Constrained in law BSDE and associated particle system

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Mean Field Models - Rennes

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Rappel du plan

1. BSDEs and Toy model
    - Backward Stochastic Differential Equations
    - Propagation of Chaos
2. Constrained in Law BSDE
    - Setting of the problem
    - Uniqueness
    - Existence of the solution
3. Associated particle system
    - Well-posedness
    - Propagation of Chaos

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad t \in [0, T], \tag{1}$$

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where $(Y_t, Z_t) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}$, W is a d-dimensional Brownian motion, and ξ and f are parameters.

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Assumptions

The terminal condition ξ belongs to L^2 and the generator f is uniformly in time Lipschitz in the space variables (y, z) and satisfies

$$\mathbb{E}\left[\int_0^T |f(s, 0, 0)|^2\,\mathrm{d}s\right] < +\infty.$$

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where $(Y_t, Z_t) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$, W is a d-dimensional Brownian motion, and ξ and f are parameters.

Theorem (Pardoux, Peng (1990))

There exists a unique solution (Y, Z) in $\mathcal{S}^2 \times \mathcal{H}^2$ to equation (1),

$$\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right] < +\infty, \quad \|Z\|^2_{\mathcal{H}^2} = \mathbb{E}\left[\int_0^T |Z_t|^2\,\mathrm{d}t\right] < +\infty.$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, Z_s^0)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s - \int_t^T Z_s^0\,\mathrm{d}W_s^0 \qquad (2)$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, Z_s^0, \mu_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s - \int_t^T Z_s^0\,\mathrm{d}W_s^0 \quad (2)$$

where $\mu_s = \mathcal{L}^1(Y_s)$.

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On $\Omega^0 \times \Omega^1$, take W^0, W Brownian motions on Ω^0 and Ω^1 respectively.

$$\mathcal{L}^1(X) : \omega^0 \in \Omega^0 \mapsto \mathcal{L}\left(X(\omega^0, \cdot)\right)$$

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The terminal condition ξ belongs to L^2 and the generator f is uniformly in time Lipschitz in the space variables (y, z) and satisfies

$$\mathbb{E}\left[\int_0^T |f(s, 0, 0, 0, \delta_0)|^2\,\mathrm{d}s\right] < +\infty.$$

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where $\mu_s = \mathcal{L}^1(Y_s)$.

Theorem

There exists a unique solution (Y, Z, Z^0) *in* $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$ *to* (2),

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2\right] < +\infty, \quad \|Z\|_{\mathcal{H}^2}^2 = \mathbb{E}\left[\int_0^T |Z_t|^2\,\mathrm{d}t\right] < +\infty.$$

Proposition

For $p \geq 2$, assume furthermore that

$$\xi \in L^p \quad \text{and} \quad \mathbb{E}\left[\int_0^T |f(s,0,0,0,\delta_0)|^p \, \mathrm{d}s\right] < +\infty.$$

Then, the solution Y belongs to $\mathcal{S}^p$, that is

$$\mathbb{E}\left[\sup_{t \leq T} |Y_t|^p\right] < +\infty.$$

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$$\mathrm{d}Y_t^{i,N} = f(t, Y_t^{i,N}, Z_t^{i,i,N}, Z_t^{0,i,N}, \mu_t^N)\,\mathrm{d}s - \sum_{k=1}^{N} Z_t^{i,k,N}\,\mathrm{d}W_t^k - Z_t^{0,i,N}\,\mathrm{d}W_t^0,$$

with $Y_T^{i,N} = \xi^i$, conditionally to $\mathcal{F}^0$ i.i.d.

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Under the same conditions, there exists a unique solution to the above system.

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Lemma

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|Y_t^{i,N}|^2 + \frac{1}{N}\sum_{i,k=1}^{N}\int_0^T |Z_s^{i,k,N}|^2\,\mathrm{d}s + \frac{1}{N}\sum_{i=1}^{N}\int_0^T |Z_s^{0,i,N}|^2\,\mathrm{d}s\right] \le C_T.$$

Theorem (A first result of conditional propagation of chaos)

For $p > 4$, assume furthermore that $\xi \in L^p$ and $(f(s,0,0,0,\delta_0)) \in \mathcal{H}^p$. Then there exists a constant C depending only on m, p, T such that

$$\mathbb{E}\left[\sup_{t\le T}\mathbb{E}^1\left[W_2^2(\mu_t^N,\mu_t)\right]\right] \le C\,\varepsilon_N = C \times \begin{cases} N^{-1/2} & \text{if } m<4, \\ N^{-1/2}\log(N) & \text{if } m=4, \\ N^{-2/m} & \text{if } m>4. \end{cases}$$

Theorem (A second result of conditional propagation of chaos)

For $p > 4$, assume that $\xi \in L^p$, $(f(s,0,0,0,\delta_0)) \in \mathcal{H}^p$ and

$$\operatorname*{ess\,sup}_{t\leq T} \mathbb{E}\left[|\tilde{Z}_t|^p + |\tilde{Z}_t^0|^p\right] < +\infty.$$

Then there exists a constant C depending only on m, p, T such that

$$\mathbb{E}\left[\sup_{s\leq T} W_2^2(\mu_s^N, \mu_s)\right] \leq C \times \begin{cases} N^{-1/2+2/p} & \text{if } m < 4, \\ N^{-1/2+2/p}\log(1+N)^{1-4/p} & \text{if } m = 4, \\ N^{-2(1-4/p)/m} & \text{if } m > 4. \end{cases}$$

$$\tilde{Y}_t^i = \xi^i + \int_t^T f\left(s, \tilde{Y}_s^i, \tilde{Z}_s^i, \tilde{Z}_s^{0,i}, \mathcal{L}^1(Y_s)\right) \mathrm{d}s - \int_t^T \tilde{Z}_s^i \,\mathrm{d}W_s^i - \int_t^T \tilde{Z}_s^{0,i} \,\mathrm{d}W_s^0$$

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Proposition

$$\mathbb{E}\left[\sup_{t\leq T}\left\{|Y_t^{i,N} - \tilde{Y}_t^i|^2 + \int_t^T |Z_s^{0,i,N} - \tilde{Z}_s^{0,i}|^2 \,\mathrm{d}s + \int_t^T \sum_{k=1}^N |Z_s^{i,k,N} - \tilde{Z}_s^i \delta_{i,k}|^2 \,\mathrm{d}s\right\}\right] \leq C_T \,\mathbb{E}\left[\sup_{s\leq T} W_2^2(\mu_s^N, \mu_s)\right]$$

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$$H(\mu_t) \geq 0, \quad t \leq T \quad \text{and} \quad \int_0^T H(\mu_s)\,\mathrm{d}K_s = 0 \tag{3}$$

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- $(Y_t, Z_t, Z_t^0) \in \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$
- W and W^0 are independent d-dimensional Brownian motions
- $\mu_t = \mathcal{L}^1(Y_t)$ is the conditional law of Y w.r.t. W^0
- $H : \mathcal{P}(\mathbb{R}^m) \to \mathbb{R}$ is the constraint function
- K is the reflection process, non-decreasing and $\mathcal{F}^0$-adapted.

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- K is the reflection process, non-decreasing and $\mathcal{F}^0$-adapted.

A solution to the problem above is a tuple (Y, Z, Z^0, K).

Assumptions

(i) $f(\cdot, 0, 0, 0, \delta_0)$ belongs to $\mathcal{H}^2$, and

$$|f(t, y, z, \tilde{z}, \mu) - f(t, y', z', \tilde{z}', \nu)| \leq C_f \left(|y - y'| + |z - z'| + |\tilde{z} - \tilde{z}'| + W_2(\mu, \nu) \right)$$

(ii) The terminal value ξ is $\mathcal{F}_T$-measurable, in L^2 and $H(\mathcal{L}^1(\xi)) \geq 0$.

(iii) The function H is fully $\mathcal{C}^2$ and

$$M_2(H) = \sup_{\mu \in \mathcal{P}_2(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_\mu H(\mu)(x)|^2 \, \mathrm{d}\mu(x) < +\infty.$$

Assumptions

(iv) $D_\mu H$ is Lipschitz: there exists $C > 0$ such that for all X, Y in L^2

$$\mathbb{E}\left[\left|D_\mu H(\mu^X)(X) - D_\mu H(\mu^Y)(Y)\right|^2\right] \le C\,\mathbb{E}\left[|X - Y|^2\right].$$

and there exists $\beta > 0$ satisfying for all μ in $\mathcal{P}_2(\mathbb{R}^m)$,

$$H(\mu) \le 0 \implies \int_{\mathbb{R}^m} |D_\mu H(\mu)(x)|^2 \,\mathrm{d}\mu(x) \ge \beta^2.$$

(v) H is concave: for X, Y in L^2 with respective laws μ^X and μ^Y

$$H(\mu^Y) - H(\mu^X) - \mathbb{E}\left[D_\mu H(\mu^X)(X) \cdot (X - Y)\right] \le 0.$$

Furthermore, we require H to be bounded above on $\mathcal{P}_2(\mathbb{R}^m)$.

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Theorem

Under the previous set of assumptions, there exists at most one tuple (Y, Z, Z^0, K) satisfying (3) such that K is continuous, non-decreasing, starting from $K_0 = 0$ and $\mathcal{F}^0$-adapted and for all t in $[0, T]$,

$$\mathbb{E}\left[|Y_t|^2 + \int_0^T |Z_s|^2 \,\mathrm{d}s + \int_0^T |Z_s^0|^2 \,\mathrm{d}s\right] < +\infty.$$

Sketch of the proof

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$$
\begin{aligned}
e^{\alpha t}|\hat{Y}_t|^2 = & \int_t^T \Big(-\alpha e^{\alpha s}|\hat{Y}_s|^2 + 2e^{\alpha s}\hat{Y}_s \cdot \Big(f(s, Y_s, Z_s, Z^0_s, \mu_s) - f(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{Z}^0_s, \tilde{\mu}_s)\Big)\Big)\,\mathrm{d}s \\
& - 2\int_t^T e^{\alpha s}\hat{Y}_s \cdot \hat{Z}_s\,\mathrm{d}W_s - 2\int_t^T e^{\alpha s}\hat{Y}_s \cdot \hat{Z}^0_s\,\mathrm{d}W^0_s \\
& - \int_t^T e^{\alpha s}|\hat{Z}_s|^2\,\mathrm{d}s - \int_t^T e^{\alpha s}|\hat{Z}^0_s|^2\,\mathrm{d}s \\
& + 2\int_t^T e^{\alpha s}\hat{Y}_s \cdot \Big(D_\mu H(\mu_s)(Y_s)\,\mathrm{d}K_s - D_\mu H(\tilde{\mu}_s)(\tilde{Y}_s)\,\mathrm{d}\tilde{K}_s\Big)\,.
\end{aligned}
$$

$$\mathbb{E}\left[e^{\alpha t}|\hat{Y}_t|^2 + \frac{1}{2}\int_t^T e^{\alpha s}\left(|\hat{Z}_s|^2 + |\hat{Z}^0_s|^2\right)\,\mathrm{d}s\right] \leq 0$$

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$$\int_s^t \mathbb{E}^1\left[|D_\mu H(\mu_u)(Y_u)|^2\right]\,\mathrm{d}K_u = \int_s^t \mathbb{E}^1\left[|D_\mu H(\mu_u)(Y_u)|^2\right]\,\mathrm{d}\tilde{K}_u.$$

$$\mathbb{E}\left[e^{\alpha t}|\hat{Y}_t|^2 + \frac{1}{2}\int_t^T e^{\alpha s}\left(|\hat{Z}_s|^2 + |\hat{Z}_s^0|^2\right)\,\mathrm{d}s\right] \le 0$$

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And for $\mathrm{d}K + \mathrm{d}\tilde{K}$-almost every u:

$$\mathbb{E}^1\left[|D_\mu H(\mu_u)(Y_u)|^2\right] \ge \beta^2 > 0, \quad a.s.$$

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Theorem

Under the previous set of assumptions, there exists a unique tuple (Y, Z, Z^0, K) satisfying (3) such that K is continuous, non-decreasing, starting from $K_0 = 0$ and $\mathcal{F}^0$-adapted and for all t in $[0, T]$,

$$\mathbb{E}\left[|Y_t|^2 + \int_0^T |Z_s|^2 \, \mathrm{d}s + \int_0^T |Z_s^0|^2 \, \mathrm{d}s\right] < +\infty.$$

Step 1 : Existence via a penalization scheme for a bounded and space independent generator

$$|f(s,y,z,z^0,\mu)| = |f(s)| \leq \kappa.$$

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$$Y_t^k = \xi + \int_t^T f(s)\,\mathrm{d}s - \int_t^T Z_s^k\,\mathrm{d}W_s - \int_t^T Z_s^{0,k}\,\mathrm{d}W_s^0 + \int_t^T D_\mu H(\mu_s^k)(Y_s^k)\,\mathrm{d}K_s^k,$$

where $\mu_s^k = \mathcal{L}^1(Y_s^k)$, $\mathrm{d}K_s^k = \psi_k(H(\mu_s^k))\,\mathrm{d}s$ and ψ_k of the form

$$\psi_k(x) = r \text{ if } x \leq -1/k,\ \ \psi_k(x) = -krx \text{ if } -1/k \leq x \leq 0,\ \ \psi_k(x) = 0 \text{ else}$$

→ $(Y^k, Z^k, Z^{0,k})$ defines a Cauchy sequence in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$

$$\mathbb{E}\left[\sup_{0\le t\le T} |\hat{Y}_t|^2 + \int_0^T \left(|\hat{Z}_s|^2 + |\hat{Z}^0_s|^2\right)\,\mathrm{d}s\right] \le C\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{l}}\right).$$

→ Deduce the uniform convergence of (K^k) with the one of (L^k) in $\mathcal{S}^2$:

$$L^k_t = \int_0^t D_\mu H(\mu^k_s)(Y^k_s)\,\mathrm{d}K^k_s$$

→ Check that the limit (Y, Z, Z^0, K) satisfies equation (3).

Step 2 : Existence via truncation for a $\mathcal{H}^2$ space independent generator

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$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^m - Y_t^l|^2 + \int_0^T(|Z_s^m - Z_s^l|^2 + |Z_s^{0,m} - Z_s^{0,l}|^2)\,\mathrm{d}s\right] \le C_T\,\mathbb{E}\left[\int_t^T \left|f(s)\mathbf{1}_{|f(s)|\le l} - f(s)\mathbf{1}_{|f(s)|\le m}\right|^2\,\mathrm{d}s\right]^{1/2}.$$

→ $(Y^k, Z^k, Z^{0,k})$ defines a Cauchy sequence in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$

→ Deduce the uniform convergence of (K^k) with the one of (L^k) in $\mathcal{S}^2$:

$$L_t^k = \int_0^t D_\mu H(\mu_s^k)(Y_s^k)\,\mathrm{d}K_s^k$$

now using the fact that $\sup_{m \geq 1} \mathbb{E}\left[(K_T^m)^2\right] < +\infty$.

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$$H(\mu_t^m) \geq 0, \quad t \in [0,T], \quad \int_0^T H(\mu_s^m)\,\mathrm{d}K_s^m = 0,$$

We can show that:

$$\mathbb{E}\left[\sup_{t\leq T}\left\{e^{\alpha t}|\hat{Y}_t^{m+1}|^2 + \int_t^T e^{\alpha s}\left(|\hat{Z}_s^{m+1}|^2 + |\hat{Z}_s^{0,m+1}|^2\right)\,\mathrm{d}s\right\}\right] \leq c_T\,\mathbb{E}\left[\int_0^T e^{\alpha s}\left(|\hat{Y}_s^m|^2 + |\hat{Z}_s^m|^2 + |\hat{Z}_s^{0,m}|^2\right)\,\mathrm{d}s\right]^{1/2}.$$

→ $(Y^k, Z^k, Z^{0,k})$ defines a Cauchy sequence in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2$

→ Deduce the uniform convergence of (K^k) with the one of (L^k) in $\mathcal{S}^2$:

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now using the fact that $\sup_{m \geq 1} \mathbb{E}\left[(K_T^m)^2\right] < +\infty$.

→ Check that the limit (Y, Z, Z^0, K) satisfies equation (3).

Proposition

For $p \geq 2$, assume that $\xi \in L^p$, that $f(\cdot, 0, 0, 0, \delta_0) \in \mathcal{H}^p$ and that

$$\sup_{\mu \in \mathcal{P}^p(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_\mu H(\mu)(x)|^p \, \mathrm{d}\mu(x) < +\infty.$$

Then, the solution Y belongs to $\mathcal{S}^p$, that is

$$\mathbb{E}\left[\sup_{t \leq T} |Y_t|^p\right] < +\infty.$$

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$$Y_t^i = \xi^i + \int_t^T f\left(s, Y_s^i, Z_s^{i,i}, Z_s^{0,i}, \mu_s^N\right) \mathrm{d}s - \int_t^T \sum_{j=1}^N Z_s^{i,j} \,\mathrm{d}W_s^j - \int_t^T Z_s^{0,i} \,\mathrm{d}W_s^0 + \int_t^T D_\mu H(\mu_s^N)(Y_s^i) \,\mathrm{d}K_s^N, \quad (4)$$

$$H(\mu_t^N) \geq 0, \quad \forall t \leq T \quad \text{and} \quad \int_0^T H(\mu_s^N) \,\mathrm{d}K_s^N = 0.$$

Lemma

Take N copies (ξ^i) of ξ and denote $\mu_T^N = N^{-1}\sum \delta_{\xi^i}$. If $\xi \in L^{2+\varepsilon}$, there exist a constant C and a family of random variables $\tilde{\xi}^i \in L^{2+\varepsilon}$ such that

$$H\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{\xi}^i}\right) \geq 0$$

and

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left|\tilde{\xi}^i - \xi^i\right|^2\right] \leq C\mathbb{E}\left[W_2^2\left(\mu_T^N, \mathcal{L}(\xi)\right)\right]^{\frac{\varepsilon}{2+\varepsilon}}.$$

$$Y_t^i = \tilde{\xi}^i + \int_t^T f\left(s, Y_s^i, Z_s^{i,i}, Z_s^{0,i}, \mu_s^N\right) \mathrm{d}s - \int_t^T \sum_{j=1}^N Z_s^{i,j} \,\mathrm{d}W_s^j$$
$$- \int_t^T Z_s^{0,i} \,\mathrm{d}W_s^0 + \int_t^T D_\mu H(\mu_s^N)(Y_s^i) \,\mathrm{d}K_s^N, \quad (4)$$
$$H(\mu_t^N) \geq 0, \quad \forall t \leq T \quad \text{and} \quad \int_0^T H(\mu_s^N) \,\mathrm{d}K_s^N = 0.$$

The system (4) is a reflected BSDE in $\mathbb{R}^{mN}$ constrained to stay in the following convex space

$$\mathcal{D} = \left\{ x = (x_1, \dots, x_N) \in (\mathbb{R}^m)^N \mid H\left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \right) \geq 0 \right\},$$

with normal vector proportional to $(D_\mu H(\mu_x^N)(x_1), \dots, D_\mu H(\mu_x^N)(x_N))$ for $x = (x_1, \dots, x_N) \in \partial\mathcal{D}$.

Proposition

Under the same assumptions, the system (4) is well-posed: there exists a unique solution $\{(Y^i, Z^i, Z^{0,i})_{1\leq i\leq N}, K^N\}$ to (4).

Proposition

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There exists a constant $C > 0$ such that this solution satisfies

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(|Y_t^i|^2 + \int_0^T \sum_{j=1}^{N} |Z_s^{i,j}|^2 \,\mathrm{d}s + \int_0^T |Z_s^{0,i}|^2 \,\mathrm{d}s\right)\right] + \mathbb{E}\left[\left(K_T^N\right)^2\right] \leq C.$$

where the constant C only depends on f and H.

Rappel du plan

1. BSDEs and Toy model
   - Backward Stochastic Differential Equations
   - Propagation of Chaos
2. Constrained in Law BSDE
   - Setting of the problem
   - Uniqueness
   - Existence of the solution
3. Associated particle system
   - Well-posedness
   - Propagation of Chaos

Theorem

For $p > 4$, assume that $\xi \in L^p$, that $f(\cdot, 0, 0, 0, \delta_0) \in \mathcal{H}^p$ and that

$$\sup_{\mu \in \mathcal{P}^p(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_\mu H(\mu)(x)|^p \, \mathrm{d}\mu(x) < +\infty.$$

Then,

$$\mathbb{E}\left[\sup_{s \leq T} \mathbb{E}^1\left[W_2^2(\mu_s^N, \mu_s)\right]\right] \leq C_T \times \begin{cases} N^{-1/2} & \text{if } m < 4, \\ N^{-1/2}\log(N) & \text{if } m = 4, \\ N^{-2/m} & \text{if } m > 4. \end{cases}$$

Theorem

For $p > 4$, assume that $\xi \in L^p$, that $f(\cdot, 0, 0, 0, \delta_0) \in \mathcal{H}^p$ and that

$$\sup_{\mu \in \mathcal{P}^p(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_\mu H(\mu)(x)|^p \, \mathrm{d}\mu(x) < +\infty.$$

If we also suppose that $\operatorname{ess\,sup}_t \mathbb{E}[|Z_t|^p + |Z_t^0|^p] < +\infty$, then there exists a constant $C_T > 0$ such that

$$\mathbb{E}\left[\sup_{s \leq T} W_2^2(\mu_s^N, \mu_s)\right] \leq C_T \times \begin{cases} N^{-1/2+2/p} & \text{if } m < 4, \\ N^{-1/2+2/p} \log(1+N)^{1-4/p} & \text{if } m = 4, \\ N^{-2(1-4/p)/m} & \text{if } m > 4. \end{cases}$$

$$\mathscr{Y}_t^i = \xi^i + \int_t^T f\left(s, \mathscr{Y}_s^i, \mathscr{Z}_s^{i,i}, \mathscr{Z}_s^{0,i}, \mu_s\right) \mathrm{d}s - \int_t^T \mathscr{Z}_s^i \,\mathrm{d}W_s^i$$

$$- \int_t^T \mathscr{Z}_s^{0,i} \,\mathrm{d}W_s^0 + \int_t^T D_\mu H(\mathscr{Y}_s^i)(\mu_s) \,\mathrm{d}K_s$$

$$H(\mu_t) \geq 0, \quad t \leq T \quad \text{and} \quad \int_0^T H(\mu_s) \,\mathrm{d}K_s = 0.$$

$$\mathscr{Y}_t^i = \xi^i + \int_t^T f\left(s, \mathscr{Y}_s^i, \mathscr{Z}_s^{i,i}, \mathscr{Z}_s^{0,i}, \mu_s\right) \mathrm{d}s - \int_t^T \mathscr{Z}_s^i \,\mathrm{d}W_s^i - \int_t^T \mathscr{Z}_s^{0,i} \,\mathrm{d}W_s^0 + \int_t^T D_\mu H(\mathscr{Y}_s^i)(\mu_s)\,\mathrm{d}K_s$$

$$H(\mu_t) \geq 0, \quad t \leq T \quad \text{and} \quad \int_0^T H(\mu_s)\,\mathrm{d}K_s = 0.$$

Proposition

$$\mathbb{E}\left[\sup_{t\leq T} \frac{1}{N}\sum_{i=1}^N \left(|\hat{Y}_t^i|^2 + \int_t^T \sum_{j=1}^N |\hat{Z}_s^{i,j}|^2\,\mathrm{d}s + \int_t^T |\hat{Z}_s^{0,i}|^2\,\mathrm{d}s\right)\right] \leq C\left(\mathbb{E}\left[\sup_{t\leq T} W_2^2(\mu_s^N, \mu_s)\right]^{1/2} + \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \left|\tilde{\xi}^i - \xi^i\right|^2\right]\right).$$

Thank You for Your Attention

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