Stochastic PDEs on Hilbert space with irregular noise coefficients

With a first part on McKean-Vlasov equations

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Part 1. McKean-Vlasov equations

- Solving McKean-Vlasov equation with various interaction coefficients via relative entropy.
- Smoothness of density in the case of an interaction kernel.
- Uniform in time propagation of chaos with a sharp rate

McKean-Vlasov SDEs with non-Lipschitz interaction

Intensive research in recent years on solving McKean-Vlasov type equations given $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$,

$$dX_t = \langle \mu_t, b(t, X_t, \cdot)\rangle dt + dW_t, \quad \mu_t = \mathrm{Law}(X_t) \tag{1}$$

given W a d-dimensional Brownian motion or stable Lévy process.

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given W a d-dimensional Brownian motion or stable Lévy process.

- When b is Lipschitz continuous in its last variable, one may solve directly via a Gronwall argument.
- When b is merely bounded measurable, one may write the drift as $B(t, X_t, \mu_t)$ with B continuous in μ in total variation distance in the sense that for any $t > 0$, $x \in \mathbb{R}^d$,

$$|B(t, x, \mu) - B(t, x, \nu)| \le C \|\mu - \nu\|_{TV}, \tag{2}$$

  Let $\Phi(\mu)$ be solution to (1) for each $\mu \in \mathcal{P}([0, T]; \mathbb{R}^d)$, then

$$H(\Phi(\mu) \mid \Phi(\nu)) = \frac{1}{2} \mathbb{E}^{P_\mu} [\int_0^t |B(s, X_s, \mu) - B(s, X_s, \nu)|^2 ds] \tag{3}$$

Solution-continued

- The right hand side becomes

$$\leq \frac{C^2}{2}\int_0^t \|\mu_s - \nu_s\|_{TV}^2 ds \leq C\int_0^T H(\mu_s \mid \nu_s) ds. \tag{4}$$

The first inequality follows from Lipschitz continuity of B in the measure component, and the second follows from Pinsker's inequality. Then the existence and uniqueness of a solution follows from a fixed point argument.

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- This method has appeared in several different papers in the 2010s. See for example [Lacker,2018].
- We can solve McKean-Vlasov equation in a much wider generality, where the assumption

$$|B(t, x, \mu) - B(t, x, \nu)| \leq C\|\mu - \nu\|_{TV}, \tag{5}$$

  is no longer valid.

Examples of new results

- Linear growth, path dependent coefficients:

$$|b(t,x,y)| \leq K(1 + \|x\|_t + \|y\|_t), \quad t \in [0,T] \tag{6}$$

we can solve (extending Lacker 2021)

$$dX_t = \langle \mu, b(t,X,\cdot)\rangle dt + dW_t, \quad \mu = \mathrm{Law}(X) \tag{7}$$

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- Singular and linear growth coefficients: given $\frac{d}{p_1} + \frac{2}{q_1} < 1$,

$$|b_1(t,x,y)| \le h_t(x-y) \text{ for some } h \in L_t^{q_1}\left([0,T], L_x^{p_1}(\mathbb{R}^d)\right),$$

$$\sup_{t,y} |b_2(t,x,y)| \le K(1+|x|^\beta) \text{ for } K>0, \beta \in [0,1), \tag{8}$$

we can solve (extending Röckner-Zhang 2019)

$$dX_t = \langle \mu_t, b_1 + b_2(t,X_t,\cdot)\rangle dt + dW_t, \quad \mu_t = \mathrm{Law}(X_t) \tag{9}$$

b_1 must be state dependent, but b_2 can be path dependent.

Examples for fractional Brownian driving noise

(Extending Galeati, Harang, Mayorcas 2021 and other works)

- Assume $H \in (0, \frac{1}{2})$ and $\alpha > 1 - \frac{1}{2H}$, and

$$\|B(t,\cdot,\mu) - B(t,\cdot,\nu)\|_{B^{\alpha}_{\infty,\infty}} \lesssim \|\mu - \nu\|_{TV}, \mu, \nu \in \mathcal{P}(\mathbb{R}^d), \quad (10)$$

we can solve, via Girsanov transform for FBMs,

$$dX_t = B(t, X_t, \mu_t)dt + dB_t^H, \quad \mathrm{Law}(X_t) = \mu_t, \quad (11)$$

and when the interaction has linear growth and path dependent, i.e. $|b(t,x,y)| \leq K(1 + \|x\|_t + \|y\|_t)$, solve

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- When $H \in (\frac{1}{2}, 1)$, and $\beta > H - \frac{1}{2} > 0$, assume

$$|b(t,x,x') - b(s,y,y')| \lesssim \left(|x-y|^{\alpha} + |x'-y'|^{\alpha} + |t-s|^{\beta}\right),$$

  we can then solve the state dependent version of (12).

Examples for SPDEs

- Stochastic heat equation on $[0,1]$, f bounded measurable,

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\begin{aligned}
dY(t,\sigma) &= dW(t) + \kappa \frac{\partial^2}{\partial \sigma^2} Y(t,\sigma)dt \\
&\quad + \int \mathcal{L}_{Y(t)}(dZ) \int_0^1 f(Y(t,\sigma), Z(\sigma'))d\sigma' dt.
\end{aligned} \tag{13}
$$

and its more abstract version, assuming G is Lipschitz in μ_t in total variation,

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\frac{\partial}{\partial t} Y(t) = \frac{\partial^2}{\partial \sigma^2} Y(t)dt + G(t, Y(t), \mu_t)dt + dW(t), \tag{14}
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- For (13), we can recover the $O(k^2/n^2)$ rate of propagation of chaos in relative entropy (Lacker 2021).

Questions

What if the diffusion coefficient also depends on the measure μ?

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- Huang, Ren and Wang, arXiv:2304.07562.

Smoothness of density for McKean-Vlasov SDEs

Given $b : \mathbb{R}^d \to \mathbb{R}^d$ bounded measurable, the SDE

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has a density but is quite irregular, while the McKean-Vlasov SDE

$$dX_t = \langle \mu_t, b(X_t - \cdot)\rangle dt + dW_t, \quad \mathrm{Law}(X_t) = \mu_t \tag{18}$$

has a much smoother density. Possible ways to see this:

- Malliavin calculus: when b is at least $\mathcal{C}^1$, we can fix μ and show μ_t has some Besov regularity via Malliavin calculus. Then bootstrap to prove μ_t has a smooth density.

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- Stochastic taylor expansion (Debussche and Fournier 2013, Debussche and Romito 2014, etc.) when b is less regular, deduce μ_t has some Besov regularity and bootstrap.

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- A more direct approach, better use of Besov space norms with exponent p, get smoothness of density for short time and very irregular b (Hao, Röckner and Zhang, arxiv 2302.04392).

Related smoothing phenomenon

This phenomenon has appeared in other situations such as

- 2D Navier-Stokes equation with (degenerate) additive white noise forcing, [Mattingly and Pardoux 2005].
- The mean field convolution structure is critical for solving

$$dX_t = \langle \mu_t, b(X_t - \cdot) \rangle dt + dW_t, \quad \mathrm{Law}(X_t) = \mu_t \tag{19}$$

for distributional b in the regularity class $b \in \mathcal{C}_b^{-1+\epsilon}$, $\epsilon > 0$. (de Raynal, Jabir, Menozzi arXiv:2205.11866).

Propagation of chaos with a sharp rate

Let $P_t^{n,k}$ be the k-marginal density of a weakly interacting diffusion process with n components,

$$dX_t^{n,i} = \frac{1}{n-1} \sum_{j \neq i} b(X_t^{n,i}, X_t^{n,j}) dt + dW_t^i. \tag{20}$$

Let also μ_t be the law of the (limiting) McKean-Vlasov equation. Then

- It is classically understood that

$$\|P_t^{(n,k)} - \mu_t^{\otimes k}\|_{TV} = O(\sqrt{k/n}). \tag{21}$$

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- In [Jabin-Wang 18] they showed this convergence rate for the vorticity formulation of 2D Navier-Stokes equation on the torus, with sufficiently smooth initial condition.
- If b is Lipschitz continuous or bounded measurable, then [Lacker, 2021] showed that we indeed have a $O(k/n)$ rate for $k << n$, which is sharp in several cases.

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- An unsuccessful approach to get the sharp $O(k/n)$ rate for singular interactions.
- May use modulated free energy instead of relative entropy. In the very recent work [De Courcel, Rosengweig, Serfaty. Arxiv: 2304.05315], they prove uniform-in-time mean-field convergence for singular periodic Riesz flows (s<d on $\mathbb{T}^d$) in the gradient case with a sharp rate in the modulated energy pseudo distance.

What makes the $O(k/n)$ rate so special?

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- Have to assume bounded or Lipschitz interactions.
- Could be relaxed if the flow has more structure (Riesz flow?) and we use modulated free energy instead.

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  - Conditioned McKean-Vlasov equations, etc.
- We explore interesting applications of the coupling method to solution theory of SPDEs.

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- Strong solutions can be proved when the solution is real valued. *"On quasi-linear stochastic partial differential equations", Gyöngy and Pardoux,1993.*

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- Weak uniqueness: X_t is a time changed Brownian motion.

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- Weak uniqueness: X_t is a time changed Brownian motion.
- Strong uniqueness: imposing Sobolev regularity conditions on σ and use PDE theory.

Irregular Noise coefficient

- For multiplicative noise, what can the noise coefficient be?
- Consider

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- Strong uniqueness: imposing Sobolev regularity conditions on σ and use PDE theory.
- None of them works in infinite dimensions.

Notion of solutions to stochastic PDEs

- Random field solutions/ martingale measures approach: JB. Walsh. Regard s and y on an equal footing. $W(A) = \int_A W(dyds)$.

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- Da Prato-Debussche technique, rough path theory, regularity structure, paracontrolled calculus.

$$\partial_t \Phi = \Delta\Phi + C\Phi - \Phi^3 + \xi$$

Only for additive noise or sufficiently regular noise coefficient.

Known results for the stochastic heat equation

- Additive noise case $\partial_t u = \Delta u + f(u) + \frac{\partial^2 W}{\partial_t \partial_x}$.
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  - Random field SPDEs: strong solutions for distributional $f(u)$. *"Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation."*
- Multiplicative, Hölder continuous noise coefficient: σ being $\frac{3}{4} + \epsilon$-Hölder continuous in $X(t,x)$, the random field case:

$$\frac{\partial}{\partial_t} X(t,x) = \frac{1}{2} \Delta X(t,x) dt + \sigma(t,x,X(t,x)) dW(t,x)$$

"Pathwise Uniqueness for Stochastic Heat Equations with Hölder Continuous Coefficients: the White Noise Case", 77 p".

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- B has linear growth.

Well-Posedness

Theorem (Well-posedness of Stochastic Heat equation)

Under the assumptions in the previous slide, there exists a unique (probabilistic weak) mild solution to

$$dX_t = AX_t dt + B(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \in H. \tag{25}$$

Given $\frac{1}{2} + \epsilon$-Hölder $F : H \to H$, unique weak-mild solution to

$$dX_t = AX_t dt + (-A)^{1/2}F(X_t)dt + B(X_t)dt + \sigma(X_t)dW_t, \tag{26}$$

- Examples: Burgers type equations, $\xi \in (0, 2\pi)$

$$du(t,\xi) = \frac{\partial^2}{\partial \xi^2}u(t,\xi)dt + \frac{\partial}{\partial \xi}h(u(t,\xi))dt + \sigma(u(t,\xi))dW_t(\xi).$$

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- Cahn-Hilliard equations in dimensions 1,2,3:

$$du(t,\xi) = -\Delta_\xi^2 u(t,\xi)dt + \Delta_\xi h(u(t,\xi))dt + \sigma(u(t,\xi))dW_t(\xi)$$

Long-time behaviour

Theorem (Exponential ergodicity)

Assume the drift $B : H \to H$ *is Hölder continuous, and the Lyapunov condition hold: for some* $V : H \to \mathbb{R}_+$ *and some* $\lambda \in (0, 1)$ *infinity at infinity,*

$$\mathbb{E}[V(X_t)] \leq \lambda V(X_0) + M \tag{27}$$

for some given $t > 0$ *and* $M > 0$*. Then there exists a unique invariant measure, and the solution converges to the invariant measure exponentially fast with respect to (some specific) Wasserstein distance on* $\mathcal{P}(H)$*.*

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- Lyapunov assumption satisfied when B, F, σ are bounded, and A is a negative operator.

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Some technical challenges in infinite dimensions:

- Derive maximal inequality for the process, when λ is large:

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- Lipschitz approximation in infinite dimensions: compactness of heat semigroup.

A bit more details

Now I outline the procedure of proof:

- Compactness reduction: for any $\epsilon > 0$ find a compact subset $K \subset H$ s.t. X_t stays in K for $t \in [0, T]$, w.p. at least $1 - \epsilon$

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- Proof of ergodic behavior follows similar lines but need an extra argument constructing the Wasserstein distance on H.

Stochastic wave equation: well-posedness

Our method works not only for the parabolic systems, but also for hyperbolic systems.
Consider the (abstract) damped stochastic wave equation

$$\mu \frac{\partial^2 u_\mu(t)}{\partial t^2} = Au_\mu(t) - \frac{\partial u_\mu(t)}{\partial t} + B(t, u_\mu(t)) + G(t, u_\mu(t))dW_t, \quad (28)$$

and the stochastic wave equation without damping term

$$\mu \frac{\partial^2 u_\mu(t)}{\partial t^2} = Au_\mu(t) + B(t, u_\mu(t)) + G(t, u_\mu(t))dW_t, \quad (29)$$

Theorem (Well-posedness of stochastic wave equation)

Under the same assumption on A, B and G as in the case of the stochastic heat equation, there exists a unique weak-mild solution to (28) *and* (29).

Stochastic wave equation: small mass limit

Theorem

Assume moreover that B is Hölder continuous in u_μ. Then as μ tends to 0, the solution to the damped stochastic wave equation

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converges in distribution on path space to the solution of the stochastic heat equation

$$\frac{\partial u(t)}{\partial t} = A u(t) + B(t, u(t)) + G(t, u(t)) dW_t.$$

"On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom", Freidlin and Cerrai, 2006.

Further discussions

Some remaining questions to be addressed:

- We only get weak well-posedness, not strong one. (Strong uniqueness is usually much harder and is now mostly proved for $\mathbb{R}$-valued, random field solutions of SPDEs, not multi-dimensional solutions when coefficients are irregular.)

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A support theorem for random field solutions

We discussed the general setting of Hilbert space valued solutions, but some better estimates hold for random field solutions.

- Consider the parabolic stochastic PDE

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + g(t,x,u) + \sigma(t,x,u)dW(t,x)$$

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- Any clue of weak uniqueness from this estimate?

Thanks