Extended Mean-Field Control Problem with Partial Observation

Ke Yan

Joint work with Prof.Tianyang Nie (Shandong University)

Summer School on Mean Field Models
Rennes, June 12-16, 2023

Preliminaries

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$: a filtered probability space.
- $\mathbb{H}_{\mathbb{F}}^{2,n}$: the space of all $\mathbb{R}^n$-valued, $\mathbb{F}$-progressively measurable processes η on $[0, T]$ such that $\mathbb{E}\int_0^T |\eta(t)|^2 dt < +\infty$.
- $\mathbb{S}_{\mathbb{F}}^{2,n}$: the set of all continuous processes $\eta \in \mathbb{H}_{\mathbb{F}}^{2,n}$ such that $\mathbb{E}\left[\sup_{t\in[0,T]} |\eta(t)|^2\right] < +\infty$.
- (E, d) separable complete metric space, $\mathcal{B}(E)$ Borel σ-field.
- $\mathcal{P}_2(E)$ the space of probability measures with finite second moments, endowed with the 2-Wasserstein distance

$$W_2(\mu, \mu^{'}) := \inf\left\{ \left(\int_{E\times E} |x-y|^2 \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{2}} ; \quad \pi \in \mathcal{P}_2(E \times E) \text{ with marginals } \mu \text{ and } \mu^{'} \right\}.$$

Note: $(\mathcal{P}_2(E), W_2)$ is a complete metric space.

Problem Formulation

- state dynamic:

$$\begin{cases} \mathrm{d}x_t = f(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}t + \sigma(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}W_t \\ \qquad\quad + \bar{\sigma}(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}\bar{W}_t^v, \\ -\mathrm{d}y_t = g(t, x_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\ x(0) = x_0, \quad y(T) = \Phi(x_T, \mathcal{L}(x_T)), \end{cases} \tag{1}$$

  where $(W(\cdot), Y(\cdot))$ is standard $\mathbb{R}^m \times \mathbb{R}^d$ valued Brownian motion.
- observation process:

$$\begin{cases} \mathrm{d}Y_t = h(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}t + \mathrm{d}\bar{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{2}$$

- Note: state dynamic can not be observed directly but only by Y.
- $\mathcal{U}_{ad}$: admissible control set ($\mathbb{F}^Y$-progressively measurable processes $v(\cdot)$ taking values in a closed-convex set $U \in \mathbb{R}^k$ such that $\sup_{t\in[0,T]} \mathbb{E}\left[|v_t|^4\right] < +\infty$.)

Problem Formulation

By inserting observation process (2) into state dynamic (1), we get

$$\begin{cases} \mathrm{d}x_t = [f(t,x_t,v_t,\mathcal{L}(x_t,v_t)) - \bar{\sigma}(t,x_t,v_t,\mathcal{L}(x_t,v_t))h(t,x_t,v_t,\mathcal{L}(x_t,v_t))]\,\mathrm{d}t \\ \qquad\quad + \sigma(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}W_t + \bar{\sigma}(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}Y_t, \\ -\mathrm{d}y_t = g(t,x_t,y_t,z_t,\bar{z}_t,v_t,\mathcal{L}(x_t,y_t,z_t,\bar{z}_t,v_t))\mathrm{d}t - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\ x(0) = x_0, \quad y(T) = \Phi(x_T,\mathcal{L}(x_T)). \end{cases} \tag{3}$$

Remark

Under suitable assumptions, equation (3) has a unique strong solution $(x(\cdot),y(\cdot),z(\cdot),\bar{z}(\cdot))$ for each given $v(\cdot)\in\mathcal{U}_{ad}$.
Moreover, one can show that $\mathbb{E}\left[\sup_{t\in[0,T]}|x_t|^p\right] < +\infty$,
$\mathbb{E}\big[\sup_{t\in[0,T]}|y_t|^p + (\int_0^T|z_t|^2\mathrm{d}t)^{p/2} + (\int_0^T|\bar{z}_t|^2\mathrm{d}t)^{p/2}\big] < +\infty$, *for* $2\le p\le 4$.
Finally, $\mathbb{E}\left[\sup_{t\in[0,T]}(|\rho_t|^p + |\rho_t|^{-p}\right] < +\infty$, *for any* $p\ge 1$.

Problem Formulation

For each $v(\cdot) \in \mathcal{U}_{ad}$, if we define stochastic process $\rho(\cdot)$:

$$\begin{cases} \mathrm{d}\rho_t = \rho_t h(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}Y_t, \\ \rho_0 = 1, \end{cases} \tag{4}$$

and define a probability measure $\mathbb{P}^v$ s.t. $\mathrm{d}\mathbb{P}^v = \rho_T \mathrm{d}\mathbb{P}$, then $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v, x(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot), Y(\cdot), W(\cdot), \bar{W}^v(\cdot))$ is a weak solution of system (1)-(2), according to Girsanov's theorem.

- cost functional

$$\begin{aligned} J(v(\cdot)) = & \mathbb{E}^v\Big[\int_0^T l(t, \rho_t, x_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t \\ & + \chi(\rho_T, x_T, \mathcal{L}(x_T)) + \gamma(y_0)\Big], \end{aligned} \tag{5}$$

  where E^v stands for the expectation w.r.t. the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}^v)$.
- **Problem** Find a $u \in \mathcal{U}_{ad}$ such that $J(u(\cdot)) = \inf_{v \in \mathcal{U}_{ad}} J(v(\cdot))$.
- Objective: establish stochastic maximum principle and verification theorem.

The novelties in our work

- partial observation structure
  - use Girsanov's transformation and the dimensional extension (Tang (1998)).
  - the variational equations and adjoint processes we obtain is a new type of mean-field FBSDEs.

- joint distribution dependence of state and control
  - use the L-derivative w.r.t. probability measure, especially the partial L-derivatives. (Lions 2013, Cardaliaguet 2013, Carmona and Delarue 2018)
  - in order to obtain the related variational inequality and establish the maximum principle under the reference probability space, we need high order estimates of variation equations.

- how to obtain the verification theorem
  - due to the partial observation structure, l and χ depend on ρ.
  - due to the existence of the joint distribution, introduce a new convexity assumption of the Hamiltonian function.

Jointly L-differentiable

Definition(see: Lions 2013, Cardaliaguet 2013, Carmona and Delarue 2018)

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which is rich enough in the sense that for every $\mu \in \mathcal{P}_2(\mathbb{R}^p)$, there is a random variable $X \in L^2(\Omega; \mathbb{R}^p)$ with law μ (i.e. $\mathbb{P}_X = \mu$),
Consider a function $f : \mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^p) \ni (x, \mu) \to f(x, \mu) \in \mathbb{R}$.
- We call f is jointly L-differentiable at (x, μ) if there exists $X \in L^2(\Omega; \mathbb{R}^p)$ with $\mathbb{P}_X = \mu$ such that the lifting $\widetilde{f} : \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p) \ni (x, X) \to f(x, \mathbb{P}_X) \in \mathbb{R}$ is jointly Fréchet differentiable at (x, X) and we denote $[D\widetilde{f}](x, X)$ as the Fréchet derivative of $\widetilde{f}$. Thanks to self-duality of L^2 spaces, $[D\widetilde{f}](x, X)$ can be viewed as an element $D\widetilde{f}(x, X)$ of $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p)$ in the sense that

$$[D\widetilde{f}](x, X)(Y) = \mathbb{E}[D\widetilde{f}(x, X) \cdot Y] \quad \text{for all} \quad Y \in \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p).$$

Jointly L-differentiable

Definition

- Then we can introduce the partial derivatives in x and μ of f, respectively as $\mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^p) \ni (x, \mu) \to \partial_x f(x, \mu) \in \mathbb{R}^q$ and $\mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^p) \ni (x, \mu) \to \partial_\mu f(x, \mu)(\cdot) \in L^2(\mathbb{R}^p, \mu; \mathbb{R}^p)$.
- The partial Fréchet derivative of $\widetilde{f}$ in the direction X is given by $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p) \ni (x, X) \to D_X \widetilde{f}(x, X) = \partial_\mu f(x, \mathbb{P}_X)(X) \in L^2(\Omega; \mathbb{R}^p)$.

  Thus the random variable $D\widetilde{f}(x, X)$ can be represented as

$$D\widetilde{f}(x, X) = (\partial_x f(x, \mathbb{P}_X)(X), \partial_\mu f(x, \mathbb{P}_X)(X)).$$

  We call the functions $\partial_x f(\cdot, \mathbb{P}_X)(\cdot)$ and $\partial_\mu f(\cdot, \mathbb{P}_X)(\cdot)$ which is defined on $\mathbb{R}^q \times \mathbb{R}^p$ and valued, respectively, on $\mathbb{R}^q, \mathbb{R}^p$, the partial L-derivatives of f at $(x, \mathbb{P}_X)$.

Assumptions

We denote η as a generic element of $\mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k)$, let $\mu_1 \in \mathcal{P}_2(\mathbb{R}^n)$, $\mu_2 \in \mathcal{P}_2(\mathbb{R}^l)$, $\mu_3 \in \mathcal{P}_2(\mathbb{R}^{l\times m})$, $\mu_4 \in \mathcal{P}_2(\mathbb{R}^{l\times d})$, $\mu_5 \in \mathcal{P}_2(\mathbb{R}^k)$ be the marginal distribution of η, and $\xi \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)$ be the joint distribution of μ_1 and μ_5.

Throughout the paper, we give the following standing assumptions (**H1**):

- $f, \sigma, \bar{\sigma}, h, g, \Phi$ are differentiable, $\bar{\sigma}$ and h are uniformly bounded.
- $\partial_x(f, \sigma, \bar{\sigma}, \Phi, h, g)$, $\partial_y g$, $\partial_z g$, $\partial_{\bar{z}} g$ and $\partial_v(f, \sigma, \bar{\sigma}, g, h)$ uniformly bounded.
- $\int_{\mathbb{R}^n} |\partial_{\mu_1}\Phi(x, \mu_1)(x')|^2 \mathrm{d}\mu_1(x')$, $\quad \int_{\mathbb{R}^{n+k}} |\partial_{\mu_i}(f, \sigma, \bar{\sigma}, h)(t, x, v, \xi)(x', v')|^2 \mathrm{d}\xi(x', v')$, and $\int_{\mathbb{R}^{n\times l\times(l\times m)\times(l\times d)\times k}} |\partial_{\mu_j} g(t, x, y, z, \bar{z}, v, \eta)(x', y', z', \bar{z}', v')|^2 \mathrm{d}\eta(x', y', z', \bar{z}', v')$ are uniformly bounded.
- $(f, \sigma, \bar{\sigma})(t, 0, 0, \delta_0)$ and $g(t, 0, 0, 0, 0, \delta_0)$ are uniformly bounded.

Assumptions

(H2): for coefficients of cost.

- $\psi = \rho, x, y, z, \bar{z}, v$, $(\rho, x, y, z, \bar{z}, v, \eta) \mapsto \partial_\psi l(\rho, x, y, z, \bar{z}, v, \eta)$ continuous.
- l is L-differentiable w.r.t. η.
- $(\rho, x, y, z, \bar{z}, v, (X, Y, Z, \bar{Z}, \beta)) \mapsto$ $\partial_{\mu_j} l(t, \rho, x, y, z, \bar{z}, v, \mathcal{L}(X, Y, Z, \bar{Z}, \beta)(X, Y, Z, \bar{Z}, \beta)$ continuous.
- $(\rho, x, \mu_1) \mapsto \partial_x \chi(\rho, x, \mu_1)$ and $y \mapsto \partial_y \gamma(y)$ continuous.
- χ is L-differentiable w.r.t. μ_1.
- $\mathbb{R}^{1+n} \times L^2(\Omega; \mathbb{R}^n) \ni (\rho, x, X) \mapsto \partial_{\mu_1} \chi(\rho, x, \mathcal{L}(X))(X) \in L^2(\Omega; \mathbb{R}^n)$ continuous.
- $\partial_\psi l$, $\partial_x \chi$, $\partial_y \gamma$ linear growth.
- $L^2(\mathbb{R}^n, \mu; \mathbb{R}^n)$ norm of $(\rho, x, y, z, \bar{z}, v, \eta) \mapsto \partial_{\mu_j} l(t, \rho, x, y, z, \bar{z}, v, \eta)(x', y', z', \bar{z}', v')$ linear growth.
- $L^\gamma(\mathbb{R}^s, \mu; \mathbb{R}^s)$ norm of $(x, \rho, \mu_1) \mapsto \partial_{\mu_1} \chi(x, \rho, \mu_1)(x')$ linear growth.
- $l(t, 0, 0, 0, 0, 0, 0, \delta_0)$ uniformly bounded.

Problem Formulation

According to Bayes' formula, the cost functional defined as in (5) can be rewritten as (noticing that $\gamma(y_0)$ is deterministic)

$$J(v(\cdot)) = \mathbb{E}\Big[\int_0^T \rho_t l(t, \rho_t, x_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t + \rho_T \chi(\rho_T, x_T, \mathcal{L}(x_T)) + \gamma(y_0)\Big]. \tag{6}$$

We mention that, under assumptions (H.1)-(H.2), we have $|J(v(\cdot))| < +\infty$, i.e. the above cost functional is well defined.

Problem Formulation

Some Notations (Tang, SIAM J. Control Optim., 1998)

We introduce the following notations for dimensional extension

$$X := \begin{pmatrix} \rho \\ x \end{pmatrix}, \quad X_0 := \begin{pmatrix} 1 \\ x_0 \end{pmatrix}, \quad X^1 := \begin{pmatrix} \rho^1 \\ x^1 \end{pmatrix},$$

$$\Sigma(t, X, v, \xi) := \begin{pmatrix} 0 \\ \sigma(t, x, v, \xi) \end{pmatrix}, \bar{\Sigma}(t, X, v, \xi) := \begin{pmatrix} \rho h(t, x, v, \xi) \\ \bar{\sigma}(t, x, v, \xi) \end{pmatrix},$$

$$F(t, X, v, \xi) := \begin{pmatrix} 0 \\ f(t, x, v, \xi) - \bar{\sigma}(t, x, v, \xi) h(t, x, v, \xi) \end{pmatrix},$$

$$G(t, X, y, z, \bar{z}, v, \eta) := g(t, x, y, z, \bar{z}, v, \eta),$$

$$L(t, X, y, z, \bar{z}, v, \eta) := \rho l(t, \rho, x, y, z, \bar{z}, v, \eta),$$

$$M(X, \mu_1) := \rho \chi(\rho, x, \mu_1).$$

Problem Formulation

Then equations (3) can be compressed into the following form

$$\begin{cases} \mathrm{d}X_t = F(t, X_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}t + \Sigma(t, X_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}W_t \\ \qquad + \bar{\Sigma}(t, X_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}Y_t \\ -\mathrm{d}y_t = G(t, X_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\ X(0) = X_0, \quad y(T) = \Phi(x_T, \mathcal{L}(x_T)), \end{cases} \tag{7}$$

and the cost functional (6) can be represented as

$$\begin{aligned} J(v(\cdot)) = & \mathbb{E}\Big[\int_0^T L(t, X_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t \\ & + M(X_T, \mathcal{L}(x_T)) + \gamma(y_0)\Big]. \end{aligned} \tag{8}$$

- minimize $J(v(\cdot))$ over $v(\cdot) \in \mathcal{U}_{ad}$ subject to (7) and (8).

Stochastic Maximum Principle

Convex perturbation

For simplicity, we set $n = l = k = m = d = 1$.
Let $u(\cdot)$ be an optimal control.

- Let $v(\cdot)$ be such that $u(\cdot) + v(\cdot) \in \mathcal{U}_{ad}$.
- Since $\mathcal{U}_{ad}$ is convex, $u^{\varepsilon}(\cdot) \triangleq u(\cdot) + \varepsilon v(\cdot)$ is also in $\mathcal{U}_{ad}$.

To simplify symbols, we set

$$\begin{aligned}
&\xi_t := \mathcal{L}(x_t, u_t), \quad \eta_t := \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, u_t), \\
&\theta_t := (X_t, u_t, \xi_t), \quad \theta_t^{'} := (x_t, u_t, \xi_t), \\
&\alpha_t := (x_t, y_t, z_t, \bar{z}_t, u_t), \quad \Theta_t := (X_t, y_t, z_t, \bar{z}_t, u_t, \eta_t), \\
&\Theta_t^{'} := (x_t, y_t, z_t, \bar{z}_t, u_t, \eta_t) = (\alpha_t, \eta_t).
\end{aligned}$$

Variational Equations

$$
\begin{cases}
\mathrm{d}X_t^1 = \Big(\partial_X F(t,\theta_t)X_t^1 + \partial_v F(t,\theta_t)v_t + \tilde{\mathbb{E}}[\partial_{\mu_1} F(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] \\
\qquad + \tilde{\mathbb{E}}[\partial_{\mu_5} F(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\mathrm{d}t \\
\qquad + \Big(\partial_X \Sigma(t,\theta_t)X_t^1 + \partial_v \Sigma(t,\theta_t)v_t + \tilde{\mathbb{E}}[\partial_{\mu_1} \Sigma(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] \\
\qquad + \tilde{\mathbb{E}}[\partial_{\mu_5} \Sigma(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\mathrm{d}W_t \\
\qquad + \Big(\partial_X \bar{\Sigma}(t,\theta_t)X_t^1 + \partial_v \bar{\Sigma}(t,\theta_t)v_t + \tilde{\mathbb{E}}[\partial_{\mu_1} \bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] \\
\qquad + \tilde{\mathbb{E}}[\partial_{\mu_5} \bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\mathrm{d}Y_t, \\
-\mathrm{d}y_t^1 = \Big(\partial_x G(t,\Theta_t)x_t^1 + \partial_y G(t,\Theta_t)y_t^1 + \partial_z G(t,\Theta_t)z_t^1 + \partial_{\bar{z}} G(t,\Theta_t)\bar{z}_t^1 + \partial_v G(t,\Theta_t)v_t \\
\qquad + \tilde{\mathbb{E}}[\partial_{\mu_1} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_2} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{y}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_3} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{z}_t^1] \\
\qquad + \tilde{\mathbb{E}}[\partial_{\mu_4} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{\bar{z}}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{v}_t]\Big)\mathrm{d}t - z_t^1\mathrm{d}W_t - \bar{z}_t^1\mathrm{d}Y_t, \\
X_0^1 = 0, \quad y_T^1 = \partial_x \Phi(x_T,\mathcal{L}(x_T))x_T^1 + \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(x_T,\mathcal{L}(x_T))(\tilde{x}_T)\tilde{x}_T^1],
\end{cases}
\tag{9}
$$

where we used the notation $X_t^1 := \begin{pmatrix} \rho_t^1 \\ x_t^1 \end{pmatrix}$, and $\tilde{\alpha}_t := (\tilde{x}_t, \tilde{y}_t, \tilde{z}_t, \tilde{\bar{z}}_t, \tilde{u}_t)$ is an independent copy of $\alpha_t := (x_t, y_t, z_t, \bar{z}_t, u_t)$.

Variational Equations

Theorem

Let assumptions (H.1)-(H.2) hold, then mean-field FBSDE (9) *admits a unique solution* $(X^1(\cdot), y^1(\cdot), z^1(\cdot), \bar{z}^1(\cdot)) \in \mathbb{S}_{\mathbb{F}}^{2,1+n} \times \mathbb{S}_{\mathbb{F}}^{2,l} \times \mathbb{H}_{\mathbb{F}}^{2,l\times m} \times \mathbb{H}_{\mathbb{F}}^{2,l\times d}$ *satisfying that for any* $2 \leq p \leq 4$ *and* $0 < \varepsilon_0 \leq p$,

$$\begin{aligned}\mathbb{E}\Big[& \sup_{t\in[0,T]} |x_t^1|^p + \sup_{t\in[0,T]} |\rho_t^1|^{p-\varepsilon_0} + \sup_{t\in[0,T]} |y_t^1|^p \\ & + \Big(\int_0^T |z_t^1|^2 dt\Big)^{p/2} + \Big(\int_0^T |\bar{z}_t^1|^2 dt\Big)^{p/2}\Big] < +\infty.\end{aligned} \tag{10}$$

- R. Buckdahn, B. Djehiche, J. Li, S. Peng (2009), R. Buckdahn, J. Li, S. Peng (2009), R. Carmona, F. Delarue (2018).

Variational Equations

Now let us denote $(X^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot), \bar{z}^\varepsilon(\cdot))$ as the trajectory corresponding to $u^\varepsilon(\cdot)$. We set

$$x_t^{\varepsilon,1} = \frac{x_t^\varepsilon - x_t}{\varepsilon} - x_t^1, \quad \rho_t^{\varepsilon,1} = \frac{\rho_t^\varepsilon - \rho_t}{\varepsilon} - \rho_t^1,$$
$$X_t^{\varepsilon,1} = \frac{X_t^\varepsilon - X_t}{\varepsilon} - X_t^1 = \begin{pmatrix} \rho_t^{\varepsilon,1} \\ x_t^{\varepsilon,1} \end{pmatrix},$$
$$y_t^{\varepsilon,1} = \frac{y_t^\varepsilon - y_t}{\varepsilon} - y_t^1, \quad z_t^{\varepsilon,1} = \frac{z_t^\varepsilon - z_t}{\varepsilon} - z_t^1, \quad \bar{z}_t^{\varepsilon,1} = \frac{\bar{z}_t^\varepsilon - \bar{z}_t}{\varepsilon} - \bar{z}_t^1, \tag{11}$$

Variational Inequality

Lemma

The functional $u(\cdot) \mapsto J(u(\cdot))$ is Gâteaux differentiable in the direction $v(\cdot)$, and its derivative is given by

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u(\cdot) + \varepsilon v(\cdot))\Big|_{\varepsilon=0} \\
=&\mathbb{E} \int_0^T \Big[\partial_X L(t,\Theta_t) X_t^1 + \partial_y L(t,\Theta_t) y_t^1 + \partial_z L(t,\Theta_t) z_t^1 + \partial_{\bar{z}} L(t,\Theta_t) \bar{z}_t^1 + \partial_v L(t,\Theta_t) v_t^1 \\
&\qquad + \tilde{\mathbb{E}}[\partial_{\mu_1} L(t,\Theta_t)(\tilde{\alpha}_t) \tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_2} L(t,\Theta_t)(\tilde{\alpha}_t) \tilde{y}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_3} L(t,\Theta_t)(\tilde{\alpha}_t) \tilde{z}_t^1] \\
&\qquad + \tilde{\mathbb{E}}[\partial_{\mu_4} L(t,\Theta_t)(\tilde{\alpha}_t) \tilde{\bar{z}}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5} L(t,\Theta_t)(\tilde{\alpha}_t) \tilde{v}_t]\Big] \mathrm{d}t \\
&\quad + \mathbb{E}\Big[\partial_X M(X_T, \mathcal{L}(x_T)) X_T^1 + \tilde{\mathbb{E}}[\partial_{\mu_1} M(X_T, \mathcal{L}(x_T))(\tilde{x}_T) \tilde{x}_T^1]\Big] + \partial_y \gamma(y_0) y_0^1 .
\end{aligned} \tag{12}
$$

Proof of Variational Inequality

For $\psi = \rho, x, y, z, \bar{z}$, $\phi = x, y, z, \bar{z}$, as $\varepsilon \to 0$,

$$\mathbb{E}\int_0^T (|\rho_t^1| + |\rho_t^{\varepsilon,1}|) \cdot |\psi_t^{\varepsilon,1}|^2 \mathrm{d}t \leq \Big(\mathbb{E}\sup_{t\in[0,T]}(|\rho_t^1| + |\rho_t^{\varepsilon,1}|)^3\Big)^{\frac{1}{3}} \Big(\mathbb{E}\Big(\int_0^T |\psi_t^{\varepsilon,1}|^2 \mathrm{d}t\Big)^{\frac{3}{2}}\Big)^{\frac{2}{3}} \to 0,$$

$$\mathbb{E}\int_0^T (|\rho_t^1| + |\rho_t^{\varepsilon,1}|) \cdot \mathbb{E}|\phi_t^{\varepsilon,1}|^2 \mathrm{d}t \leq \mathbb{E}\sup_{t\in[0,T]}(|\rho_t^1| + |\rho_t^{\varepsilon,1}|) \cdot \int_0^T \mathbb{E}|\phi_t^{\varepsilon,1}|^2 \mathrm{d}t \to 0,$$

$$\mathbb{E}\int_0^T (|\rho_t| + |\rho_t^1| + |\rho_t^{\varepsilon,1}|)^2 |\psi_t^{\varepsilon,1}| \mathrm{d}t \leq \Big(\mathbb{E}\sup_{t\in[0,T]}(|\rho_t| + |\rho_t^1| + |\rho_t^{\varepsilon,1}|)^3\Big)^{\frac{2}{3}} \Big(\mathbb{E}\Big(\int_0^T |\psi_t^{\varepsilon,1}|^2 \mathrm{d}t\Big)^{\frac{3}{2}}\Big)^{\frac{1}{3}} \to 0,$$

$$\mathbb{E}\int_0^T (|\rho_t| + |\rho_t^1| + |\rho_t^{\varepsilon,1}|)^2 \cdot \mathbb{E}|\phi_t^{\varepsilon,1}| \mathrm{d}t \leq \mathbb{E}\sup_{t\in[0,T]}(|\rho_t| + |\rho_t^1| + |\rho_t^{\varepsilon,1}|)^2 \cdot \int_0^T \mathbb{E}|\phi_t^{\varepsilon,1}| \mathrm{d}t \to 0,$$

$$\mathbb{E}\int_0^T |\rho_t^{\varepsilon,1}| \cdot (1 + |\psi_t|^2 + |\psi_t^{\varepsilon,1}|^2) \mathrm{d}t \leq \Big(\mathbb{E}\sup_{t\in[0,T]}|\rho_t^{\varepsilon,1}|^3\Big)^{\frac{1}{3}} \Big(\mathbb{E}\Big(\int_0^T (1 + |\psi_t|^2 + |\psi_t^{\varepsilon,1}|^2) \mathrm{d}t\Big)^{\frac{3}{2}}\Big)^{\frac{2}{3}} \to 0,$$

$$\mathbb{E}\int_0^T |\rho_t^{\varepsilon,1}|^2 \cdot (1 + |\psi_t| + |\psi_t^{\varepsilon,1}|) \mathrm{d}t \leq \Big(\mathbb{E}\sup_{t\in[0,T]}|\rho_t^{\varepsilon,1}|^3\Big)^{\frac{2}{3}} \Big(\mathbb{E}\Big(\int_0^T (1 + |\psi_t| + |\psi_t^{\varepsilon,1}|) \mathrm{d}t\Big)^{\frac{3}{2}}\Big)^{\frac{1}{3}} \to 0.$$

Remark

Due to the application of Girsanovs transformation, the coefficients l and χ in cost functional will be multiplied by ρ, so we need high order estimates and high order convergence results of variational equations when we derive the variational inequality.

Some Lemmas

The follwing lemmas are about high order estimates and high order convergence results of variational equations.

Lemma

Suppose assumptions (H.1) and (H.2) hold, then we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}|X_t^{\varepsilon,1}|^2=\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}\left(|\rho_t^{\varepsilon,1}|^2+|x_t^{\varepsilon,1}|^2\right)=0.$$

Moreover, for any $2\le p\le 4$ *and* $0<\varepsilon_0\le p$*, it follows that*

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}(|x_t^{\varepsilon,1}|^p+|\rho_t^{\varepsilon,1}|^{p-\varepsilon_0})=0.$$

Some Lemmas

Lemma

Suppose assumptions (H.1) and (H.2) hold, then we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^2+\int_0^T(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right]=0. \tag{13}$$

Lemma

Suppose assumptions (H.1) and (H.2) hold, then for any $2\le p\le 4$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p+\left(\int_0^T(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}\right]=0. \tag{14}$$

- see Theorem 4.4.4 of Zhang(2017) for the case without mean-field term.

Proof of Lemma

We need first establish the L^2 estimates of the variational equations and then use it to obtain the desired L^p estimates. Due to the existence of mean-field term, it requires some skills, especially for the higher order estimates of the mean-filed backward equation. We mention that since g depends on the law of $z(\cdot), \bar{z}(\cdot)$, it will be a little complicated to obtain the related L^p estimate.

Proof of Lemma

applying Itô's formula to $|y_t^{\varepsilon,1}|^p$, where $2 \le p \le 4$, we have for any $0 \le r \le T$,

$$
\begin{aligned}
&|y_r^{\varepsilon,1}|^p + \frac{p(p-1)}{2}\int_r^T |y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t \\
=&|v_T^{\Phi}|^p + p\int_r^T |y_t^{\varepsilon,1}|^{p-2} y_t^{\varepsilon,1} v_t^G \mathrm{d}t - p\int_r^T |y_t^{\varepsilon,1}|^{p-2} y_t^{\varepsilon,1}(z_t^{\varepsilon,1}\mathrm{d}W_t + \bar{z}_t^{\varepsilon,1}\mathrm{d}Y_t),
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
v_t^G := &\frac{G(t,(\Theta_t)^{\varepsilon}) - G(t,\Theta_t)}{\varepsilon} - \partial_x G(t,\Theta_t)x_t^1 - \partial_y G(t,\Theta_t)y_t^1 - \partial_z G(t,\Theta_t)z_t^1 - \partial_{\bar{z}} G(t,\Theta_t)\bar{z}_t^1 \\
&- \partial_v G(t,\Theta_t)v_t - \tilde{\mathbb{E}}[\partial_{\mu_1} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{x}_t^1] - \tilde{\mathbb{E}}[\partial_{\mu_2} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{y}_t^1] \\
&- \tilde{\mathbb{E}}[\partial_{\mu_3} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{z}_t^1] - \tilde{\mathbb{E}}[\partial_{\mu_4} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{\bar{z}}_t^1] - \tilde{\mathbb{E}}[\partial_{\mu_5} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{v}_t]
\end{aligned}
$$

and

$$
v_T^{\Phi} := \frac{\Phi(x_T^{\varepsilon},\mathcal{L}(x_T^{\varepsilon})) - \Phi(x_T,\mathcal{L}(x_T))}{\varepsilon} - \partial_x \Phi(x_T,\mathcal{L}(x_T))x_T^1 - \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(x_T,\mathcal{L}(x_T))(\tilde{x}_T)\tilde{x}_T^1].
$$

Then we have

$$
\mathbb{E}|y_r^{\varepsilon,1}|^p + \frac{p(p-1)}{2}\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t = \mathbb{E}|v_T^{\Phi}|^p + p\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-2} y_t^{\varepsilon,1} v_t^G \mathrm{d}t. \tag{16}
$$

Proof of Lemma

From (15), and with the help of BDG inequality, we have

$$\begin{aligned}\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^p &\le \mathbb{E}|v_T^{\Phi}|^p + C\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-1}|v_t^G|\mathrm{d}t \\ &+ C\mathbb{E}\left(\int_r^T |y_t^{\varepsilon,1}|^{2p-2}|z_t^{\varepsilon,1}|^2\mathrm{d}t\right)^{1/2} + C\mathbb{E}\left(\int_r^T |y_t^{\varepsilon,1}|^{2p-2}|\bar{z}_t^{\varepsilon,1}|^2\mathrm{d}t\right)^{1/2}.\end{aligned} \tag{17}$$

Then the term $\mathbb{E}\int_0^T |y_t|^{p-1}(\mathbb{E}|z_t|^2)^{\frac{1}{2}}\mathrm{d}t$ and $\mathbb{E}\int_0^T |y_t|^{p-1}(\mathbb{E}|z_t|^2)^{\frac{1}{2}}\mathrm{d}t$ will appear.

Hamiltonian Function

$$H : [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{l} \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^{k} \times \mathcal{P}_2(\mathbb{R}^{n} \times \mathbb{R}^{l} \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^{k}) \times \mathbb{R}^{n+1} \times \mathbb{R}^{l} \times \mathbb{R}^{(n+1)\times m} \times \mathbb{R}^{(n+1)\times d} \mapsto \mathbb{R},$$

$$\begin{aligned} &H(t, X, y, z, \bar{z}, v, \eta, p, q, k, \bar{k}) \\ &= \langle F(t, X, v, \xi), p\rangle - \langle G(t, X, y, z, \bar{z}, v, \eta), q\rangle \\ &\quad + \mathrm{tr}[k^\top \Sigma(t, X, v, \xi)] + \mathrm{tr}[\bar{k}^\top \bar{\Sigma}(t, X, v, \xi)] + L(t, X, y, z, \bar{z}, v, \eta). \end{aligned} \tag{18}$$

where $\xi \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)$ is the joint margin distribution of $\eta \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k)$ on the first and the fifth components.

Adjoint Equation

Using the definition of Hamiltonian function H, we can give the adjoint equation as

$$\begin{cases} -\mathrm{d}p_t = \left[\partial_X H(t,\Theta_t;p,q,k,\bar{k}) + \begin{pmatrix} 0 \\ \tilde{\mathbb{E}}[\partial_{\mu_1} H(t,\tilde{\Theta}_t;\tilde{p},\tilde{q},\tilde{k},\tilde{\bar{k}})(\alpha_t)] \end{pmatrix}\right]\mathrm{d}t \\ \qquad - k_t \mathrm{d}W_t - \bar{k}_t \mathrm{d}Y_t \\ \mathrm{d}q_t = -\left[\partial_y H(t,\Theta_t;p,q,k,\bar{k}) + \tilde{\mathbb{E}}[\partial_{\mu_2} H(t,\tilde{\Theta}_t;\tilde{p},\tilde{q},\tilde{k},\tilde{\bar{k}})(\alpha_t)]\right]\mathrm{d}t \\ \qquad - \left[\partial_z H(t,\Theta_t;p,q,k,\bar{k}) + \tilde{\mathbb{E}}[\partial_{\mu_3} H(t,\tilde{\Theta}_t;\tilde{p},\tilde{q},\tilde{k},\tilde{\bar{k}})(\alpha_t)]\right]\mathrm{d}W_t \\ \qquad - \left[\partial_{\bar{z}} H(t,\Theta_t;p,q,k,\bar{k}) + \tilde{\mathbb{E}}[\partial_{\mu_4} H(t,\tilde{\Theta}_t;\tilde{p},\tilde{q},\tilde{k},\tilde{\bar{k}})(\alpha_t)]\right]\mathrm{d}Y_t, \\ p_T = \partial_X^\top M(X_T,\mathcal{L}(x_T)) + \begin{pmatrix} 0 \\ \tilde{\mathbb{E}}[\partial_{\mu_1} M(\tilde{X}_T,\mathcal{L}(x_T))(x_T)] \end{pmatrix} \\ \qquad - \begin{pmatrix} 0 \\ \partial_x \Phi(x_T,\mathcal{L}(x_T)) \end{pmatrix} q_T - \begin{pmatrix} 0 \\ \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(\tilde{x}_T,\mathcal{L}(x_T))(x_T)\cdot\tilde{q}_T] \end{pmatrix}, \\ q_0 = -\partial_y \gamma(y_0). \end{cases} \tag{19}$$

- E. Pardoux, A. Rascanu (2014), R. Carmona, F. Delarue (2018).

Stochastic Maximum Principle

Theorem

Let (H1) and (H2) hold, if $u(\cdot)$ *is an optimal control and* $(X(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot))$ *is the corresponding trajectory, and* $(p(\cdot), q(\cdot), k(\cdot), \bar{k}(\cdot))$ *is corresponding adjoint process satisfying* (19), *we have*

$$\mathbb{E}\left[\left(\partial_v H(t, \Theta_t, p_t, q_t, k_t, \bar{k}_t) + \tilde{\mathbb{E}}[\partial_{\mu_5} H(t, \tilde{\Theta}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{\bar{k}}_t)(\alpha_t)]\right)(v - u_t)\Big|\mathcal{F}_t^Y\right] \geq 0, \quad \forall v \in U, \ a.s. \ a.e. \tag{20}$$

where we recall

$$\begin{aligned} \alpha_t &:= (x_t, y_t, z_t, \bar{z}_t, u_t) \quad \eta_t := \mathcal{L}(\alpha_t), \\ \Theta_t &:= (X_t, y_t, z_t, \bar{z}_t, u_t, \eta_t) = (\rho_t, \alpha_t, \eta_t). \end{aligned} \tag{21}$$

Verification Theorem

Convexity Assumptions

(H3) γ is convex and M is convex in the sense that

$$M(\check{X},\check{\mu}_1) - M(X,\mu_1) \geq \langle \partial_X M(X,\mu_1), \check{X} - X\rangle + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_1} M(X,\mu_1)(\tilde{x}), \tilde{\check{x}} - \tilde{x}\rangle\big]$$

(H4) Hamiltonian H satisfies the following convexity condition:

$$\begin{aligned} & H(t,\Lambda',\eta',\Pi) - H(t,\Lambda,\eta,\Pi) \\ \geq & \langle \partial_X H(t,\Lambda,\eta,\Pi), \check{X} - X\rangle + \langle \partial_y H(t,\Lambda,\eta,\Pi), \check{y} - y\rangle \\ & + \langle \partial_z H(t,\Lambda,\eta,\Pi), \check{z} - z\rangle + \langle \partial_{\bar{z}} H(t,\Lambda,\eta,\Pi), \check{\bar{z}} - \bar{z}\rangle \\ & + \langle \partial_v H(t,\Lambda,\eta,\Pi), \check{v} - v\rangle + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_1} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{x}} - \tilde{x}\rangle\big] \\ & + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_2} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{y}} - \tilde{y}\rangle\big] + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_3} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{z}} - \tilde{z}\rangle\big] \\ & + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_4} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{\bar{z}}} - \tilde{\bar{z}}\rangle\big] + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_5} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{v}} - \tilde{v}\rangle\big], \end{aligned}$$

where $\Pi := (p,q,k,\bar{k})$, $\Lambda := (X,y,z,\bar{z},v)$.

Verification Theorem

Remark

In this formulation, the cost functional (6) contains ρ. If the coefficients l and χ in the cost functional (6) do not depend on ρ, then one can check that the mappings $(\rho, x, y, z, \bar{z}, v, \eta) \mapsto \rho l(t, x, y, z, \bar{z}, v, \eta)$ and $(\rho, x) \mapsto \rho\chi(x, \mu_1)$ are usually not convex. Fortunately, if we allow l and χ depend on ρ, then it is possible to make sense that the convexity assumptions (H.3)-(H.4) hold (one simple example is that $l(\rho, x, y, z, \bar{z}, \eta) = \frac{1}{\rho}(|x|^2 + |y|^2 + |z|^2 + |\bar{z}|^2 + |v|^2 + |\eta|^2)$ and $\chi(\rho, x, \mu_1) = \frac{1}{\rho}(|x|^2 + |\mu|^2)$), then we can get the verification theorem.

Verification Theorem

Theorem

Suppose (H1)-(H4) are satisfied. Let $u(\cdot) \in \mathcal{U}_{ad}$ *be an admissible control,* $(X(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot))$ *be the corresponding trajectory, and* $\Pi(\cdot) := (p(\cdot), q(\cdot), k(\cdot), \bar{k}(\cdot))$ *be the corresponding adjoint process satisfying* (19)*. If* (20) *holds, then* $u(\cdot)$ *is an optimal control, i.e.* $J(u(\cdot)) = \inf_{v(\cdot) \in \mathcal{U}_{ad}} J(v(\cdot))$*.*

Examples

- Scalar interactions
- Linear quadratic case

Reference

- B. Acciaio, J. Backhoff-Veraguas, R. Carmona. Extended mean field control problems: stochastic maximum principle and transport perspective. *SIAM J. Control Optim.*, **57**, 3666-3693 (2019).
- R. Buckdahn, J. Li, J. Ma. A mean-field stochastic control problem with partial observations. *Ann. Appl. Probab.*, **27**, 3201-3245 (2017).
- R. Buckdahn, J. Li, S. Peng. Mean-field backward stochastic differential equations and related partial differential equations. *Stochatic Process. Appl.*, **19**, 3133-3154 (2009).
- R. Carmona, F. Delarue, Forward-backward stochastic differential equations and controlled Mckean-Vlasov dynamics. *Ann. Probab.*, **43**, 2647-2700 (2015).
- R. Carmona, F. Delarue, Probabilistic theory of mean-field games with applications, Springer, 2018.
- S. Tang. The maximum principle for partially observed optimal control of stochastic differential equations. *SIAM J. Control Optim.*,**36**, 1596-1617 (1998).
- G. Wang, Z. Wu, J. Xiong. Maximum principles for forward-backward stochastic control systems with correlated state and observation noises. *SIAM J. Control Optim.*,**51**, 491-524 (2013).
- J. Zhang. Backward stochastic differential equations. From linear to fully nonlinear theory. Springer, New York, 2017.

Thank you for your attention!