Lecture. I : The Shafarevich hyperbolicity conjecture

Wednesday, May 21^{st} 2014

Rennes

„Using MMP to extend range of applicability of Hodge theory"

- see arXiv: 1103.5630

§ 1 Shafarevich hyperbolicity

Thm (Arakelov / Parshin, Shafarevich hyperbolicity conj. ICM '62)

Let $f: X \to Y$ be a proper, smooth family of curves, $g \geq 2$. If $Y \simeq \mathbb{P}^1$, elliptic curve, $\mathbb{C}$, $\mathbb{C}^*$ $\Rightarrow$ all fibres are isomorphic.

Equiv. any morphism $Y \to$ (Moduli stack of curves of genus g) is constant if Y is ...

Recall: A complex manifold X is Brody-hyperbolic if any hol. map $\mathbb{C} \to X$ is constant.

Goal Understand idea of proof, extend results to higher-dim. fibres and higher-dimensional base manifolds.

§ 2 Idea of proof : overview

Assume $f: X \longrightarrow Y$ is any family as in thm, not all fibres isomorphic.
Assume Y compact.
$\mu: Y \longrightarrow \mathfrak{M}$ = Moduli space the moduli map
$y \mapsto [X_y]$

Two main steps

Ⓐ Show that $f_* \omega_{X/Y}^{\otimes r}$ is ample, whenever non-zero.

Recall: $\mathcal{E}$ vector bundle on a curve C
nef $\Leftrightarrow$ any line bundle quotient has degree ≥ 0.
ample $\Leftrightarrow$ —————— " —————— > 0

Ⓑ Show that $f_* \omega_{X/Y}^{\otimes 2}$ maps non-trivially into Ω^1_Y.

§ 3 Part B

Recap: Kodaira - Spencer map

Given $y \in Y$, $F := f^{-1}(y)$

$T_{\mathcal{M}}|_{[F]} \cong H^1(F, T_F)$

Derivative of moduli map is $T_Y|_y \longrightarrow H^1(F, T_F)$

is constructed as follows ...

... doing this for all $y \in Y$ simultaneously, get

$$\mathcal{K} : T_Y \longrightarrow \underbrace{R^1 f_* (T_{X/Y})}$$

dual to $R^0 f_* (T_{X/Y}^* \otimes \omega_{X/Y}) = f_* \, \omega_{X/Y}^{\otimes 2}$

... so get dual map

$\mathcal{K}^* : f_* \, \omega_{X/Y}^{\otimes 2} \longrightarrow \Omega^1_Y$ ⟵ If $Y = \mathbb{P}^1$, this is $\mathcal{O}_{\mathbb{P}^1}(-2)$, so negative

§ 4 Part A

I will only show:

Proposition $f_* \omega_{X/Y}^{\otimes \nu}$ is locally free and nef. for any $\nu > 0$

[That shows: any family over $\mathbb{P}^1$ is isotrivial]

Remark Positivity results for (higher) direct images have a long history

- Griffiths 1970: explicit curvature comp. on period domain
- Hodge-metric computations [Fujita '77, Kawamata '81, Moriwaki, Fujino, Berndsson, Mourougane, Takayama ...]
- Vanishing theorems [Viehweg, Kollár, ...]

I sketch a proof of the Prop. following Viehweg

— see Viehweg's ICTP Lecture Notes.

Step 1 / central observation

Observation $\phi : W \longrightarrow Z$ any smooth map of proj. manifolds, where Z a curve, $\mathcal{H}$ any ample bundle on X, $z \in Z$ any point

$$\Rightarrow \quad f_*(\omega_{W/Z} \otimes \mathcal{H}) \otimes \omega_Z \otimes \mathcal{O}_Z(2\cdot z) \quad \text{is locally free, nef.}$$

Idea of application if $\mathcal{H} = \omega_{W/Z}$ were ample, then we'd be in business...

Proof Look at ideal sheaf sequence on W, write $F := \phi^{-1}(z)$

$$0 \to \underbrace{\omega_{W/Z} \otimes \mathcal{H} \otimes \phi^*(\omega_Z \otimes \mathcal{O}_Z(z))}_{A} \to \underbrace{\omega_{W/Z} \otimes \mathcal{H} \otimes \phi^*(\omega_Z \otimes \mathcal{O}_Z(2z))}_{\mathcal{B}}$$

$$\longrightarrow \mathcal{B}|_F \longrightarrow 0$$

Note $H^1(X, A) = H^1(X, \omega_W \otimes \text{ample} \otimes \text{nef}) = 0$

$$\Rightarrow \quad H^0(X, \mathcal{B}) \to H^0(F, \mathcal{B}|_F) \quad \text{surjective}$$

Note: $h^0(F, \mathcal{B}|_F)$ is independent of F

$\Rightarrow$ $f_*(\mathcal{B})$ is locally free, glob. generated and in part. nef.

$\square$

Crucial: Number 2 in Observation does not depend on anything.

<u>Step 2</u> Given ν, set

$$\mu := \min \{ \eta > 0 \mid f_* \omega_{X/Y}^{\otimes \nu} \otimes \mathcal{O}_Y((\eta \nu - 1) y) \text{ nef.}$$

Then

- $f_* \omega_{X/Y}^{\otimes \nu} \otimes \mathcal{O}_Y((\mu \nu - 1) y)$ nef
- $f_* \omega_{X/Y}^{\otimes \nu} \otimes \mathcal{O}_Y(\mu \nu \cdot y)$ ample

<u>Claim</u> $\mathcal{L} := \omega_{X/Y}^{\otimes \nu - 1} \otimes f^* \mathcal{O}_Y(\mu(\nu-1) \cdot y)$ is ample

<u>Proof of semi-ampleness</u> $\mathcal{L}^{\nu} = [\underbrace{\omega_{X/Y}^{\nu} \otimes f^* \mathcal{O}_Y(\mu \cdot \nu \cdot y)}_{\mathcal{F}}]^{\nu - 1}$

Know: $f_* \mathcal{F}$ is ample, $\mathrm{Sym}^m f_* \mathcal{F}$ is glob. gen. for $m \gg 0$

Look at map

$$f^* \mathrm{Sym}^m f_* \mathcal{F} = \mathrm{Sym}^m f^* f_* \mathcal{F} \longrightarrow \mathrm{Sym}^m \mathcal{F} = \mathcal{F}^{\otimes m}$$

and use that $\mathcal{F}^{\otimes m}$ is base point-free on fibres. $\square$

Step 3

We know by Step 1 that

$$f_*(\omega_{X/Y} \otimes \mathcal{L}) \otimes \omega_Y \otimes \mathcal{O}_Y(2_Y) \quad \text{is nef.}$$

We know more, set

$$\begin{array}{ccc} X^{(n)} := \underbrace{X \times_Y \cdots \times_Y X}_{n\times} & \xrightarrow{\pi_i} & X \\ \Big\downarrow f^{(n)} & & \\ Y & & \end{array}$$

$\mathcal{L}^{(n)} := \bigotimes \pi_i^*(\mathcal{L})$ ... ample on $X^{(n)}$

Observe: $\omega_{X^{(n)}/Y} = \bigotimes \pi_i^*(\omega_{X/Y})$

Then

$$f^{(n)}_*(\omega_{X^n/Y} \otimes \mathcal{L}^{(n)}) \otimes \omega_Y \otimes \mathcal{O}_Y(2_Y) \quad \text{nef}$$

$$\shortparallel$$

$$\left[f_*(\omega_{X/Y} \otimes \mathcal{L}) \right]^n \otimes \omega_Y \otimes \mathcal{O}_Y(2_Y)$$

Consequence

$$f_*(\omega_{X/Y} \otimes \mathcal{L}) = f_*(\omega_{X/Y}^{\otimes \nu}) \otimes \mathcal{O}_Y(\mu(\nu-1)_Y) \quad \text{nef.}$$

In particular:

$$\mu \cdot (\nu - 1) > (\mu - 1)\cdot \nu - 1$$

$$\Leftrightarrow \quad \mu \cdot \nu - \mu > \mu\nu - \nu - 1$$

$$\Leftrightarrow \quad \nu \geq \mu$$ ⟵ universal bound on μ that does not depend on anything!

Step 4 Now, assume $f_* \omega_{X/Y}^{\otimes \nu}$ was not nef, i.e. has negative quotient Q. Do a 10.000 : 1 cover

$$\begin{tikzcd} \tilde{X} := X \times_Y \tilde{Y} \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d] \\ \tilde{Y} \arrow[r, "\gamma", "10.000:1"'] & Y \end{tikzcd}$$

Then $\tilde{f}_* \omega_{\tilde{X}/\tilde{Y}}^{\otimes \nu} = \gamma^* f_* \omega_{X/Y}^{\otimes \nu}$ has quotient $\gamma^* Q$ of

neg. degree $10.000 \cdot \deg Q$

But $\tilde{f}_* (\omega_{\tilde{X}/\tilde{Y}}^{\otimes \nu}) \otimes \mathcal{O}_{\tilde{Y}} (\nu(\nu-1) \cdot \tilde{y})$ is nef $\lightning$

$\square$

§ 5 What needs to be done in general?

- In case Y compact, need to show $f_* \omega_{X/Y}^{\otimes \nu}$ is ample, not just nef.

  → better vanishing results: Kollár injectivity instead of Kodaira vanishing.

- In case Y non-compact: logarithmic differentials.

Covering tricks

→ lead to singular spaces → multiplier ideals

→ lead to fractional divisors → Kawamata-Viehweg vanishing.

Detour: logarithmic differentials

Setting: X complex manifold, $\mathcal{D} \subset X$ a simple-normal-crossing divisor

Then $T_X(-\log \mathcal{D})$ ... vector fields stab. $\mathcal{D}$
$\Omega^1_X(\log \mathcal{D})$ ... dual to that, log differentials

Alternate description:

- $\Omega^1_X(\log \mathcal{D})$ = diff' forms σ with at most single pole at $\mathcal{D}$ s.t. $d\sigma \in \Omega^2_X \otimes \mathcal{O}_X(\mathcal{D})$
- If $x_0 \ldots x_{n-1}$ local coord. where $\mathcal{D} = \{x_0 \cdots x_k = 0\}$, then

$$\Omega^1_X(\log \mathcal{D}) = \left\langle \frac{1}{x_0} dx_0, \ldots, \frac{1}{x_k} dx_k, dx_{k+1}, \ldots, dx_{n-1} \right\rangle$$

Main properties:

- $\Omega^1_X(\log \mathcal{D})$ is locally free, $\det \Omega^1_X(\log \mathcal{D}) = K_X + \mathcal{D}$
- $X \to Y$ cover, $\mathcal{D}_X$ snc divisor on X s.t. $\gamma^* \mathcal{D}_Y \subset \mathcal{D}_X$
  $\mathcal{D}_Y$ snc divisor on Y

  have pull-back $\gamma^* \Omega^1_Y(\log \mathcal{D}_Y) \to \Omega^1_X(\log \mathcal{D}_X)$
- T a tensor operation $\Rightarrow h^0(X, T\,\Omega^1_X(\log \mathcal{D}))$ is an invariant of $X \setminus \mathcal{D}$.

  In particular, can define Kodaira dimension for quasi-proj. manifolds.

- Hodge theory works with $\Omega^1_X(\log D)$. Have Hodge-to de Rham spectral sequences, decomposition ...

  - Any global form $\sigma \in H^0(X, \Omega^p_X(\log D))$ is closed.

    $\rightsquigarrow$ Bogomolov - Sommese vanishing :

$$\mathcal{A} \subset \Omega^p_X(\log D) \text{ invertible } \Rightarrow \kappa(\mathcal{A}) \leq p$$

- Restriction. If $E \subset X$ is smooth and intersects D transversally ($D|_E$ is snc in E), then can restrict

$$\Omega^p_X(\log D) \to \Omega^p_E(\log D|_E)$$

- Residue

$$0 \to \Omega^p_X(\log D - D_1) \to \Omega^p_X(\log D) \to \Omega^{p-1}_{D_1}(\log (D - D_1)|_{D_1}) \to 0$$

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Lecture II: Hyperbolicity in higher dimensions

Thursday, May 22^{nd} 2014

Rennes

§ 1 Intro

The Shafarevich Hyperbolicity has been generalized by many people, including Migliorini, Kovács, Viehweg - Zuo

<u>Thm:</u> ~~If f: X → Y~~ is If $f^\circ: X^\circ \to Y^\circ$ is a smooth, proper family of canonically polarised manifolds, if $Y \simeq \mathbb{P}^1$, ellipt., $\mathbb{C}$, $\mathbb{C}^*$, then all fibres are isomorphic

<u>Aim:</u> Generalise this to families over higher-dimensional base

<u>Setup:</u> $f^\circ: X^\circ \longrightarrow Y^\circ$ a proj. family of canon. polarized manifolds over smooth base manifold.

<u>Two main invariants:</u>

- Variation of the family $\mathrm{Var}(f)$
- (logarithmic) Kodaira dimension $\kappa(Y^\circ)$

$\rightsquigarrow$ explain $\kappa(Y^\circ)$.

<u>Shafarevich:</u> $\dim Y = 1$, then $\left| \begin{array}{ll} \kappa(Y^\circ) = -\infty, 0 & \text{implies } \mathrm{Var}(f) = 0 \\ \mathrm{Var}(f) \geq 1 & \text{implies } \kappa(Y^\circ) = 1 \end{array} \right.$

Aim Relate κ and Variation

Theorem (K, Kovács) if $\dim Y^\circ \leq 3$, then either

- $\kappa(Y^\circ) = -\infty$ and $\mathrm{Var}(f^\circ) < \dim Y^\circ$
- $\kappa(Y^\circ) \geq 0$ and $\mathrm{Var}(f^\circ) \leq \kappa(Y^\circ)$ $\square$

„Any surface or threefold in the moduli stack is of gen'l type"

Aim Give an idea of proof

Note: There are other, stronger results, (sometimes with problematic proofs)

Main Ingredients:

A. Results of Viehweg-Zuo

B. Miyaoka semipositivity and generalisations

C. Minimal model program & Analysis of singularities.

§ 2 Results of Viehweg-Zuo

Thm (Viehweg-Zuo, 2000) Setting as above, choose a compactification $Y \supseteq Y^\circ$ s.t. $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings.

Then there exists $m \gg 0$, a line bundle $\mathcal{A} \in \mathrm{Pic}(Y)$ with

$\kappa(\mathcal{A}) \geq \mathrm{Var}(f)$ and an embedding $\mathcal{A} \hookrightarrow \mathrm{Sym}^m \Omega^1_Y(\log D)$

Recap: differential form w/ log poles.

Addendum: at the general point of Y, $\mathcal{A}$ is contained in $\mathrm{Sym}^m \mu^* \Omega^1_{\mathfrak{M}}$

Synopsis of VZ's proof Look only at case where $Y = Y^\circ$ is compact, $D = 0$. Have pull-back-sequence

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$$

This induces a filtration

$$\Omega^p_X = F^0 \supset F^1 \supset \cdots \supset F^p \supset F^{p+1} = 0$$

s.t. $$F^r / F^{r+1} = f^*\Omega^r_Y \otimes \Omega^{p-r}_{X/Y}$$

look at sequence

$$0 \to F^1 \to F^0 \to F^0/F^1 \to 0$$

modulo F^2:

$$0 \to F^1/F^2 \to F^0/F^2 \to F^0/F^1 \to 0$$

$$= \quad 0 \to f^*\Omega^1_Y \otimes \Omega^{p-1}_{X/Y} \to F^0/F^2 \to \Omega^p_{X/Y} \to 0 \qquad | \otimes \omega^{-1}_{X/Y}$$

Twist with $\omega^{-1}_{X/Y}$, ~~and~~ take push-forward and look at conn. morphisms

$$\tau^0_{p,q} : \underbrace{R^q f_*\left(\Omega^p_{X/Y} \otimes \omega^{-1}_{X/Y}\right)}_{F^0_{p,q}} \longrightarrow \underbrace{R^{q+1} f_*\left(\Omega^{p-1}_{X/Y} \otimes \omega^{-1}_{X/Y}\right)}_{F^0_{p-1,q+1}} \otimes \Omega^1_Y$$

<u>Fundamental fact:</u> $\mathcal{N}_{p,q} := \ker(\tau^0_{p,q})$ has the following property

$\exists\ \mathcal{A} \in \mathrm{Pic}(Y)$ with $\kappa(\mathcal{A}) \geq \mathrm{Var}(f)$ and $m \in \mathbb{N}$ s.t.

$(\mathcal{A} \otimes \mathrm{Sym}^m \mathcal{N}_{p,q})^*$ is globally generated.

Write $F_{p,q} := F^0_{p,q} \otimes (\Omega^1_Y)^{\otimes q}$ get morphisms

$$\tau_{p,q} = \tau^0_{p,q} \otimes \mathrm{Id}_{(\Omega^1_Y)^{\otimes q}}$$

$$F_{n,0} \xrightarrow{\tau_{n,0}} F_{n-1,1} \xrightarrow{\tau_{n-1,1}} \cdots \quad F_{0,n} \longrightarrow 0$$

Note: $F_{n,0} = R^0 f_* \left(\Omega^n_{X/Y} \otimes \omega^{-1}_{X/Y} \right) = f_* \mathcal{O}_Y = \mathcal{O}_Y$

is not contained in $\ker(\tau_{n,0})$!

However, there is q s.t. we get a map

$$\mathcal{O}_Y \longrightarrow \ker(\tau_{n-q,q}) = \mathcal{N}_{n-q,q} \otimes (\Omega^1_Y)^{\otimes q} =$$

$$= \mathrm{Hom}\left(\mathcal{N}_{n-q,q}{}^*, (\Omega^1_Y)^{\otimes q} \right)$$

$$= \mathrm{Hom}\left(\underbrace{\mathcal{N}^*_{n-q,q} \otimes \mathcal{A}^*}_{\text{glob. gen.}}, \underbrace{\mathcal{A}^* \otimes (\Omega^1_Y)^{\otimes q}}_{\text{has a section}} \right)$$

$\Rightarrow$

Questions

- is this the right construction ?
- What are universal properties ? Does failure of universal property give rise to structures on moduli space ?