On the (non)-singularity of Kac-Rice densities

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Zeros of Gaussian fields

$f : \mathbb{R} \to \mathbb{R}$ smooth stationary centered Gaussian field.

Correlation function $\kappa : x \mapsto \mathbb{E}[f(0)f(x)]$ s.t. $\kappa(0) = 1$ and $\kappa(x) \xrightarrow[x \to +\infty]{} 0$.

For any $x_1, \ldots, x_m \in \mathbb{R}$ distinct and $k_1, \ldots, k_m \in \mathbb{N}$,

$$\left(f(x_1), \ldots, f^{(k_1)}(x_1), \ldots, f(x_m), \ldots, f^{(k_m)}(x_m)\right)$$

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Linear statistics

Almost surely $f^{-1}(0)$ is locally finite and we denote $\mu = \sum_{x \in f^{-1}(0)} \delta_x$

$$\langle \mu \, , \phi \rangle = \sum_{x \in f^{-1}(0)} \phi(x).$$

Kac–Rice formula for factorial moment measures

Let $p \geqslant 1$, we denote $\underline{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p$ and $\phi^{\otimes p}(\underline{x}) = \prod_{i=1}^{p} \phi(x_i)$.

$\mu^{\otimes p} = \sum_{\underline{x} \in f^{-1}(0)^p} \delta_{\underline{x}}$ random Radon measure on $\mathbb{R}^p$ and

$$\mathbb{E}[\langle \mu \, , \phi \rangle^p] = \left\langle \mathbb{E}\left[\mu^{\otimes p}\right] , \phi^{\otimes p} \right\rangle.$$

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$$\mathbb{E}[\langle \mu\,, \phi \rangle^p] = \left\langle \mathbb{E}\left[\mu^{\otimes p}\right], \phi^{\otimes p} \right\rangle.$$

Denote $\Delta_p = \{\underline{x} \in \mathbb{R}^p \mid \exists i \neq j \text{ s.t. } x_i = x_j\}$ and $\mu^{[p]} = \sum_{\underline{x} \in f^{-1}(0)^p \setminus \Delta_p} \delta_{\underline{x}}$.

Moments of linear statistics can be expressed in terms of $\left(\mathbb{E}\left[\mu^{[p]}\right]\right)_{p \geqslant 1}$.

Kac–Rice formula

$\mathbb{E}\left[\mu^{[p]}\right] = \rho_p(\underline{x})\, \mathrm{d}\underline{x}$, where $\rho_p(\underline{x}) = \dfrac{\mathbb{E}\left[\prod_{i=1}^{p} |f'(x_i)| \;\middle|\; \forall i,\ f(x_i) = 0\right]}{(2\pi)^{\frac{p}{2}} \det \operatorname{Var}\left(f(x_1), \ldots, f(x_p)\right)^{\frac{1}{2}}}$.

Computation of central moments of $\langle \mu , \phi \rangle$

General strategy

Step 1 (local integrability): understand the singularity of ρ_p along Δ_p.

Step 2 (clustering): $\rho_{p+q}(\underline{x}, \underline{y}) = \rho_p(\underline{x})\rho_q(\underline{y}) + \varepsilon_\kappa\left(\min_{i,j}|y_j - x_i|\right).$

Step 3 (combinatorics): write the p-th central moment as the integral of a polynomial in $(\rho_k)_{1\leqslant k\leqslant p}$, and compute.

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Ancona–L., 19: N_d number of real roots of a degree d Kostlan polynomial

$$\mathbb{E}\left[\left(\frac{N_d - \mathbb{E}[N_d]}{\mathrm{Var}(N_d)^{\frac{1}{2}}}\right)^p\right] \xrightarrow[d\to+\infty]{} \mathbb{E}[\mathcal{N}(0,1)^p].$$

Ancona–L., 20: $N_R = \mathrm{card}\left(f^{-1}(0) \cap [0, R]\right) = \langle \mu , \mathbf{1}_{[0,R]} \rangle$ as $R \to +\infty$, same result if $\kappa^{(k)}(x) = o\left(x^{-4p}\right)$ for all $0 \leqslant k \leqslant p$.

Gass, 21: Same result for N_R if $\kappa^{(k)} \in L^2(\mathbb{R})$ for all $0 \leqslant k \leqslant 2p$.

A conceptual proof that ρ_p is continuous

$F(\underline{x}) = (f(x_1), \ldots, f(x_p)) = \mathrm{ev}_{\underline{x}}(f)$ smooth Gaussian field on $\mathbb{R}^p$.

Non-degenerate on $\mathbb{R}^p \setminus \Delta_p$ and $\qquad \mu^{[p]} = \sum_{\underline{x} \in F^{-1}(0) \setminus \Delta_p} \delta_{\underline{x}}$.

Kac–Rice for $\mu^{[p]}$ is Kac–Rice for the expectation applied to $F_{|\mathbb{R}^p \setminus \Delta_p}$ and:

$$\rho_p(\underline{x}) = \frac{\mathbb{E}\left[|\det(D_{\underline{x}} F)| \;\middle|\; F(\underline{x}) = 0\right]}{(2\pi)^{\frac{p}{2}} \det \mathrm{Var}(F(\underline{x}))^{\frac{1}{2}}}.$$

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Main point

- We can replace F by *any* smooth Gaussian field G such that $F^{-1}(0) \setminus \Delta_p \overset{\text{law}}{=} G^{-1}(0) \setminus \Delta_p$.
- If G is non-degenerate on $\mathbb{R}^p$ then ρ_p extends continuously to $\mathbb{R}^p$.

Hermite interpolation and multi-jets

We can build a natural G that "looks like":

$$G(\overbrace{x_1,\ldots,x_1}^{(k_1+1)\text{ times}},\ldots,\overbrace{x_m,\ldots,x_m}^{(k_m+1)\text{ times}}) = \underbrace{\left(f(x_1),\ldots,f^{(k_1)}(x_1),\ldots,f(x_m),\ldots,f^{(k_m)}(x_m)\right)}_{\text{"multi-jet of } f \text{ at } \underline{x}\text{"}}.$$

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Hermite polynomial of f at $\underline{x} = (x_1,\dots,x_1,\dots,x_m,\dots,x_m)$

There exists a unique $P \in \mathbb{R}_{p-1}[X]$ s.t. $(P^{(k)}(x_i))_{\substack{1\leqslant i\leqslant m\\0\leqslant k\leqslant k_i}} = (f^{(k)}(x_i))_{\substack{1\leqslant i\leqslant m\\0\leqslant k\leqslant k_i}}$.

Denote this polynomial by $j(f,\underline{x})$.

Hermite interpolation and multi-jets

- $j(\cdot, \underline{x})$ is linear $\rightsquigarrow$ $j(f, \underline{x})$ is Gaussian, and non-degenerate.
- $j(f, \cdot)$ is smooth $\rightsquigarrow$ $j(f, \cdot) : \mathbb{R}^p \to \mathbb{R}_{p-1}[X]$ smooth Gaussian field.
- $\forall \underline{x} \notin \Delta_p$, $j(f, \underline{x}) = 0 \iff F(\underline{x}) = 0$ $\rightsquigarrow$ same zeros as F on $\mathbb{R}^p \setminus \Delta_p$.

Replacing F by $G = j(f, \cdot)$, we immediatly obtain that $\rho_p \in \mathcal{C}^0(\mathbb{R}^p)$.

A geometric perspective

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}) & \xrightarrow{j(\cdot,\underline{x})} & \mathbb{R}_{p-1}[X] \simeq \mathcal{J}_{\underline{x}} \\ & \searrow^{\mathrm{ev}_{\underline{x}}} & \downarrow \mathrm{ev}_{\underline{x}} \\ & & \mathbb{R}^p \end{array}$$

$$\mathcal{J}_{\underline{x}} := \mathbb{C}^\infty(\mathbb{R}) / \ker j(\cdot, \underline{x})$$

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As vector bundles over $\mathbb{R}^p$, $\mathcal{C}^\infty(\mathbb{R}) \times \mathbb{R}^p \xrightarrow{j} \mathbb{R}_{p-1}[X] \times \mathbb{R}^p \simeq \mathcal{J}$.

$j(f, \cdot)$ is a section of $\mathcal{J} \to \mathbb{R}^p$.

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Take home message

- Building a multi-jet bundle $\mathcal{J} \to \mathbb{R}^p$ and a smooth bundle map j sending $(f, \underline{x})$ to its multi-jet is enough to prove $\rho_p \in \mathcal{C}^0(\mathbb{R}^p)$.
- This is done using polynomial interpolation.

Can we do the same for fields on $\mathbb{R}^n$?

$f : \mathbb{R}^n \to \mathbb{R}$ nice Gaussian field and $\langle \mu , \phi \rangle = \int_{f^{-1}(0)} \phi \, \mathrm{d}\mathcal{H}^{n-1}$.

Kac–Rice formula

$$\mathbb{E}\left[\mu^{[p]}\right] = \rho_p(\underline{x})\,\mathrm{d}\underline{x}, \quad \text{where} \quad \rho_p(\underline{x}) = \frac{\mathbb{E}\left[\prod_{i=1}^{p} \|D_{x_i} f\| \,\middle|\, \forall i,\ f(x_i) = 0\right]}{(2\pi)^{\frac{p}{2}} \det \mathrm{Var}\big(f(x_1), \dots, f(x_p)\big)^{\frac{1}{2}}}.$$

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Bad news

- Polynomial interpolation is ill-posed in $\mathbb{R}^n$.
- $\rho_2(x, y) \propto \|y - x\|^{2-n}$ as $y \to x$. Can hope that ρ_p is L^1_{loc} but not $\mathcal{C}^0$.

Yes, we can!

Theorem (Ancona–L.)

For any ambient dimension $n \geqslant 1$ and any $p \geqslant 1$ we have $\rho_p \in L^1_{loc}((\mathbb{R}^n)^p)$.

Corollary

The linear statistics of μ have finite moments of all order.

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Theorem (Ancona–L.)

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Corollary

The linear statistics of μ have finite moments of all order.

- Recovers a result by Armentano–Azaïs–Ginsbourger–Leòn.
- Works for $f : \mathbb{R}^n \to \mathbb{R}^{n-d}$, proved independently by Gass–Stecconi.
- Works on manifolds.

Divided differences in $\mathbb{R}^n$

$\sigma_k = \left\{ \underline{t} \in [0,1]^{k+1} \;\middle|\; \sum_{i=0}^k t_i = 1 \right\}$ and ν_k its Lebesgue measure.

Let $\underline{x} = (x_0, \dots, x_k) \in (\mathbb{R}^n)^{k+1}$ we set $\sigma(\underline{x}) = \left\{ \sum_{i=0}^k t_i x_i \;\middle|\; \underline{t} \in \sigma_k \right\}$ and

$$\int_{\sigma(\underline{x})} \phi(y) \, \mathrm{d}\nu(y) := \int_{\sigma_k} \phi\left(\sum_{i=0}^k t_i x_i \right) \mathrm{d}\nu_k(\underline{t}).$$

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Definition

If $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, we define $f[x_0, \ldots, x_k] := \int_{\sigma(\underline{x})} D_y^k f \;\mathrm{d}\nu(y) \in \mathrm{Sym}^k(\mathbb{R}^n)$.

- $f[x_0, \ldots, x_k]$ linear w.r.t. f;
- It is smooth and symmetric w.r.t. $\underline{x}$;
- If $k = 0$, then $f[x] = f(x)$.

Kergin interpolation

Kergin polynomial of $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ at $\underline{x} \in (\mathbb{R}^n)^p$

There exists a unique $K(f, \underline{x}) \in \mathbb{R}_{p-1}[X]$ s.t $f[(x_i)_{i \in I}] = K(f, \underline{x})[(x_i)_{i \in I}]$ for all non-empty $I \subset \{1, \dots p\}$.

$$K(f, \underline{x}) = \sum_{i=1}^{p} f[x_1, \dots, x_i](X - x_1, \dots, X - x_{i-1})$$

In particular, $f(x_i) = K(f, \underline{x})(x_i)$ for any $i \in \{1, \dots, p\}$.

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The multi-jet of f at $\underline{x} \notin \Delta_p$ is defined by p linear conditions.

$\mathbb{R}_{p-1}[X]$ is too large to be the space of multi-jets at $\underline{x}$.

Multi-jet bundle over $(\mathbb{R}^n)^p \setminus \Delta_p$

Let $\underline{x} \notin \Delta_p$, then $\mathrm{ev}_{\underline{x}} : \mathbb{R}_{p-1}[X] \to \mathbb{R}^p$ is surjective.

Denote $\mathcal{G}(\underline{x}) = \ker \mathrm{ev}_{\underline{x}} \in \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$, then

$$(f(x_1), \ldots, f(x_p)) = (g(x_1), \ldots, g(x_p)) \iff K(f, \underline{x}) - K(g, \underline{x}) \in \mathcal{G}(\underline{x}).$$

$$\begin{array}{c} \mathcal{C}^\infty(\mathbb{R}^n) \times (\mathbb{R}^n)^p \setminus \Delta_p \\ \Big\downarrow K \\ \mathbb{R}_{p-1}[X] \times (\mathbb{R}^n)^p \setminus \Delta_p \\ \Big\downarrow \mathrm{ev} \\ \mathbb{R}^p \times (\mathbb{R}^n)^p \setminus \Delta_p \end{array}$$

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$$\begin{array}{ccccc} & & \mathcal{C}^\infty(\mathbb{R}^n) \times (\mathbb{R}^n)^p \setminus \Delta_p & & \\ & & \Big\downarrow K & & \\ \ker(\mathrm{ev}) = \mathcal{G} & \hookrightarrow & \mathbb{R}_{p-1}[X] \times (\mathbb{R}^n)^p \setminus \Delta_p & \longrightarrow & \mathcal{J} \\ & & \Big\downarrow \mathrm{ev} & & \\ & & \mathbb{R}^p \times (\mathbb{R}^n)^p \setminus \Delta_p & & \end{array}$$

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$$\mathcal{J}_{\underline{x}} = \mathbb{R}_{p-1}[X]/\mathcal{G}(\underline{x}) \qquad \text{and} \qquad j(f, \underline{x}) = K(f, \underline{x}) \mod \mathcal{G}(\underline{x}).$$

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Does this extend over $(\mathbb{R}^n)^p$?

More bad news

Can we extend $\mathcal{G}$ as a smooth map on $(\mathbb{R}^n)^p$?

No, we can't!

Take $n = p = 2$, $x = (0,0)$ and $y = R(\cos(\theta), \sin(\theta))$ then

$$\mathcal{G}(0,y) = \mathrm{Span}\big(X_1 \sin(\theta) - X_2 \cos(\theta)\big) \subset \mathbb{R}_1[X_1, X_2]$$

and $\mathcal{G}(0,\cdot) : \mathbb{R}^2 \setminus \{0\} \to \mathrm{Gr}_2(\mathbb{R}_1[X])$ does not extend $\mathcal{C}^0$ to $\mathbb{R}^2$.

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But $(R,u) \mapsto \mathcal{G}(0,Ru)$ extends $\mathcal{C}^0$ from $(0,+\infty) \times \mathbb{S}^1$ to $[0,+\infty) \times \mathbb{S}^1$.

That is $\mathcal{G}(0,\cdot)$ extends to $\mathrm{Bl}_0(\mathbb{R}^2) \simeq \big(\mathbb{R}^2 \setminus \{0\}\big) \sqcup \mathbb{S}^1$.

“Compactification” of configuration spaces

We need to build a smooth manifold $C_p[\mathbb{R}^n]$ such that:

- $C_p[\mathbb{R}^n]$ contains $(\mathbb{R}^n)^p \setminus \Delta_p$ as a dense open subset;
- $\mathcal{G}$ extends smoothly to $C_p[\mathbb{R}^n]$.

Example

For $p = 2$ and $n \geqslant 2$, $C_2[\mathbb{R}^n] = \mathrm{Bl}_{\Delta_2}((\mathbb{R}^n)^2)$ works.

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Things that we tried and don't work

- Olver multispaces;
- Compactifications by Fulton–MacPherson, Axelrod–Singer, Ulyanov;
- Hilbert schemes;
- Various sequences of blow-ups;
- An explicit construction with nice global coordinates.

Construction of $C_p[\mathbb{R}^n]$

Recall that $\mathcal{G} : (\mathbb{R}^n)^p \setminus \Delta_p \to \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ is smooth.

$$(\mathbb{R}^n)^p \setminus \Delta_p \simeq \{(x, \mathcal{G}(\underline{x})) \mid \underline{x} \in (\mathbb{R}^n)^p \setminus \Delta_p\} =: \Sigma.$$

$\mathcal{G}$ extends to $\overline{\Sigma} \subset (\mathbb{R}^n)^p \times \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ by projecting onto $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$.

Problem

$\overline{\Sigma}$ is not smooth unless $p = 1$ or $n = 1$.

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Recall that $\mathcal{G} : (\mathbb{R}^n)^p \setminus \Delta_p \to \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ is smooth.

$$(\mathbb{R}^n)^p \setminus \Delta_p \simeq \{(x, \mathcal{G}(\underline{x})) \mid \underline{x} \in (\mathbb{R}^n)^p \setminus \Delta_p\} =: \Sigma.$$

$\mathcal{G}$ extends to $\overline{\Sigma} \subset (\mathbb{R}^n)^p \times \mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$ by projecting onto $\mathrm{Gr}_p(\mathbb{R}_{p-1}[X])$.

Problem

$\overline{\Sigma}$ is not smooth unless $p = 1$ or $n = 1$.

Theorem (Hironaka, 1964)

$\overline{\Sigma}$ *admits a resolution of singularities, obtained by a sequence of blow-ups.*

Any such resolution $C_p[\mathbb{R}^n] \twoheadrightarrow \overline{\Sigma}$ solves our problem.

The end

Thank you for your attention.