. The isotypical component of type τ of a representation (π, V) of G is the set of vectors $v \in V$ such that for some continuous map $L\colon W \to V$ we have $v = Lw$ and $\pi(k)L = L\tau(k)$ for all $k \in K$.

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Example

If ϵ^n is the character of $\mathbb{R}/2\pi Z$ defined by $\epsilon(x) = e^{ix}$ then for any $v \in V$ the vector

$$v_n := \hat{L}v = \int_{\mathbb{R}/2piZ} \overline{\epsilon^n(\theta)}\pi(k_\theta)v \,\mathrm{d}\theta$$

satisfies $\pi(k_\theta)v_n = \epsilon^n(\theta)v_n$. Thus $\hat{L}(V)$ is the the isotypical component of type ϵ^n (consider the identical embedding of $\hat{L}(V)$ in V).

Admissible representations

Notation

The set of equivalence classes of the unitary representations of the group G is denoted by $\widehat{G}$.

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Theorem

Every K-finite vector in an admissible representation is C^∞. If V is a Banach space every K-finite vector is analytic.

$(\mathfrak{g}, K)$-modules

and Harish-Chandra modules

Theorem

Let (π, V) be admissible. The space of K-finite vectors, i.e. the algebraic sum $V_{(K)} = \bigoplus_{\tau \in \widehat{K}} V_\tau$ is a stable under $\mathfrak{g}_0$, hence a $\mathfrak{U}(\mathfrak{g})$-module.

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Then if (π, V) is an admissible representation of G, the vector space of K-finite vectors is a $(\mathfrak{g}, K)$-module

A suitable basis for $\mathfrak{sl}_2(\mathbb{R})$

Raising and lowering elements

We choose as generators for $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{C}$ the elements

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \eta^+ = \frac{1}{2}(H - iH^\perp) = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}, \quad \eta^- = \frac{1}{2}(H + iH^\perp) = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$$

which satisfy the following commutations rules:

$$[\kappa, \eta^\pm] = \pm 2i\eta^\pm, \qquad [\eta^+, \eta^-] = -i\kappa. \tag{6}$$

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The *Casimir* operator, up to a rescaling, is the element

$$\Omega := -\frac{1}{4}(H^2 + {H^\perp}^2 - \Theta^2) = \Delta + \Theta^2/2. \tag{7}$$

which may be rewritten as

$$\Omega = -\frac{1}{4}\kappa^2 + \frac{i}{2}\kappa + \eta^+\eta^- = -\frac{1}{4}\kappa^2 - \frac{i}{2}\kappa + \eta^-\eta^+.$$

Irreducible $(\mathfrak{g}, K)$-modules for $SL_2(\mathbb{R})$

Structure

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \otimes \mathbb{C}$, $K = SO(2)$. Suppose V is an irreducible admissible $(\mathfrak{g}, K)$-module. By the Dixmier-Schur Lemma the Casimir operator acts as a scalar λ.

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- Set for all $k \in \mathbb{N}$

$$v_{n+2k} := (\eta^+)^k v_n, \quad v_{n-2k} := (\eta^-)^k v_n, , \quad \tilde{V} = \operatorname{span}_{\mathbb{C}}\{v_{n+2j}\}$$

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  Then $\tilde{V} = V$ and the spectrum of κ (the set of j such that $V_j \neq 0$) is an interval of integers with the same parity as n. In fact for any v_ℓ

$$\eta^+\eta^- v_\ell = \left(\lambda - \tfrac{1}{4}\ell^2 + \tfrac{1}{2}\ell\right) v_\ell, \qquad \eta^-\eta^+ v_\ell = \left(\lambda - \tfrac{1}{4}\ell^2 - \tfrac{1}{2}\ell\right) v_\ell$$

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Classification

Four possibilities for the spectrum of $-i\kappa$

- $$\{-\ell, -\ell+2, \dots, \ell-2, \ell\}, \quad \text{and} \quad \Omega = \frac{(\ell+1)^2-1}{4}.$$

  This $(\mathfrak{g}, K)$-module is of dimension $\ell+1$ and denoted $F(\ell+1)$.

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- $$\{\ell, \ell+2, \ldots, \ell+2j, \ldots\}, \quad \ell \geq 1 \quad \text{and} \quad \Omega = \frac{(\ell-1)^2-1}{4}.$$

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  This $(\mathfrak{g}, K)$-module is denoted $D^-_{\ell-1}$.
- $S = 2\mathbb{Z}$ or $S = 2\mathbb{Z}+1$.
  In this case $\lambda \neq \frac{(m-1)^2-1}{4}$ for any $m \in S$.