The Ginzburg-Landau model in the surface superconductivity regime

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Joint work with Michele Correggi (Rome 3).
1. Ginzburg-Landau theory of type II superconductors
2. Surface superconductivity
3. Leading order results between $H_{c2}$ and $H_{c3}$
4. Elements of proof
5. Expansion beyond the leading order
Superconductors in magnetic fields

- Superconductivity = absence of resistivity at low temperature in some materials
- Peculiar response to applied magnetic fields = small fields do not penetrate (Meissner effect)
- Ginzburg-Landau 50: phenomenological theory, order parameter
- Bardeen-Cooper-Schrieffer 57: microscopic theory, Cooper pairing
- Gor’kov 59: BCS ⇒ GL, mathematically rigorous derivation
  Frank-Hainzl-Seiringer-Solovej 12

Superconductor levitating above a magnet
Ginzburg-Landau theory

Sample = infinite cylinder of smooth cross-section $\Omega \subset \mathbb{R}^2$, in a uniform external magnetic field perpendicular to $\Omega$.

- **Order parameter** $\Psi : \mathbb{R}^2 \to \mathbb{C}$. $|\Psi|^2 = $ relative density of superconducting electrons (bound in Cooper pairs)

- **Induced magnetic field** $h \neq$ applied magnetic field $h_{\text{ex}}$

- **Induced magnetic vector potential** $A$ with $\text{curl } A = h$.

- $\kappa =$ penetration depth. $\kappa \sigma =$ strength of applied magnetic field

- **Type II superconductor** : $\kappa > 1/\sqrt{2}$, “extreme type II”: $\kappa \to \infty$

Energy functional to be minimized:

$$G_{\kappa,\sigma}^{GL}[\Psi, A] = \int_{\Omega} \left| (\nabla + i \kappa \sigma A) \Psi \right|^2 - \kappa^2 |\Psi|^2 + \frac{1}{2} \kappa^2 |\Psi|^4 + (\kappa \sigma)^2 |\text{curl } A - 1|^2$$

**Gauge invariance**: energy invariant under

$$\Psi \to \Psi e^{-i \kappa \sigma \varphi}, \quad A \to A + \nabla \varphi$$
Phenomenology of type II superconductors

For minimizers $|\Psi| \leq 1$.

- $|\Psi| = 1$: purely superconducting state, all electrons in Cooper pairs.
- $|\Psi| = 0$: normal state, no Cooper pairs.
- Low magnetic field, $\kappa\sigma \leq H_{c1}$: superconducting state $|\Psi| \approx 1$ a.e.
- First critical field:
  \[
  \kappa\sigma = H_{c1} \approx C_\Omega \log \kappa
  \]
  isolated normal regions (vortices) start to appear.
- $H_{c1} \leq \kappa\sigma \leq H_{c2}$: vortex lattice state, Abrikosov lattice.
- Second critical field:
  \[
  \kappa\sigma = H_{c2} \approx \kappa^2
  \]
  superconductivity disappears uniformly in the bulk.
- $H_{c2} \leq \kappa\sigma \leq H_{c3}$: surface superconductivity state, $|\Psi| \approx 0$ in the bulk, $|\Psi| > 0$ close to the boundary.
- Normal state $|\Psi| \equiv 0$ above the third critical field:
  \[
  \kappa\sigma > H_{c3} \approx \Theta_0^{-1}\kappa^2, \quad \Theta_0 < 1.
  \]
Mixed state: Abrikosov lattice

- Theoretical prediction: Abrikosov 57, first observation 67.
- External magnetic field penetrates in small normal regions.

Vortex lattice in a type II superconductor, Hess-et al-Waszczak 89.
Mixed state: surface superconductivity

- Theoretical prediction: Saint-James and de Gennes 63, observed 64.
- Bulk is normal, magnetic field penetrates.
- A thin superconducting layer survives along the boundary.
1. Ginzburg-Landau theory of type II superconductors
2. **Surface superconductivity**
3. Leading order results between $H_{c2}$ and $H_{c3}$
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Transition from the normal state in decreasing fields

\[ G_{GL}^{\varepsilon}[\Psi, A] = \int_{\Omega} \left| \left( \nabla + i\varepsilon^{-2}A \right) \Psi \right|^2 + \frac{1}{2b\varepsilon^2} \left( |\Psi|^4 - 2|\Psi|^2 \right) + \frac{b}{\varepsilon^4} |\text{curl } A - 1|^2. \]

- New parameters: \( \sigma = b\kappa, \ b \text{ fixed, } \varepsilon = (\sigma\kappa)^{-1/2} \ll 1. \)
- Correspondence: \( H_{c2} \leftrightarrow b = 1, \ H_{c3} \leftrightarrow b = \Theta_0^{-1} \)
- St-James/de Gennes 63: Start at large \( b, \) normal state \( |\Psi| \equiv 0, \text{curl } A \equiv 1. \) When does this become unstable?
- At first, curl \( A \) stays fixed \( \equiv 1. \) Choice of gauge \( A \approx F \)
  \[
  \begin{cases}
    \text{curl } F = 1 \text{ in } \Omega \\
    \text{div } F = 0 \text{ in } \Omega \\
    \nu.F = 0 \text{ on } \partial\Omega
  \end{cases}
  \]
- Close to transition, for small values of \( \Psi, \) energy to leading order
  \[
  \int_{\Omega} \left| \left( \nabla + i\varepsilon^{-2}F \right) \Psi \right|^2 - \frac{1}{b\varepsilon^2} |\Psi|^2
  \]
- Can one make this \(< 0, \) smaller than energy of the normal state?
The critical fields $H_{c2}$ and $H_{c3}$

$$\mathcal{E}[\psi] = \left\langle \psi \left| H_\varepsilon - \frac{1}{b\varepsilon^2} \right| \psi \right\rangle$$

- $H_\varepsilon = -\left( \nabla + i\varepsilon^{-2} \mathbf{F} \right)^2$, magnetic Laplacian, uniform field $= \varepsilon^{-2}$.
- When does $H_\varepsilon$ have an eigenvalue strictly less than $1/(b\varepsilon^2)$?
- Eigenfunctions of $H_\varepsilon$ are localized over length scales of order $\varepsilon$:
  - Localization in the bulk $\leadsto$ magnetic Laplacian in the plane
  - Localization close to boundary $\leadsto$ magnetic Laplacian in a half-plane
- First eigenvalues for small $\varepsilon$ (semi-classics, e.g. Helffer-Morame):
  - Magnetic Laplacian in the plane $\rightarrow \lambda_1 \sim \varepsilon^{-2}$
  - Magnetic Laplacian in a half-plane $\rightarrow \lambda_1 \sim \Theta_0\varepsilon^{-2} < \varepsilon^{-2}$

- **Third critical field**: if $1 < b < \Theta_0^{-1}$, favorable to put mass close to the boundary, but only there.
- **Second critical field**: if $b < 1$, favorable to also put mass in the bulk.
More precise effective model between $H_{c2}$ and $H_{c3}$

- $1 < b < \Theta_0^{-1}$, $\Psi$ concentrated close to boundary on length scale $\varepsilon$.
- Magnetic field penetrates $\text{curl } \mathbf{A} \approx 1$, choose a convenient gauge.
- In scaled boundary coordinates $(s, t)$ (units of $\varepsilon^{-1}$), curvature $k(s)$

$$
\int_{s=0}^{\varepsilon^{-1}} \int_{t=0}^{c_0 |\log \varepsilon|} \left( 1 - \varepsilon k(s,t) \right) \left\{ |\partial_t \Psi|^2 + \frac{1}{(1 - \varepsilon k(s,t))^2} |(\varepsilon \partial_s + i a_\varepsilon(s, t)) \Psi|^2 \right. \\
\left. + \frac{1}{2b} [ |\Psi|^4 - 2 |\Psi|^2 ] \right\}
$$

- To leading order in $\varepsilon$, after scaling $s$:

$$
\mathcal{E}_{hp}[\Psi] = \int_{s=0}^{\varepsilon^{-1}} \int_{t=0}^{+\infty} \left\{ |(\nabla - it \mathbf{e}_s) \Psi|^2 + \frac{1}{2b} |\Psi|^4 - \frac{1}{b} |\Psi|^2 \right\}.
$$

- Natural ansatz $\Psi(s, t) = f(t)e^{-i\alpha s}$ (exact in the linear case) leads to

$$
\mathcal{E}_{0,\alpha}^{1D}[f] := \int_0^{+\infty} |\partial_t f|^2 + (t + \alpha)^2 f^2 + \frac{1}{2b} (f^4 - 2f^2)
$$
1. Ginzburg-Landau theory of type II superconductors
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Previously known results

- $b \to \Theta_0^{-1}$ “easier case”, cf Lu-Pan, Fournais-Helffer ...
- $b \to 1^+$: transition boundary to bulk behavior, Fournais-Kachmar 09
- $b \to 1^-$, cf Almog, Sandier-Serfaty, Aftalion-Serfaty, circa 07
- X.B. Pan 02, if $1 < b < \Theta_0^{-1}$, for some implicit constant $E_b < 0$

$$E_{\varepsilon}^{GL} = \frac{|\partial \Omega| E_b}{\varepsilon} + o(\varepsilon^{-1})$$

- Minimize $E_{0,\alpha}^{1D}[f] \Rightarrow$ optimal energy $E_0^{1D}$, phase $\alpha_0$, density $f_0$. Almog-Helffer 07, Fournais-Helffer-Persson 11, for $1.25 \leq b < \Theta_0^{-1}$

$$E_{\varepsilon}^{GL} = \frac{|\partial \Omega| E_0^{1D}}{\varepsilon} + o(\varepsilon^{-1}), \quad |\Psi_{GL}|^2 \approx f_0^2(t) \text{ in } L^2(\Omega)$$

Methods (cf Fournais-Helffer’s book)

- Decay estimates à la Agmon + Magnetic field estimates (elliptic PDEs methods) $\sim$ boundary problem
- Linear problem has unique non degenerate ground state
- Treat non linearity “perturbatively”
New energy and density estimates

The simplified 1D limit problem gives the leading order for all field strengths between $H_{c2}$ and $H_{c3}$.

**Theorem (Correggi-NR 13)**

Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 < b < \Theta^{-1}_0$, in the limit $\varepsilon \to 0$, it holds

$$E^{\text{GL}}_\varepsilon = \frac{|\partial \Omega|E^{1\text{D}}_0}{\varepsilon} + \mathcal{O}(1),$$

and

$$\| |\Psi^{\text{GL}}|^2 - f_0^2(t) \|_{L^2(\Omega)} \leq C\varepsilon \ll \| f_0^2(t) \|_{L^2(\Omega)}.$$

- Idea of proof: don’t think perturbatively around the linear problem
- Use the physics of the problem: “quantum fluid mechanics”
Uniform density estimates and degree estimates

Conjecture by Pan 02: $|\Psi_{GL}|^2 \to C(b) > 0$ pointwise on $\partial \Omega$.

Theorem (Correggi-NR 14)
For any $r \in \Omega$ with $\text{dist}(r, \partial \Omega) \lesssim \varepsilon$ we have

$$\left| |\Psi_{GL}(r)| - f_0(t) \right| \to 0$$

- No defects (e.g. vortices) in the surface superconductivity layer.
- Phase is well-defined along $\partial \Omega$: $\Psi_{GL} = \sqrt{\rho} e^{i\varphi}$.
- Phase circulation along $\partial \Omega \leftrightarrow$ number of vortices in the bulk.

Theorem (Correggi-NR 14)
Any GL minimizer $\Psi_{GL}$ satisfies in the limit $\varepsilon \to 0$

$$\frac{1}{2\pi} \int_{\partial B_R} \partial_T \varphi = \text{deg} (\Psi_{GL}, \partial \Omega) = \frac{|\Omega|}{\varepsilon^2} + \frac{|\alpha_0|}{\varepsilon} (1 + o(1)).$$
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Preliminary reductions

- Agmon estimates → exponential decay of order parameter away from the boundary (distances $\gg \varepsilon$).
- Magnetic field replacement, induced field $\approx$ applied field. $\mathbf{A} \rightarrow \mathbf{F}$
- Clever choice of gauge to represent the field.
- Mapping to boundary coordinates

$\Rightarrow$ all this previously known, cf Fournais-Helffer’s book

Model problem in scaled boundary coordinates, gives the original energy in units of $\varepsilon^{-1}$:

$$
\mathcal{E}_{hp}[\psi] = \int_{s=0}^{\partial\Omega} \int_{t=0}^{+\infty} \left\{ |(\nabla - it\mathbf{e}_s)\psi|^2 + \frac{1}{b} |\psi|^4 - \frac{2}{b} |\psi|^2 \right\}.
$$

- $s =$ tangential coordinate, impose periodicity of $\psi$ in $s$
- $t =$ normal coordinate
- Only large parameter: length of the domain in $s$-direction
The boundary problem

- Insert (formally) the ansatz $\psi(s, t) = f(t)e^{-i\alpha s}$

$$E_{1D}^{0,\alpha}[f] := \int_0^{+\infty} |\partial_t f|^2 + (t + \alpha)^2 f^2 + \frac{1}{2b} (f^4 - 2f^2)$$

- Minimize in $f$ and $\alpha \rightsquigarrow$ energy $E_{1D}^{0,\alpha}$, phase $\alpha_0$, density $f_0$

Proposition

Let $E_{hp}$ be the infimum of $E_{hp}$ under periodic boundary conditions in the $s$-direction. Assume $1 \leq b < \Theta_0^{-1}$, then

$$\frac{|\partial\Omega|}{\varepsilon} E_{1D}^{0} + O(\varepsilon|\log \varepsilon|) \geq E_{hp} \geq \frac{|\partial\Omega|}{\varepsilon} E_{1D}^{0}.$$

- Trivial upper bound, take trial state of the form

$$\psi(s, t) = f_0(t) \exp \left( -i\varepsilon \left\lfloor \frac{\alpha_0}{\varepsilon} \right\rfloor s \right)$$

- Lower bound is the main part.

- For a lower bound, think of the case where only $|\psi|$ is periodic.
Sketch of the lower bound 1

Inspired by earlier works (Correggi-Pinsker-NR-Yngvason) on the Gross-Pitaevskii theory of rotating superfluids (cf book by Aftalion).

1. **State decoupling**: since $f_0 > 0$, to any $\psi$ associate a $v$ by setting

   \[
   \psi(s, t) = f_0(t)e^{-i\alpha_0 s}v(s, t).
   \]

2. **Energy decoupling**: Variational equation for $f_0 \Rightarrow$ reduced energy

   \[
   \mathcal{E}_{hp}[\psi] = \frac{|\partial \Omega|}{\varepsilon} E_0^{1D} + \mathcal{E}_0[v],
   \]

   \[
   \mathcal{E}_0[v] = \int_{s=0}^{1} \int_{t=0}^{+\infty} f_0^2(t) \left\{ |\nabla v|^2 - 2(t + \alpha_0) e_s \cdot j(v) + \frac{1}{2b} f_0^2(t) (1 - |v|^2)^2 \right\},
   \]

   with the superconducting current

   \[
   j(v) = \frac{i}{2} (v \nabla v^* - v^* \nabla v) = \rho \nabla \phi \text{ if } v = \sqrt{\rho} e^{i\phi}
   \]

3. Suffices to prove that the reduced energy is positive for any $v$

   \[
   \mathcal{E}_0[v] \geq 0.
   \]
Sketch of the lower bound 2

4. Write \(2(t + \alpha_0)f_0^2(t)e_s = \nabla \perp F_0\) with a potential function

\[F_0(t, s) = F_0(t) = 2 \int_0^t d\eta (\eta + \alpha_0)f_0^2(\eta).\]

5. By definition \(F_0 \leq 0\), \(F_0(0) = F_0(+\infty) = 0\).

6. Stokes’ formula

\[\mathcal{E}_0[v] := \int_{s=0}^{+\infty} \int_{t=0}^{+\infty} f_0^2(t)|\nabla v|^2 + F_0(t)\mu(v) + \frac{1}{2b}f_0^4(t)(1 - |v|^2)^2,\]

with the vorticity

\[\mu(v) = \text{curl } j(v), \quad |\mu(v)| \leq |\nabla v|^2,\]

7. Then, setting \(K_0(t) := f_0^2(t) + F_0(t)\)

\[\mathcal{E}_0[v] \geq \int_{s=0}^{+\infty} \int_{t=0}^{+\infty} K_0(t)|\nabla v|^2.\]

8. Lemma: the cost function \(K_0(t) \geq 0\) for any \(t \in \mathbb{R}^+\) and \(1 \leq b < \Theta_0^{-1}\).
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Motivation

Local density deviations:

- **Pan’s conjecture** $|\psi^{GL}|^2 \rightarrow C(b) > 0$ on $\partial \Omega$ does not follow from leading order energy considerations.

- Optimal bound $|\nabla|\psi^{GL}|| \propto \varepsilon^{-1}$: holes in the density are repaired over a length scale $O(\varepsilon)$.

- Density terms come multiplied by $\varepsilon^{-2}$ $\Rightarrow$ potential energy cost of a hole $\sim \varepsilon^{-2} \times \text{length}^2 = O(1)$

- Local density deviations are controled by the $O(1)$ remainder in previous estimates. Normal inclusions are not ruled out yet.

Role of the curvature:

- Known to play a role in corrections to $H_{c3}$: Helffer-Morame, Fournais-Helffer, Raymond ...

- Superconductivity starts to appear where curvature is maximum.

- Special behavior of domains with corners (infinite curvature): Bonnaillie-Noël with Dauge, Fournais, Martin-Vial.

- For smooth domains, when $1 < b < \Theta_0^{-1}$, curvature appears only at subleading order.
Reintroducing curvature: case of the disc

- Effective functional in boundary coordinates, including corrections due to curvature $s \mapsto k(s)$:

\[
\int_{s=0} \mid \partial \Omega \mid \int_{t=0} c_0 \mid \log \epsilon \mid (1 - \epsilon k(s) t) \left\{ \mid \partial_t \psi \mid^2 
\right.
\]
\[
\left. + \frac{1}{(1 - \epsilon k(s) t)^2} \mid (\epsilon \partial_s + i a_\epsilon(s, t)) \psi \mid^2 \right.
\]
\[
\left. + \frac{1}{2b} \left[ \mid \psi \mid^4 - 2 \mid \psi \mid^2 \right] \right\}
\]

with

\[
a_\epsilon(s, t) := -t + \frac{1}{2} \epsilon k(s) t^2 + \epsilon \delta_\epsilon, \quad \delta_\epsilon = O(1)
\]

- Easier case: disc sample, constant curvature $k$.

- Keep the same ansatz $\psi(s, t) = f(t) e^{-i\alpha s}$, obtain ($c_0 = \text{cst}$)

\[
\mathcal{E}_{k,\alpha}^{1D}[f] := \int_0^{c_0} \mid \log \epsilon \mid dt \left( 1 - \epsilon k t \right) \left\{ \mid \partial_t f \mid^2 + \frac{(t + \alpha - \frac{1}{2} \epsilon k t^2)^2}{(1 - \epsilon k t)^2} f^2 \right. 
\]
\[
\left. + \frac{1}{2b} \left( f^4 - 2 f^2 \right) \right\}
\]
Refined results in the disc case

Minimize $\mathcal{E}_{k,\alpha}^{1D}[f] \rightsquigarrow$ energy $E_{*}^{1D}(k)$, phase $\alpha(k)$, density $f_k$.

**Theorem (Correggi-NR 13)**

Let $\Omega$ be a disc of radius $R = k^{-1}$. For any fixed $1 < b < \Theta^{-1}_0$

$$E_{\varepsilon}^{\text{GL}} = \frac{2\pi E_{*}^{1D}(k)}{\varepsilon} + \mathcal{O}(\varepsilon |\log \varepsilon|),$$

and

$$|||\Psi^{\text{GL}}|^2 - f_k^2 \left( \frac{R-r}{\varepsilon} \right)|||_{L^2(\Omega)} = \mathcal{O}(\varepsilon^{3/2} |\log \varepsilon|^{1/2}).$$

▶ Does contain the subleading order:

$$E_{*}^{1D}(k) = E_{0}^{1D} + \mathcal{O}(\varepsilon), \quad \alpha(k) = \alpha_0 + \mathcal{O}(\varepsilon), \quad f_k = f_0 + \mathcal{O}(\varepsilon).$$

▶ Method similar as before, second order cost function.

▶ Significant but technical additional difficulties.
Refined results in the general case

- Associate $E_{1D}^*(k(s)), \alpha_{k(s)}, f_{k(s)}$ to smooth curvature $k(s)$
- Approximate locally the boundary by a disc: think of
  $$\psi_{GL}(r) = \psi_{GL}(s, t) \approx f_{k(s)}\left(\frac{t}{\varepsilon}\right) \exp\left(-i\alpha_{k(s)}\frac{s}{\varepsilon}\right)$$

Theorem (Correggi-NR 14)

For any fixed $1 < b < \Theta_0^{-1}$,

$$E_{GL}^\varepsilon = \frac{1}{\varepsilon} \int_0^{|\partial\Omega|} E_{1D}^*(k(s)) \, ds + O(\varepsilon |\log \varepsilon|^{\infty}).$$

and

$$\left\|\left|\psi_{GL}\right|^2 - f_{k(s)}\left(\frac{t}{\varepsilon}\right)^2\right\|_{L^2(\Omega)} \leq C\varepsilon^{3/2}|\log \varepsilon|^{\infty}.$$

- Curvature $k(s) \rightarrow$ approximate by constants in cells of side length $\varepsilon$
- Use the disc analysis locally in each cell
- Patch things together and control unphysical boundary terms
- Requires a fine analysis of the $k$-dependence of the model problem
Effect of curvature on surface superconductivity

- It was previously known (Pan, Fournais-Kachmar ...) that

\[
\frac{1}{\varepsilon} |\psi_{GL}|^4 dr \xrightarrow{\varepsilon \to 0} C(b) ds.
\]

- \(C(b) > 0\) identified by previous theorems, \(ds = 1D\) Lebesgue measure along the boundary.

*Superconductivity density is (roughly) uniform along the boundary.*

- Corollary of the previous results: estimate of subleading order

\[
\frac{1}{\varepsilon} \left( \frac{1}{\varepsilon} |\psi_{GL}|^4 dr - C(b) ds \right) \xrightarrow{\varepsilon \to 0} C_2(b) k(s) ds.
\]

- \(k(s) = \text{curvature.}\)

- \(C_2(b) > 0\) (not so) explicitly identified.

*Superconductivity density is (slightly) larger in regions of larger curvature.*
Thank You !