# Multilinear compressive sensing and an application to convolutional linear networks

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# Statement, without technicality

- $f_{\mathbf{h}}$ a family of functions parameterized by **h** (e.g. linear networks)
- $I, X$ matrix containing input-output pairs

## Informal statement

Under a certain **condition** on the family $f$ (e.g. on the topology of the network):
There exists $C$ such that for $\eta$ small and for any

$$\overline{\mathbf{h}}, \mathbf{h}^* \in \{\mathbf{h} | \ \|f_{\mathbf{h}}(I) - X\| \leq \eta\}$$

we have

$$d(\overline{\mathbf{h}}, \mathbf{h}^*) \leq C\,\eta$$

- If the condition is satisfied we have stably defined features

$\Rightarrow$ **interpretable learning**

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# Deep linear networks

## Problem formulation

Let $K \in \mathbb{N}^*$, $m_1 \dots m_{K+1} \in \mathbb{N}$, write $m_1 = m$, $m_{K+1} = n$. We assume that we know the matrix $X \in \mathbb{R}^{m \times n}$ which is (approximatively) the product of factors $X_k \in \mathbb{R}^{m_k \times m_{k+1}}$:

$$X = X_1 \cdots X_K.$$

We investigate models/constraints imposed on the factors $X_k$ for which we can (up to obvious scale rearrangement) stably recover the factors $X_k$ from $X$.

# Deep linear networks

## Structure of the factors

- For $k = 1 \dots K$, we know

$$\begin{array}{rcl} M_k : \mathbb{R}^S & \longrightarrow & \mathbb{R}^{m_k \times m_{k+1}}, \\ h & \longmapsto & M_k(h) \end{array}$$

- We know models

$$\mathcal{M} = (\mathcal{M}^L)_{L\in\mathbb{N}} \qquad \text{with ,} \qquad \mathcal{M}^L \subset \mathbb{R}^{K\times S}, \forall L.$$

- Assume there exists $\overline{L}$, $L^*$ and $(\overline{\mathbf{h}}_k)_{k=1..K} \in \mathcal{M}^{\overline{L}}$ and $(\mathbf{h}_k^*)_{k=1..K} \in \mathcal{M}^{L^*}$ such that

$$\|M_1(\overline{\mathbf{h}}_1)\cdots M_K(\overline{\mathbf{h}}_K) - X\| \leq \delta,$$

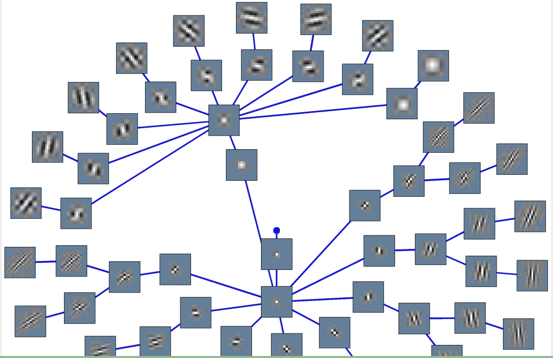
$$\|M_1(\mathbf{h}_1^*)\cdots M_K(\mathbf{h}_K^*) - X\| \leq \eta,$$

Is $(\overline{\mathbf{h}}_k)_{k=1..K}$ close to $(\mathbf{h}_k^*)_{k=1..K}$ ?

# Examples

- $K = 1$: Compressed sensing problem: Recovering $\mathbf{h}_1$ from $M_1(\mathbf{h}_1)$ is linear inverse problem.
- $K = 2$:
  - **Dictionary learning:** $M_1(\mathbf{h}_1)$ is a dictionary of atoms, $M_2(\mathbf{h}_2)$ is sparse
  - **Non-negative matrix factorization:** $M_1(\mathbf{h}_1) \geq 0$ and $M_2(\mathbf{h}_2) \geq 0$
  - **Low rank approximation:** $M_1(\mathbf{h}_1)$ is rectangular "vertical" ($m_1 \gg m_2$), $M_2(\mathbf{h}_2)$ is rectangular "horizontal" ($m_2 \ll m_3$).
  - **Phase recovery:** $M_1(\mathbf{h}_1) = diag(F\mathbf{h}_1)$, $M_2(\mathbf{h}_2) = (F\mathbf{h}_2)^*$, with $F$ the Fourier matrix and $\mathbf{h}_1 = \mathbf{h}_2$.
  - **Blind deconvolution:** $M_1(\mathbf{h}_1)$ is circulant, $M_2(\mathbf{h}_2)$ is a signal
  - **Blind-demixing, self-calibration, Internet of things**...

- $K$ large :
  - Fast Fourier, Discrete Cosine, Discrete Wavelet, Jacobi eigenvalue Algorithm
  - Tsiligkaridis, Hero, Zhou: **Kronecker graphical lasso** (IEEE SP 2013)
  - Lyu, Wang: **Multi-layer NMF** (NIPS'13)
  - Kondor, Tevena, Garg: **Multiresolution Matrix fatorization** (ICML 2014)
  - Chabiron, Malgouyres, Wendt, Tourneret: **Fast Transform Learning** (IJCV, 2015)
  - Le Magoarou, Gribonval: **Faust** (IEEE STSP, 2016)
  - Rusu, Thomson: **Transforms based on Householder reflectors** (IEEE SP 2016) and **Givens rotations** (IEEE SP 2017)
  - Sulam, Papyan, Romano, Elad : **Multi-layer Convolutional Sparse Coding** (IEEE SP 2018)



# Link with Deep learning

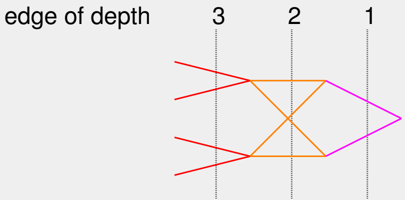



Figure: Deep network

$$\mathcal{N}(\mathbf{h}, I) = U_1 M_1'(\mathbf{h}_1) U_2 M_2'(\mathbf{h}_2) U_3 M_3'(\mathbf{h}_3) I$$

- $M_k'(\mathbf{h}_k)$ : is a linear operator, depending linearly on $\mathbf{h}_k$
  - Feed-forward : $M_3'(\mathbf{h}_3) = \begin{pmatrix} \mathbf{h}_{3,1} & \mathbf{h}_{3,2} & 0 & 0 \\ 0 & 0 & \mathbf{h}_{3,3} & \mathbf{h}_{3,4} \end{pmatrix}$
  - Convolutional : $M_3'(\mathbf{h}_3) = \begin{pmatrix} C_1(\mathbf{h}_3) & C_2(\mathbf{h}_3) & 0 & 0 \\ 0 & 0 & C_3(\mathbf{h}_3) & C_4(\mathbf{h}_3) \end{pmatrix}$
    where $C_i(.)$ convolution+sampling matrices.

# Link with Deep learning

- With ReLU : $U_k : \mathbb{R}^{n_k \times L} \longmapsto \mathbb{R}^{n_k \times L}$ (where $n_k$ is the size of the layer $k$) is such that :

$$(U_k M)_{n,l} = a_k(\mathbf{h})_{n,l} M_{n,l} \qquad , \text{with } a_k(\mathbf{h}) \in \{0,1\}^{n_k \times L}.$$

and

$$a_k(\mathbf{h})_{n,l} = \begin{cases} 1 & , \text{if } \left( M'_k(\mathbf{h}_k) U_{k+1} M'_{k+1}(\mathbf{h}_{k+1}) \cdots U_K M'_K(\mathbf{h}_K) X \right)_{n,l} \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

The function

$$\begin{array}{rcl} a_k : \mathbb{R}^{K \times S} & \longrightarrow & \{0,1\}^{n_k \times L} \\ \mathbf{h} & \longmapsto & a_k(\mathbf{h}) \end{array}$$

is piecewise constant.

**As a function of h, the neural network is a piecewise linear network**

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we have

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$\Rightarrow$ **interpretable learning**

# Notations

- $\mathbb{N}_k = \{1, \ldots, k\}$
- $\mathbf{h} \in \mathbb{R}^{K \times S}$, $\mathbf{h}_k \in \mathbb{R}^S$, $\mathbf{h}_{k,\mathbf{i}_k} \in \mathbb{R}$

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- $\mathbb{R}_*^{K \times S} = \{\mathbf{h} \in \mathbb{R}^{K \times S}, \forall k \in \mathbb{N}_K, \|\mathbf{h}_k\| \neq 0\}$
- For $\mathbf{h}$ and $\mathbf{g} \in \mathbb{R}_*^{K \times S}$, $\mathbf{h} \sim \mathbf{g}$ if and only if there exists $(\lambda_k)_{k \in \mathbb{N}_K} \in \mathbb{R}^K$ such that

$$\prod_{k=1}^{K} \lambda_k = 1 \qquad \text{and} \qquad \mathbf{h}_k = \lambda_k \mathbf{g}_k, \forall k \in \mathbb{N}_K.$$

  We say $\mathbf{g} \in [\mathbf{h}]$.

## Remark

Since for any $\mathbf{g} \in [\overline{\mathbf{h}}]$

$$M_1(\overline{\mathbf{h}}_1) \cdots M_K(\overline{\mathbf{h}}_K) = M_1(\mathbf{g}_1) \cdots M_K(\mathbf{g}_K)$$

Recovering $[\overline{\mathbf{h}}]$ is the best we can hope for.

- Tensors $T \in \mathbb{R}^{\overbrace{S \times \cdots \times S}^{\kappa \text{ times}}} = \mathbb{R}^{S^\kappa}$

- Tensors $T \in \mathbb{R}^{\overbrace{S \times \cdots \times S}^{K \text{ times}}} = \mathbb{R}^{S^K}$
- Tensor value $T_{i_1,\ldots,i_K}$ or $T_{\mathbf{i}}$, for $\mathbf{i} = (i_1, \ldots, i_K) \in \mathbb{N}_S^K$

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- Tensor value $T_{i_1,\ldots,i_K}$ or $T_{\mathbf{i}}$, for $\mathbf{i} = (i_1, \ldots, i_K) \in \mathbb{N}_S^K$
- $T \in \mathbb{R}^{S^K}$ is of rank 1 if and only if there exists $\mathbf{h} \in \mathbb{R}^{K \times S}$ s.t.:

$$T_{\mathbf{i}} = \mathbf{h}_{1,i_1} \cdots \mathbf{h}_{K,i_K} \qquad , \forall \mathbf{i} \in \mathbb{N}_S^K.$$

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  We say $T \in \Sigma_1$.
- Segre embedding: Parameterize $\Sigma_1 \subset \mathbb{R}^{S^K}$ by the map

$$\begin{array}{rcl} P : \mathbb{R}^{K \times S} & \longrightarrow & \Sigma_1 \subset \mathbb{R}^{S^K} \\ \mathbf{h} & \longmapsto & (\mathbf{h}_{1,i_1} \mathbf{h}_{2,i_2} \cdots \mathbf{h}_{K,i_K})_{\mathbf{i} \in \mathbb{N}_S^K} \end{array}$$

## Remark

Since for any $\mathbf{g} \in [\overline{\mathbf{h}}]$

$$P(\overline{\mathbf{h}}) = P(\mathbf{g})$$

Recovering $[\overline{\mathbf{h}}]$ from $P(\overline{\mathbf{h}})$ is the best we can hope for.
Recovering $[\overline{\mathbf{h}}]$ from $P(\overline{\mathbf{h}})$ is easy. (By extracting lines in $P(\overline{\mathbf{h}})$.)

# Tensorial Lifting

## Theorem

*There exists a unique linear map*

$$\mathcal{A} : \mathbb{R}^{S^K} \longrightarrow \mathbb{R}^{m \times n},$$

*such that for all* $\mathbf{h} \in \mathbb{R}^{K \times S}$

$$M_1(\mathbf{h}_1) M_2(\mathbf{h}_2) \cdots M_K(\mathbf{h}_K) = \mathcal{A} P(\mathbf{h}).$$

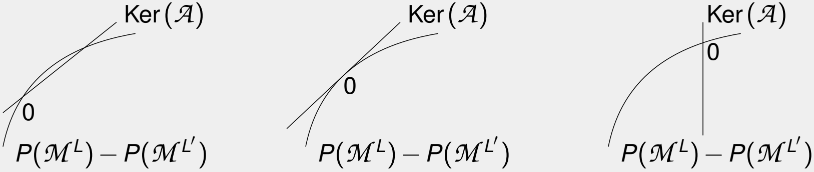
- Changing $M_1, M_2, \ldots, M_K$ only modifies $\mathcal{A}$
- The properties of
  - $M_1(\mathbf{h}_1) M_2(\mathbf{h}_2) \cdots M_K(\mathbf{h}_K)$
  - $\mathbf{h} \longmapsto \|M_1(\mathbf{h}_1) M_2(\mathbf{h}_2) \cdots M_K(\mathbf{h}_K) - X\|^2$

  relate to the geometry of $\mathcal{A}$ and $\Sigma_1$ (or $\Sigma_2$).

## Deep-Null Space Property

Let $\gamma > 0$ and $\rho > 0$, we say that $\mathrm{Ker}\,(\mathcal{A})$ satisfies the *deep-Null Space Property (deep-NSP )* with respect to the collection of models $\mathcal{M}$ with constants $(\gamma, \rho)$ if for any $L$ and $L' \in \mathbb{N}$, any $T \in P(\mathcal{M}^L) - P(\mathcal{M}^{L'})$ satisfying $\|\mathcal{A}T\| \leq \rho$ and any $T' \in \mathrm{Ker}\,(\mathcal{A})$, we have

$$\|T\| \leq \gamma \|T - T'\|. \qquad (1)$$




NO (deep-NSP ) NO (deep-NSP ) (deep-NSP )

$$\|M_1(\overline{\mathbf{h}}_1)\cdots M_K(\overline{\mathbf{h}}_K) - X\| \leq \delta,$$

and

$$\|M_1(\mathbf{h}_1^*)\cdots M_K(\mathbf{h}_K^*) - X\| \leq \eta,$$

for $\delta$ and $\eta$ small.

## Theorem : **Sufficient condition for interpretability**

Assume Ker$(\mathcal{A})$ satisfies the deep-NSP with respect to the collection of models $\mathcal{M}$ and with the constant $(\gamma, \rho)$. If $\delta + \eta \leq \rho$, we have

$$\|P(\mathbf{h}^*) - P(\overline{\mathbf{h}})\| \leq \frac{\gamma}{\sigma_{min}} (\delta + \eta),$$

where $\sigma_{min}$ is the smallest non-zero singular value of $\mathcal{A}$. Moreover, if $\overline{\mathbf{h}} \in \mathbb{R}_*^{K \times S}$ and $\frac{\gamma}{\sigma_{min}} (\delta + \eta) \leq \frac{1}{2} \max\left(\|P(\overline{\mathbf{h}})\|_\infty, \|P(\mathbf{h}^*)\|_\infty\right)$ then

$$d_p([\mathbf{h}^*], [\overline{\mathbf{h}}]) \leq \frac{7(KS)^{\frac{1}{p}}\gamma}{\sigma_{min}} \min\left(\|P(\overline{\mathbf{h}})\|_\infty^{\frac{1}{K}-1}, \|P(\mathbf{h}^*)\|_\infty^{\frac{1}{K}-1}\right)(\delta + \eta). \qquad (2)$$

## Theorem : **Necessary condition for interpretability**

Assume the interpretability holds: There exists $C$ and $\delta > 0$ such that for any $\overline{L} \in \mathbb{N}$, $\overline{\mathbf{h}} \in \mathcal{M}^{\overline{L}}$, any $X = \mathcal{A}P(\overline{\mathbf{h}}) + e$, with $\|e\| \leq \delta$, any $L^* \in \mathbb{N}$ and any $\mathbf{h}^* \in \mathcal{M}^{L^*}$ such that

$$\|\mathcal{A}P(\mathbf{h}^*) - X\|^2 \leq \|e\|$$

we have

$$d_2([\mathbf{h}^*], [\overline{\mathbf{h}}]) \leq C \min\left( \|P(\overline{\mathbf{h}})\|_\infty^{\frac{1}{K}-1}, \|P(\mathbf{h}^*)\|_\infty^{\frac{1}{K}-1} \right) \|e\|.$$

Then, Ker$(\mathcal{A})$ satisfies the deep-NSP with respect to the collection of models $\mathcal{M}$ with constants

$$(\gamma, \rho) = (CS^{\frac{K-1}{2}} \sqrt{K}\, \sigma_{max}, \delta)$$

where $\sigma_{max}$ is the spectral radius of $\mathcal{A}$.

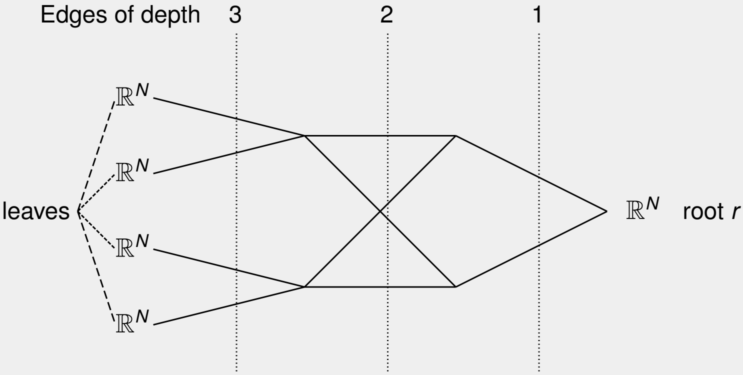



Figure: Example of the **convolutional linear network**. To every edge is attached a convolution kernel. The network does not involve non-linearities or sampling.

$$X = M_1(\mathbf{h}_1)M_2(\mathbf{h}_2)M_3(\mathbf{h}_3) = [X_1 X_2 X_3 X_4] \in \mathbb{R}^{N\times N|\mathcal{F}|}$$

$X_1, ..., X_4$ are convolution matrix

## Proposition : **Necessary condition of interpretability**

If some of the entries of $M_1(\mathbb{1})\cdots M_K(\mathbb{1})$ do not belong to $\{0,1\}$ :

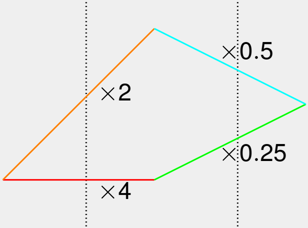
$$\mathbb{R}^{K\times S} \text{ is not interpretable.}$$

The condition "all the entries of $M_1(\mathbb{1})\cdots M_K(\mathbb{1})$ belong to $\{0,1\}$" can be computed by applying the network $|\mathcal{F}|$ times to a dirac delta function.

## Proposition

*If the network is a branch and all the entries of $M_1(\mathbb{1})\cdots M_K(\mathbb{1})$ belong to $\{0,1\}$, then* $\mathrm{Ker}(\mathcal{A}) = \{0\}$ *and* $\mathrm{Ker}(\mathcal{A})$ *satisfies the deep-NSP with respect to any model collection $\mathcal{M}$ with constant $(\gamma,\rho) = (1,+\infty)$. Moreover, we have $\sigma_{min} = \sqrt{N}$.*

Edges of depth 2 1




$\mathbf{h}$ and $\mathbf{g} \in \mathbb{R}^{K\times S}$ are equivalent if and only if

$$\forall \mathbf{p} \in \mathcal{P}, \exists (\lambda_e)_{e\in\mathbf{p}} \in \mathbb{R}^{\mathbf{p}}, \text{ such that } \prod_{e\in\mathbf{p}} \lambda_e = 1 \text{ and } \forall e \in \mathbf{p}, \mathcal{T}_e(\mathbf{g}) = \lambda_e \mathcal{T}_e(\mathbf{h}).$$

The equivalence class of $\mathbf{h} \in \mathbb{R}^{K\times S}$ is denoted by $\{\mathbf{h}\}$. For any $p \in [1, +\infty]$, we define

$$\delta_p(\{\mathbf{h}\}, \{\mathbf{g}\}) = \left( \sum_{\mathbf{p}\in\mathcal{P}} d_p([\mathbf{h}^{\mathbf{p}}], [\mathbf{g}^{\mathbf{p}}])^p \right)^{\frac{1}{p}},$$

where $\mathbf{h}^{\mathbf{p}}$ (resp $\mathbf{g}^{\mathbf{p}}$) is the restriction of $\mathbf{h}$ (resp $\mathbf{g}$) to the path $\mathbf{p}$.

$$\|M_1(\overline{\mathbf{h}}_1)\cdots M_K(\overline{\mathbf{h}}_K) - X\| \leq \delta,$$

and

$$\|M_1(\mathbf{h}_1^*)\cdots M_K(\mathbf{h}_K^*) - X\| \leq \eta,$$

for $\delta$ and $\eta$ small.

## Theorem : **Sufficient condition of interpretability**

If all the entries of $M_1(\mathbb{1})\cdots M_K(\mathbb{1})$ belong to $\{0,1\}$, if there exists $\varepsilon > 0$ such that for all $e \in \mathcal{E}$, $\|\mathcal{T}_e(\overline{\mathbf{h}})\|_\infty \geq \varepsilon$, and if $\delta + \eta \leq \frac{\sqrt{N}\varepsilon^K}{2}$ then

$$\delta_p(\{\mathbf{h}^*\},\{\overline{\mathbf{h}}\}) \leq 7(KS')^{\frac{1}{p}}\varepsilon^{1-K}\ \frac{\delta+\eta}{\sqrt{N}}$$

where $S' = \max_{e\in\mathcal{E}} |\mathcal{S}_e|$.

**Rks :**

- The condition "$M_1(\mathbb{1})\cdots M_K(\mathbb{1})$ belong to $\{0,1\}$" is not satisfied by most network structure encountered in practice.
- The action of the activation function favors interpretability.

Thank you for your attention !

**Papers available on**

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**Coming soon**

Characterization of good properties of the **landscape** of the objective function for deep linear networks.