Statistique pour des processus longue-mémoire

Jean-Marc Bardet
bardet@univ-paris1.fr

Ecole de Statistique d'Angers

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Famous historic data

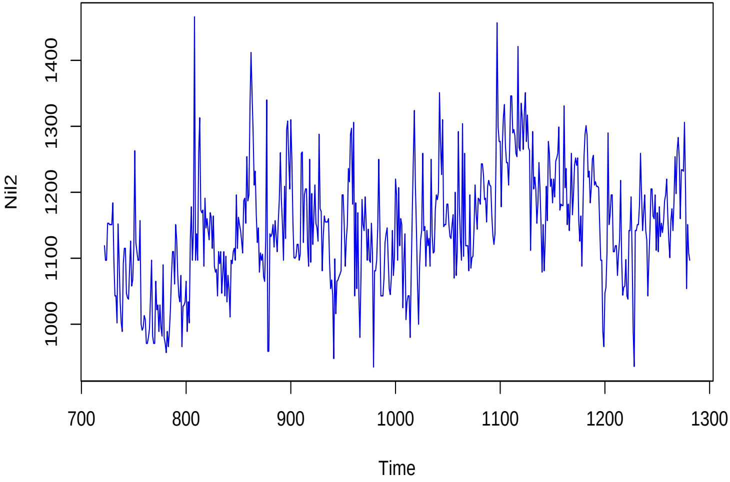

Figure: Lowest annual level of the Nile River from 722 to 1281

Increments of Nile data

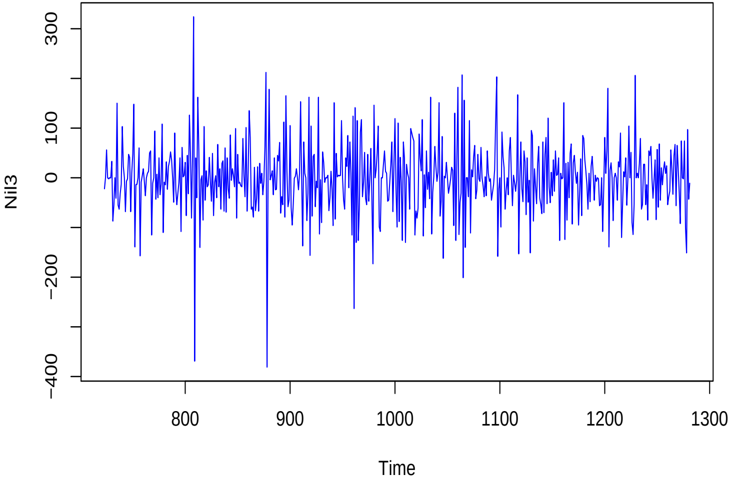

Figure: Increments of Nile data

Correlogram of Nile data

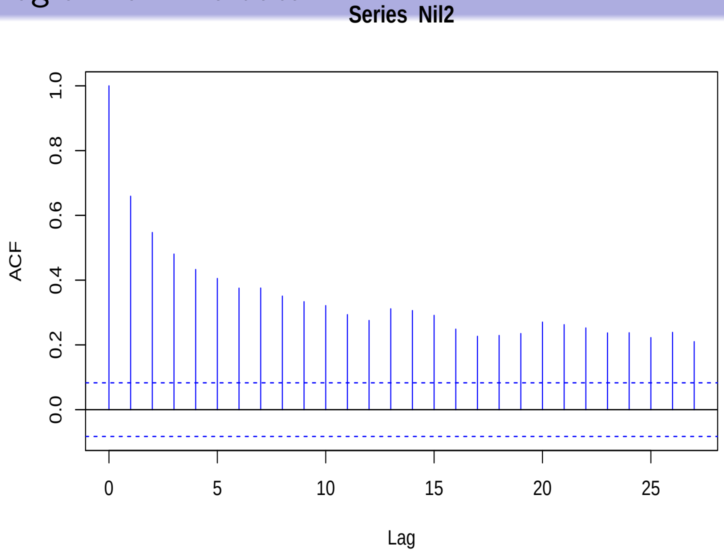

Figure: Correlogram of Nile data

Consequences and objectives

The correlogram decreases very slowly

$$\widehat{\rho}(k) \simeq C\,|k|^{-D}, \text{ with } \; D > 0?$$

- Behavior different to usual ARMA processes
- Long Memory or Long Range Dependent process

$\Longrightarrow$ Define such a process

$\Longrightarrow$ Estimate the parameters of this process

Outline

1. Long Memory Processes
   - Basic definitions
   - Definition(s) of LM process
   - Two famous examples of LM processes
   - Limit theorems for LM processes
2. Estimation of the LM parameter
   - The estimation problem
   - Parametric estimators
   - Semi-parametric estimators of LM parameter
   - Results of simulations

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Basic definitions

Definition

Let $X = (X_k)_{k\in\mathbb{Z}}$ be a sequence of r.v. on $(\Omega, \mathcal{A}, \mathbb{P})$. Then X stationary:

$$\forall m \in \mathbb{N}^*,\ \forall (k_1, \cdots, k_m) \in \mathbb{Z},\ \forall c \in \mathbb{Z},\ (X_{k_1}, \cdots, X_{k_m}) \stackrel{\mathcal{D}}{=} (X_{k_1+c}, \cdots, X_{k_m+c})$$

Examples: X i.i.d.r.v., ARMA(p,q), GARCH(p,q),...

Other definition: X is a second order stationary process then:

- $\mathbb{E}X_k = C$ and autocovariance $r(k) = \mathrm{Cov}(X_j, X_j + k)$ for any j, $k \in \mathbb{Z}$

$\Longrightarrow$ If it exists, spectral density $f(\lambda) = \dfrac{1}{2\pi} \displaystyle\sum_{k\in\mathbb{Z}} r(k) e^{-ik\lambda}$, $\lambda \in [-\pi, \pi]$

Example of i.i.d.r.v.

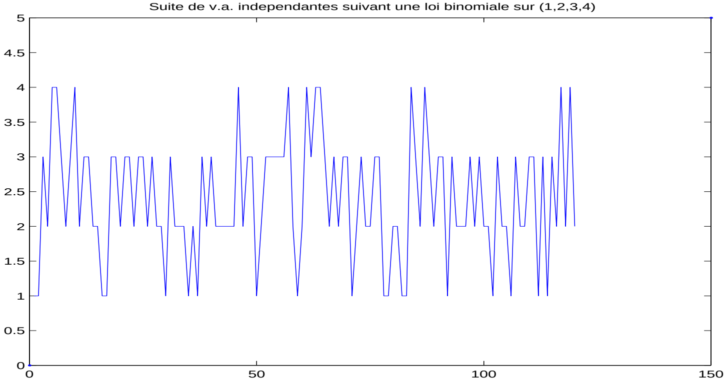

Figure: Sequence of $\mathcal{B}(4, 1/2)$ r.v.

Example of an ARMA process trajectory

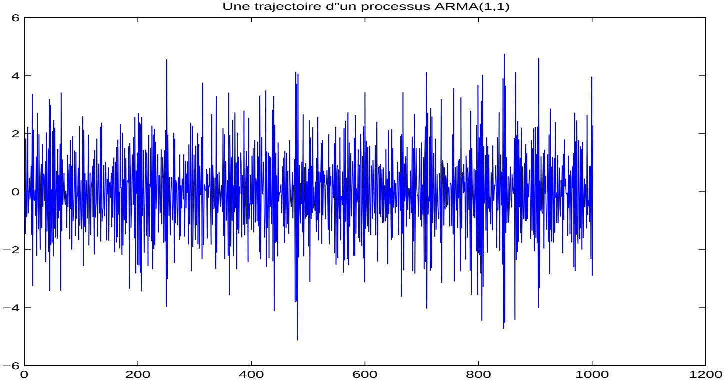

Figure: ARMA$(1, 1)$ process

Example of a GARCH process trajectory

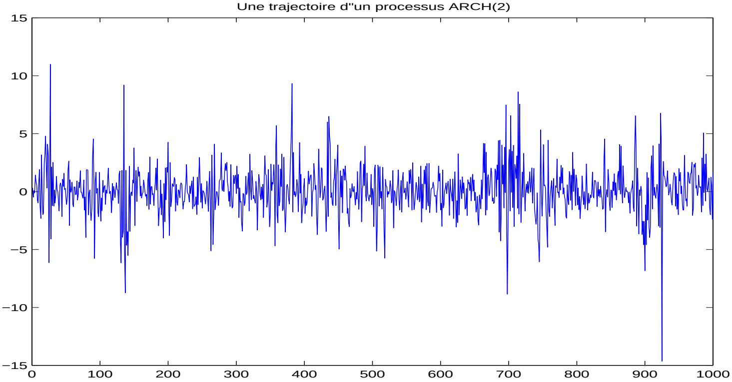

Figure: ARCH(2) process

Definition 0

Definition

Let $X = (X_k)_{k\in\mathbb{Z}}$ be a second order stationary process

$$X \text{ is a Long Memory Process} \Longleftrightarrow \sum_{k\in\mathbb{Z}} |r(k)| = \infty$$

Exemple: $X = (X_k)_{k\in\mathbb{Z}}$ with $X_k = X_0$ for all $k \in \mathbb{Z}$.

Consequences: If X LM process:

- The spectral density of X, if it exists, is not continuous;
- This definition is not really satisfying

Definition 1

Definition (1)

$X = (X_k)_{k \in \mathbb{Z}}$ is a LM stationary second order process

$$r(k) = |k|^{-D} L(|k|), \quad \text{for } k \neq 0, \quad \text{with}$$

- $D \in]0,1[$ LM parameter
- L slowly varying function in ∞, *i.e.*: $\forall t > 0, \lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$

Counter-example: $X = (X_k)_{k \in \mathbb{Z}}$ with $X_k = X_0$ for all $k \in \mathbb{Z}$ is not LM.

Definition (2)

$X = (X_k)_{k \in \mathbb{Z}}$ is a LM stationary second order process

$$f(\lambda) = |\lambda|^{D-1} M\left(\frac{1}{|\lambda|}\right) \quad , \quad \text{for } \lambda \to 0, \quad \text{with}$$

- $D \in]0,1[$ LM parameter
- M a slowly varying function in ∞

Remarks:

- Definition (1) implies Definition (2) (Abélian Theorem);
- Definition (2) + decreasing of $r(\cdot)$ implies Definition (1) (Tauberian Theorem).

Other possible definitions [Samorodnitsky and Taqqu (1994)]

Definition

$X = (X_t)_{t \in \mathbb{R}}$ *H*-self similar process with stationary increments (*H*-SSSI)

$$\begin{cases} (X_{c\,s})_s \overset{\mathcal{D}}{=} c^H (X_s)_s & \text{for any} \quad c > 0 \\ (X_{t+s} - X_t)_t \ \text{ stationary process} & \text{for any} \quad s \in \mathbb{R} \end{cases}$$

Definition

$X = (X_k)_{k \in \mathbb{Z}}$ is a LM stationary process when

$$X_k = Y_{k+1} - Y_k, \quad \text{for } k \in \mathbb{Z}, \quad \text{with} \quad (Y_k) \ H\text{-SSSI}$$

Fractional Gaussian Noise

Definition (Kolmogorov, 1940, Lévy, 1965)

$Y = \{Y_t, t \in R\}$ is a Fractional Brownian Motion

$$\Longleftrightarrow$$

Y is a centered Gaussian process with stationary increments

such as $\mathbb{E} Y_t^2 = \sigma^2 |t|^{2H}$, $\sigma^2 > 0$, $H \in (0, 1]$

Consequences:

1. Y is the only Gaussian H-SSSI process
2. $X = (Y_{t+1} - Y_t)_{t \in \mathbb{Z}}$, Fractional Gaussian noise.

$$\Longrightarrow X \text{ is LM if } H > 1/2\text{: } r(k) \sim \sigma^2 H(2H-1)\,|k|^{2H-2} \quad |k| \to \infty$$

Other definitions of FBM

- Harmonizable representation:

$$Y_t = \sigma^2 \, C_1(H) \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{2H+1}} \, \widehat{W}(d\xi) \quad t \in \mathbb{R}$$

- Temporal representation:

$$Y_t = \sigma^2 \, C_2(H) \int_{\mathbb{R}} \left(\int_0^t (u-y)_+^{H-\frac{3}{2}} du \right) dW(y) \quad t \in \mathbb{R}$$

$\Longrightarrow$ Existence of FBM and FGN

Simulations of a FGN trajectory

How to generate a FGN path $(X_1, \cdots, X_n)$?

1. Natural idea: Cholesky decomposition of $\Sigma = \big(r(|j-i|)\big)_{1\leq i,j\leq n} = R\,R'$

$$X = R\,Z, \quad \text{with } Z \text{ a sample of Gaussian i.i.d.r.v.}$$

2. Best choice: plug Σ in a circulant matrix and use the spectral decomposition of a circulant matrix $\Longrightarrow$ spectral decomposition of $\Sigma^{1/2}$

Example of FGN

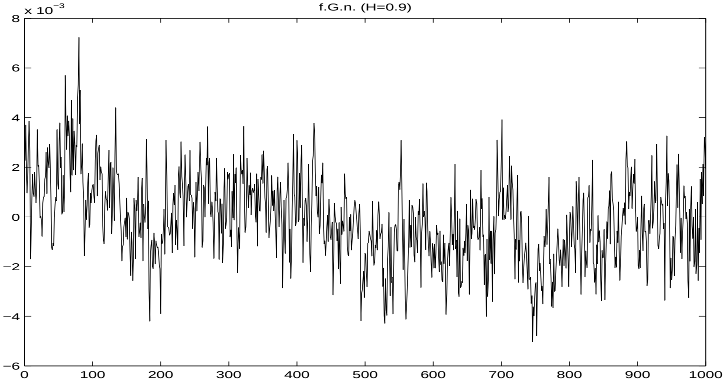

Figure: FGN with $H = 0.9$

FARIMA(p, d, q) processes

Definition (Granger et Joyeux, 1980)

If $\varepsilon = (\varepsilon_t)_{t \in \mathbb{Z}}$ white noise, $X = \{X_t, t \in R\}$ FARIMA(p, d, q) process when

$$\iff \quad (1-B)^d P(B)(X) = Q(B)(\varepsilon) \ \text{ avec } P \in \mathbb{R}_p[X],\ Q \in \mathbb{R}_q[X]$$

$$\iff \quad X_k = \sum_{j=0}^{\infty} \Big(\frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \Big) \eta_{k-j} \ \text{ with } (\eta_t) \text{ ARMA}(p,q):$$

$$\eta_k + \theta_1 \eta_{k-1} + \cdots + \theta_p \eta_{k-p} = \varepsilon_k + \phi_1 \varepsilon_{k-1} + \cdots + \phi_q \varepsilon_{k-q}$$

Consequence: $f(\lambda) = \frac{\sigma^2}{\pi} \Big| \frac{Q(e^{i\lambda})}{P(e^{i\lambda})} \Big|^2 \frac{1}{|1 - e^{i\lambda}|^{2d}}$ for $\lambda \neq 0$

$\Longrightarrow$ Existence of X since $f \geq 0$ measurable function

$\Longrightarrow$ X is LM if $0 < d < 1/2$: $f(\lambda) \sim C\,|\lambda|^{-2d}$ when $\lambda \to 0$

Simulations of a FARIMA trajectory

How to generate a FARIMA path $(X_1, \cdots, X_n)$?

1. Best Gaussian idea: use also the circulant matrix method...
2. Non Gaussian idea: truncation of $\sum_{j=0}^{\infty} \left(\frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \right) \eta_{k-j}$

$$\Longrightarrow \quad X_k = \sum_{j=0}^{M} \left(\frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \right) \eta_{k-j} \qquad \text{with } M \text{ large number}$$

Example of FARIMA process

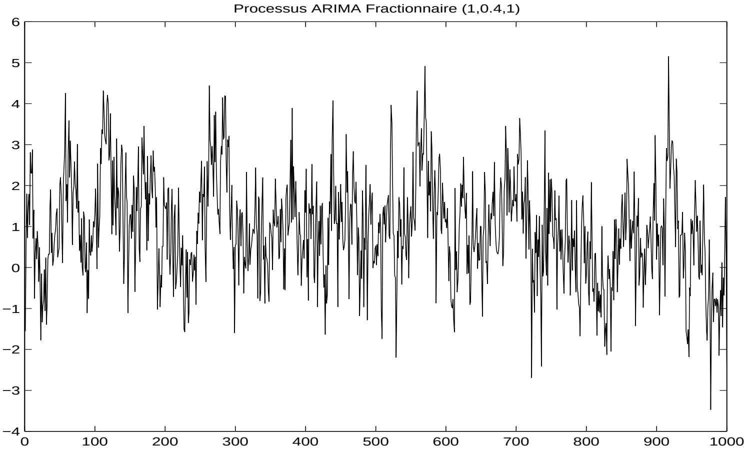

Figure: FARIMA(1,d,1) with $d = 0.4$

Other example of FARIMA process

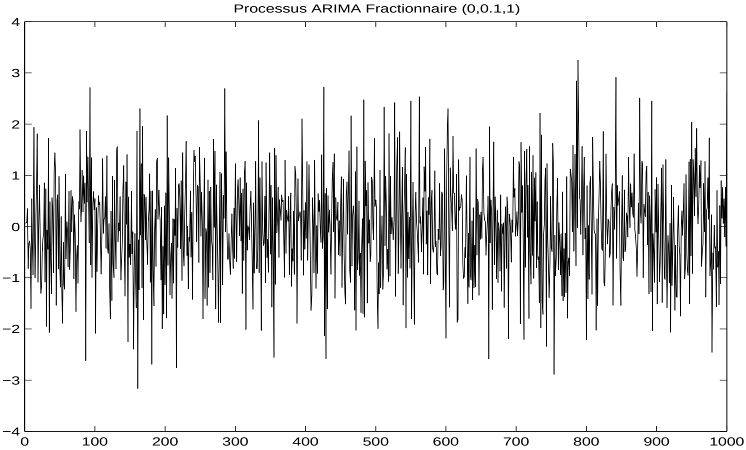

Figure: FARIMA(0,d,1) with $d = 0.1$

Limit theorem for the sum of Short Memory processes

Definition

Let $\varepsilon = (\varepsilon_k)_{k\in\mathbb{Z}}$ be a $\mathbb{L}^2$-white noise and $(a_k)_{k\geq 0} \in \ell^2(\mathbb{R})$. A one-sided linear process $X = (X_k)_{k\in\mathbb{Z}}$ is defined by

$$X_k = \sum_{j\geq 0} a_j\, \varepsilon_{k-j} \quad \text{for } k \in \mathbb{Z}$$

Consequence: If $|a_j| = j^{-\beta} L(j)$, $1/2 < \beta < 1$, X LM with $D = 2\beta - 1$.

Theorem (Ibragimov, 1962)

If X is a one-sided linear process with $\sum_{k\in\mathbb{Z}} r(k) = \sigma^2 \neq 0$, then

$$\Big(N^{-1/2} \sum_{j=1}^{[Nt]} X_t\Big)_{t\geq 0} \xrightarrow[N\to\infty]{D([0,1])} \sigma^2\,\big(W_t\big)_{t\geq 0}.$$

A general limit theorem for the sum of LM processes

Theorem (Rosenblatt, 1961, Davydov 1970)

If X is a Gaussian or one-sided linear process, $\mathbb{E}(X_0) = 0$, $Var(X_0) = 1$, *LM (Definition 1) with parameter* $D \in]0,1[$,

$$\Big(\frac{1}{N^{1-D/2} L^{1/2}(N)} \sum_{j=1}^{[Nt]} X_t\Big)_{t \geq 0} \xrightarrow[N \to \infty]{D([0,1])} \big(B_{1-D/2}(t)\big)_{t \geq 0}$$

Limit theorem for functionals of Gaussian LM processes

H_n be the Hermite polynomial of degree n: $H_1(X) = X$, $H_2(X) = X^2 - 1$,..

Call Hermite rank m of $f \in \mathbb{L}^2(\mathcal{N}(0,1))$ if $\exists (c_j)_j$ such as $f = \sum_{j=m}^{\infty} c_j H_j$.

Theorem (Rosenblatt, 1961, Taqqu, 1975, Dobrushin and Major, 1979)

If the Hermite rank of f is m, if X Gaussian process, $\mathbb{E}(X_0) = 0$, $Var(X_0) = 1$, LM with parameter $D \in]0, 1/m[$,

$$\Big(\frac{1}{N^{1-\frac{mD}{2}} L^{1/2}(N)} \sum_{j=1}^{[Nt]} \big[f(X_t) - \mathbb{E}(f(X_0)) \big] \Big)_t \xrightarrow[N\to\infty]{D([0,1])} \frac{\mathbb{E}(f(X_0) H_m(X))}{m!} \big(Z_{m,D}(t) \big)_t$$

1. If $m = 1$, $0 < D < 1$, the limit process is $Z_{1,D}(t) = B_{1-D/2}(t)$,

$$\text{ex:} \quad \Big(\frac{1}{N^{1-D/2}L^{1/2}(N)} \sum_{j=1}^{[Nt]} X_t\Big)_{t\geq 0} \xrightarrow[N\to\infty]{D([0,1])} \big(B_{1-D/2}(t)\big)_{t\geq 0}$$

2. If $m = 2$, $0 < D < 1/2$, limit process $Z_{2,D}(t)$, Rosenblatt process,

$$\text{ex:} \quad \Big(\frac{1}{N^{1-D}L^{1/2}(N)} \sum_{j=1}^{[Nt]} \big[X_t^2 - 1\big]\Big)_{t\geq 0} \xrightarrow[N\to\infty]{\mathcal{L}} \frac{1}{2}\big(Z_{2,D}(t)\big)_{t\geq 0}$$

Limit theorems for linear processes

Theorem (Surgailis, 1980, Giraitis, 1985, Ho and Hsin, 1997, Surgailis, 2000)

K function and $K_\infty^r = \Big(\frac{\partial^r}{\partial x^r} \int K(x+y) d\mu_X(y) \Big)_0$. If $K_\infty^r = 0$ for $r < k$ and $|K_\infty^k| \neq 0 < \infty$, if X LM one-sided linear process, with parameter $D \in]0, 1/k[$,

$$\frac{1}{N^{1-\frac{kD}{2}} L^{2k}(N)} \sum_{j=1}^{N} \big[K(X_t) - \mathbb{E}(K(X_0)) \big] \xrightarrow[N\to\infty]{\mathcal{L}} C(k,\mu)\, Z_{k,D}(1)$$

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Problem

Suppose that

- X is LM process with parameter $D \in (0,1)$ unknown;
- $(X_1, \cdots, X_N)$ is an observed trajectory of X.

Aim:

- Propose a consistant estimator $\widehat{D}_N$ of D;
- Study the asymptotic behavior of $\widehat{D}_N$.

Hurst (1953) propose the R/S estimator of D based on:

$$\text{Hurst effect: } \frac{R(N))}{S(N)} \xrightarrow[N\to\infty]{\mathcal{P}} C(H)\, N^H \text{ where } H = 1 - D/2$$

with
$$\begin{cases} R(N) = \max_{0\leq i\leq N}\{i(\overline{X}_i - \overline{X}_N)\} - \min_{0\leq i\leq N}\{i(\overline{X}_i - \overline{X}_N)\} \\ S^2(N) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} X_j^2 - \overline{X}_N^2 \end{cases}$$

$\Longrightarrow$ H estimated by log-log regression of $\log\left(\dfrac{R(N_i)}{S(N_i)}\right)$ onto $\log(N_i)$

But not really an accurate estimator... (see Mandelbrot and Taqqu, 1979)

Maximum Likelihood Estimator (MLE)

Natural estimator: estimate D by maximum likelihood

For example, for zero mean stationary Gaussian process:

$$-2\log\big(L_\theta(X_1,\cdots,X_n)\big) = N\log(2\pi)+\log(|\Sigma_\theta^{(N)}|) + (X_1,\cdots,X_N)\big(\Sigma_\theta^{(N)}\big)^{-1}(X_1,\cdots,X_N)'$$

with $\Sigma_\theta^{(N)} = (r_\theta(|j-i|))_{1\le i,j\le N}$.

$$\Longrightarrow \widehat{\theta}_N = \operatorname{Arg\,min}_{\theta\in\Theta}\Big\{-2\log\big(L_\theta(X_1,\cdots,X_n)\big)\Big\}$$

Maximum Likelihood Estimator (2)

Numerous drawbacks:

- Requires the knowledge of the exact distribution
- Even in the Gaussian case, the study of asymptotic behavior is difficult
- Numerically impossible to be computed for $N \geq 10^4$

An approximation: Whittle estimator

Theorem (Szegö Theorem, Whittle, 1953, Dahlhaus, 1989)

Under conditions, for stationary Gaussian process X,

$$-\frac{1}{N}\log\big(L_\theta(X_1,\cdots,X_N)\big)-\frac{1}{2}\log(2\pi)$$

$$\underset{N\to\infty}{\overset{\mathcal{D}}{\simeq}}\ \widehat{U}_N(\theta)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\Big(\log(f_\theta(\lambda))+\frac{\widehat{I}_N(\lambda)}{f_\theta(\lambda)}\Big)\,d\lambda$$

where
- $\widehat{I}_N(\lambda)=\frac{1}{2\pi N}\Big|\sum_{k=1}^N X_k e^{-ik\lambda}\Big|^2$ *is the periodogram*
- $f_\theta(\lambda)$ *is the spectral density of X*

$\Longrightarrow\ \widetilde{\theta}_N=\mathrm{Arg}\min_{\theta\in\Theta}\widehat{U}_N(\theta)$ Whittle estimator of θ

Theorem (Fox and Taqqu, 1987, Dahlhaus, 1989)

If X LM stationary Gaussian, with spectral density f_θ satisfying conditions (derivatives...),

$$\sqrt{N}(\widetilde{\theta}_N - \theta) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}\big(0\,,\, I^{-1}(\theta)\big)$$

où $I(\theta) = \dfrac{1}{4\pi} \displaystyle\int_{-\pi}^{\pi} \Big(\frac{\partial}{\partial\theta} \log f_\theta(\lambda)\Big)\Big(\frac{\partial}{\partial\theta} \log f_\theta(\lambda)\Big)' d\lambda$

$\Longrightarrow$ Could be applied to X FGN or Gaussian FARIMA(p, d, q).

1. The proof uses limit theorems for quadratic forms of Gaussian LM processes + usual arguments for M-estimator
2. By Slutsky Lemma, the central limit theorem implies asymptotic confidence intervals
3. Quite surprising result since convergence rate $\sqrt{N}$

Extensions

Theorem (Dahlhaus, 1989)

If X LM stationary Gaussian, with spectral density f_θ satisfying conditions (derivatives...), with $\widehat{\theta}_N$ MLE,

$$\sqrt{N}\big(\widehat{\theta}_N - \theta\big) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}\big(0\,,\, I^{-1}(\theta)\big)$$

$\Longrightarrow$ $\widehat{\theta}_N$ et $\widetilde{\theta}_N$ are asymptotic efficient estimator in Gaussian case

Theorem (Giraitis and Surgailis, 1990)

If X is a stationary LM linear process, with $\mathbb{E}(\varepsilon_0^4) < \infty$ spectral density f_θ satisfying conditions (derivatives...),

$$\sqrt{N}\big(\widetilde{\theta}_N - \theta\big) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}\big(0\,,\, I^{-1}(\theta)\big)$$

Extension to LM functionals of Gaussian process

Let $Y = (Y_k)_k$ stationary LM Gaussian process and $X_k = G(Y_k)$ LM such as:

$$f(\lambda) = \lambda^{-D(\theta)} L_\theta(1/|\lambda|)$$

Theorem (Giraitis and Taqqu, 1999)

Under certain conditions on G,

- *If* $1/2 < D(\theta) < 1$, $N^{1-D(\theta)}(\widetilde{\theta}_N - \theta) \xrightarrow[N\to\infty]{\mathcal{L}} C_G(\theta)\, R_1(D(\theta))$ *with* $R_1(D(\theta))$ *a Rosenblatt r.v.*
- *If* $0 < D(\theta) < 1/2$, $N^{1/2}(\widetilde{\theta}_N - \theta) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0\,,\, \sigma_G^2(\theta))$.

Extension to Rosenblatt process

Temporal representation of FBM:

$$B_t^H = \sigma^2 \, C_2(H) \int_{\mathbb{R}} \left(\int_0^t (u-y)_+^{H-\frac{3}{2}} du \right) dW(y) \quad t \in \mathbb{R}$$

Temporal representation of the Rosenblatt process:

$$Z_t^H = \sigma^2 \, c_Z(H) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_0^t (u-y_1)_+^{\frac{H}{2}-1} (u-y_2)_+^{\frac{H}{2}-1} du \right) dW(y_1) dW(y_2)$$

where $H \in (1/2, 1)$ and $c_Z^2(H) = \frac{2H(2H-1)}{\beta^2(1-H, \frac{H}{2})}$.

Propriété

Z^H is a H-self-similar process with same second order properties than B^H

Whittle estimator of Rosenblatt process increments

$(X_1^H, \cdots, X_N^H)$ where $X_t^H = Z_{t+1}^H - Z_t^H$ increments of Rosenblatt process

Théorème

$$\widetilde{H}_N \xrightarrow[N\to\infty]{p.s.} H \qquad \textit{and} \qquad N^{1-H}\left(\widetilde{H}_N - H\right) \xrightarrow[N\to\infty]{\mathcal{L}} \gamma(H)\, R_1^H$$

where R_1^H is a Rosenblatt r.v. and $\gamma(H)$ defined by:

$$\gamma(H) := 16\pi \sqrt{\frac{2(2H-1)}{H(1+H)^2}} \left(\int_{-\pi}^{\pi} \frac{f_{(H+1)/2,1}(\lambda)}{g_H(\lambda)}\, d\lambda\right) \left(\int_{-\pi}^{\pi} f_{H,1}(\lambda) \frac{\partial^2}{\partial H^2} \left(\frac{1}{g_H(\lambda)}\right) d\lambda\right)^{-1}$$

Note: we have $f_{H,C}(\lambda) = \frac{C^2 H\Gamma(2H)\sin(\pi H)}{2\pi}(1-\cos\lambda)\sum_{k\in\mathbb{Z}} |\lambda + 2k\pi|^{-1-2H}$.

Sketch of proof

1. Renormalization of parameters for only minimizing $\int_{-\pi}^{\pi} \frac{\widehat{I}_N(\lambda)}{f_\theta(\lambda)}\, d\lambda$

2. Limit theorems for $\widehat{J}_N(g) = \int_{-\pi}^{\pi} g(\lambda)\widehat{I}_N(\lambda)\, d\lambda$ Integrated periodogram
$\Longrightarrow N^{1-H}\left(\widehat{J}_N(g) - \mathbb{E}\widehat{J}_N(g)\right) \xrightarrow[N\to\infty]{\mathcal{L}} \gamma_{H,C}\, R_1^H$ (Malliavain calculus...)

3. Application to $\frac{\partial}{\partial H}\left(\frac{1}{g_H}\right)$ + Taylor expansion:

$$\Longrightarrow \quad \widehat{J}_N\Big(\frac{\partial}{\partial H}\big(\frac{1}{g_H}\big)\Big) \simeq -(\widehat{H}_N - H) \times \widehat{J}_N\Big(\frac{\partial^2}{\partial H^2}\big(\frac{1}{g_{\widetilde{H}_N}}\big)\Big)$$

4. Prove that $N^{1-H}\,\mathbb{E}\widehat{J}_N\Big(\frac{\partial}{\partial H}\big(\frac{1}{g_H}\big)\Big) \xrightarrow[N\to\infty]{\mathcal{L}} 0$

Semi-parametric estimation problem

For $d \in (-1/2, 1/2)$ and $\beta > 0$, unknown parameters:

Assumption $I(d,\beta)$: There exist $c_0 > 0$, $c_1 \in \mathbb{R}$, spectral density f satisfies

$$f(\lambda) = |\lambda|^{-2d}\big(c_0 + c_1|\lambda|^{\beta} + o(|\lambda|^{\beta})\big) \quad \text{when } \lambda \to 0.$$

$\Longrightarrow$ Estimate d from an observed trajectory $(X_1 \cdots, X_N)$

Remark: only the behavior around 0 of f is known

Local Whittle estimator

Whittle estimator:

$$\widetilde{\theta}_N = \mathrm{Arg}\min_{\theta \in \Theta} \int_{-\pi}^{\pi} \Big(\log(f_\theta(\lambda)) + \frac{\widehat{I_N}(\lambda)}{f_\theta(\lambda)} \Big) \, d\lambda$$

$\Longrightarrow$ Can not be applied if X satisfies $I(d,\beta)$ since f_θ unknown

$\Longrightarrow$ Work around 0 instead of $[-\pi,\pi]$.

Local Whittle estimator (2)

Robinson (1995) define the local Whittle contrast

$$W_N(d,m) = \log\Big(\frac{1}{m}\sum_{k=1}^{m}\big(\frac{k}{m}\big)^{2d} I_N(\lambda_k)\Big) - \frac{2\,d}{m}\sum_{k=1}^{m}\log(k/m),$$

$$\text{with}\quad \lambda_k = 2\pi\,\frac{k}{N} \quad\text{and}\quad I_N(\lambda) = \frac{1}{2\pi\,N}\Big|\sum_{k=1}^{N} X_k e^{-i\,k\,\lambda}\Big|^2.$$

Define the local Whittle estimator:

$$\widetilde{d}_N^{(LW)} = \text{Arg}\min_{d\in(-1/2,1/2)}\big\{W_N(d,m)\big\}$$

Local Whittle estimator (3)

Theorem (Robinson, 1995)

Assume $I(d,\beta)$ *where* X *one-sided linear process with* $\mathbb{E}(\varepsilon_0^4)<\infty$ *and*

- $\sum_{j=0}^{\infty} a_j^2 < \infty$ *and*
- $\frac{\partial}{\partial\lambda}\alpha(\lambda) = O\big(\big|\lambda^{-1}\,\alpha(\lambda)\big|\big)$ *when* $\lambda \to 0^+$ *with* $\alpha(\lambda)=\sum_{j=0}^{\infty} a_j e^{ij\lambda}$.

Then if $m = o\big(N^{2\beta/(1+2\beta)}(\log N)^{-2}\big)$,

$$\sqrt{m}\,\big(\widetilde{d}_N^{(LW)} - d\big) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}\big(0\,,\,1/4\big)$$

Dalla, Giraitis and Hidalgo (2005) improves this result: $m = o\big(N^{2\beta/(1+2\beta)}\big)$

$\Longrightarrow$ Optimal convergence rate: $\simeq N^{\beta/(1+2\beta)}(\log N)^{-1}$.

An optimal result

Theorem (Giraitis, Robinson and Samarov, 1997)

An estimator $\widehat{d}_N$ is rate optimal in the minimax sense if

$$\limsup_{N\to\infty} \sup_{d\in(0.5,0.5)} \sup_{f\in I(d,\beta)} N^{\frac{2\beta}{1+2\beta}} \cdot \mathbb{E}[(\widehat{d}_N - d)^2] < \infty.$$

$\Longrightarrow$ If β is known, $\widetilde{d}_N^{(LW)}$ is optimal in the minimax sense

$\Longrightarrow$ If β is unknown, build an adaptive estimator?

Monte-Carlo experiments for Rosenblatt process

$N = 1000$	$H = 0.55$	$H = 0.65$	$H = 0.75$	$H = 0.85$	$H = 0.95$
mean $\widehat{H}_N$	0.570	0.653	0.736	0.815	0.917
std $\widehat{H}_N$	0.030	0.041	0.047	0.053	0.050
mean $\widehat{H}_{ADG}$	0.570	0.634	0.708	0.795	0.906
std $\widehat{H}_{ADG}$	0.072	0.084	0.094	0.105	0.102
mean $\widehat{H}_{Wa}$	0.499	0.542	0.619	0.685	0.766
std $\widehat{H}_{Wa}$	0.104	0.116	0.115	0.129	0.119

$N = 5000$	$H = 0.55$	$H = 0.65$	$H = 0.75$	$H = 0.85$	$H = 0.95$
mean $\widehat{H}_N$	0.582	0.655	0.743	0.837	0.929
std $\widehat{H}_N$	0.014	0.019	0.029	0.033	0.035
mean $\widehat{H}_{ADG}$	0.575	0.627	0.723	0.824	0.919
std $\widehat{H}_{ADG}$	0.041	0.052	0.062	0.067	0.072
mean $\widehat{H}_{Wa}$	0.550	0.610	0.698	0.800	0.891
std $\widehat{H}_{Wa}$	0.055	0.062	0.072	0.079	0.075

Estimated density of $\widehat{H}_N$

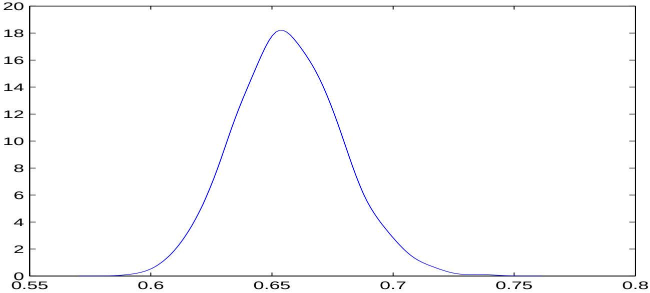

Figure: Estimation (Silverman method) of $\widehat{H}_N$ for $H = 0.65$, $N = 5000$ from 1000 independent replications