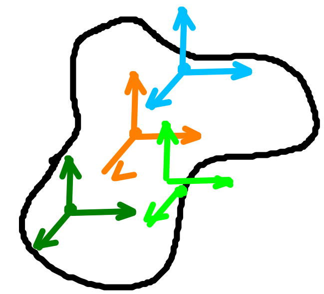
Sur quelques invariants des 3-variétés

On some 3-manifold invariants

Vera Vértesi

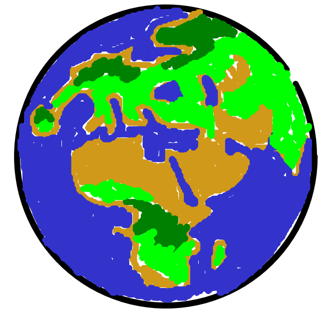
LM∃ UN

3- & 4- Manifolds

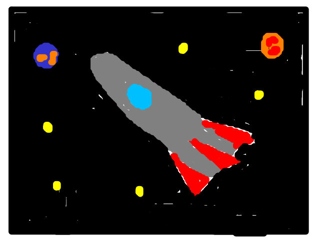


Def: Manifolds are spaces that are locally Euclidian:

e.g. - the Earth is a 2-dimensional manifold



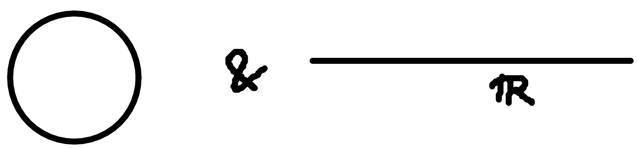
- the World is a 3-dimensional manifold



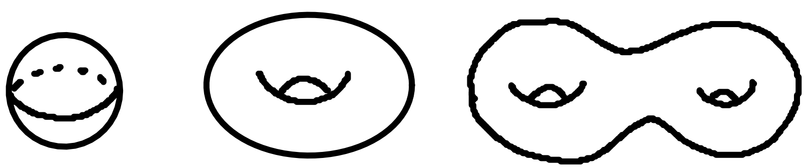
- spacetime is a 4-dimensional manifold

Classification of manifolds

• 1-dimension:



• 2-dimensions:



. . .

• higher dimensions:

Poincaré conjecture: M^n simply connected, $H_*(S^n) \cong H_*(M)$ $\Rightarrow$ M^n homeomorphic to S^n

Perelman $(n=3)$, Freedman $(n=4)$, Smale $(n \geq 5)$

Many Equivalent Theories

Donaldson invariants (1982) $\boxed{4}$

Instanton Floer homology $\boxed{3}$
(Floer, 1988)

Seiberg-Witten invariants (1994)
monopole Floer homology $\boxed{3}$
(Kronheimer-Mrowka, 2007)

Taubes 2008

Gromov-Taubes invariant $\boxed{4}$
(Taubes 199?)
embedded contact homology $\boxed{3}$
(Hutchings, 2010)

Kutluhan-Lee-Taubes
2010

Colin-Ghiggini-Honda
2011

Heegaard Floer homology $\boxed{3}$, $\boxed{4}$
(Ozsváth-Szabó, 2001)

What is Heegaard Floer Theory ?

$\mathbb{F} = \mathbb{Z}$ or $\mathbb{Z}/(2)$

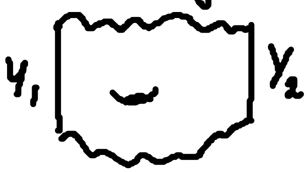
Y 3-manifold $\rightsquigarrow$ $\widehat{HF}(Y)$ graded $\mathbb{F}$-module

$HF^-(Y)$ $\mathbb{F}[U]$ - module

$HF^\infty(Y)$ $\mathbb{F}[U,U^{-1}]$ - module (determined by $H^*(Y)$)

$HF^+(Y)$ $\mathbb{F}[U]$ - module

W cobordism btwn 3-manifolds $\rightsquigarrow$ $\hat{F}_W^{+,-,\infty} : \widehat{HF}^{+,-,\infty}(Y_1) \to \widehat{HF}^{+,-,\infty}(Y_2)$



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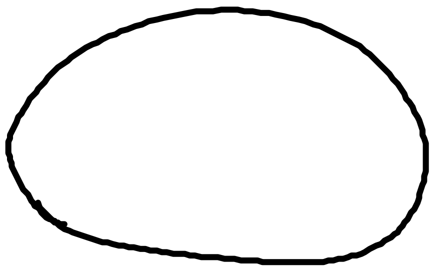

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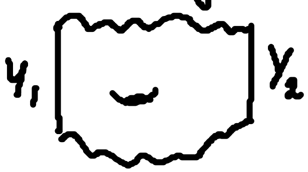

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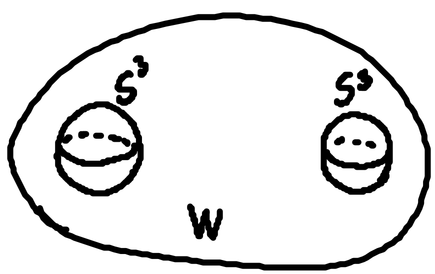

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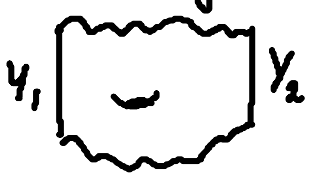

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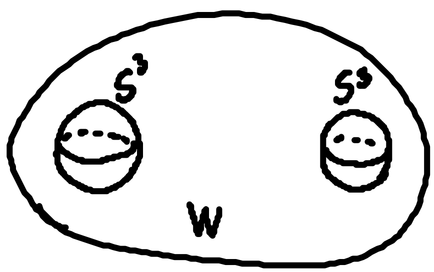

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! This does not work !

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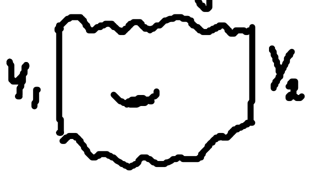


X 4-manifold $\rightsquigarrow$ a number for every spin^c-structure

$K \subset Y$ knot $\rightsquigarrow$ $\widehat{HFK}(Y,K)$ bigraded $\mathbb{F}$-module

$HFK^-(Y,K)$ $\mathbb{F}[U]$-module

Focus: $\widehat{HFK}(S^3, K)$ over $\mathbb{Z}/(2)$

Why is Heegaard Floer Theory Useful?

Geometric content

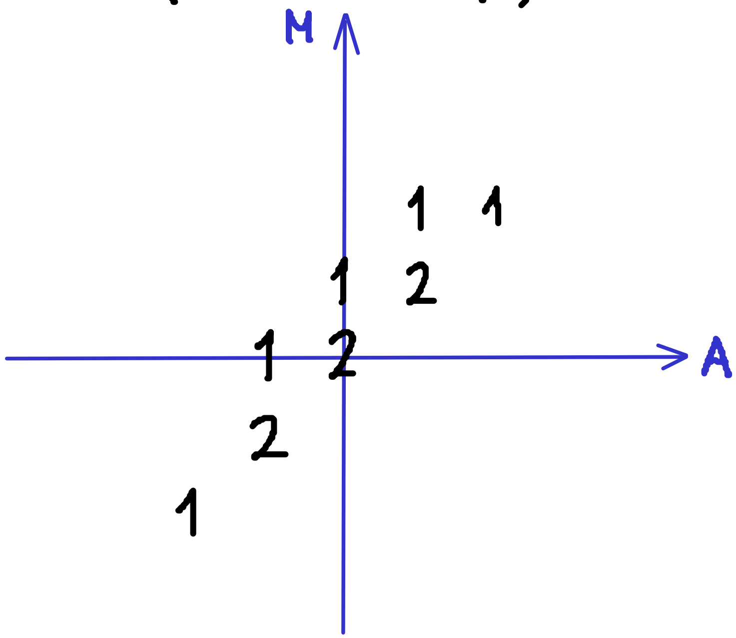
- Ozsváth-Szabó: Detects smooth structures on 4-manifolds
- Ozsváth-Szabó, Ni: Detects the genus of knots
  Thurston norm of 3-manifolds
- Ozsváth-Szabó, Ghiggini, Ni, Juhász,...
  Detects fiberedness of knots and 3-manifolds
- Ozsváth-Szabó,... Bounds the slice genus
  minimal class representatives of homology classes

Computability

- defined using a PDE but sometimes can be combinatorial:
- Manolescu-Ozsváth-Sarkar: $\widehat{HFK}^-$ for knots
- Sarkar-Wang, Ozsváth-Stipsicz-Szabó: $\widehat{HF}(Y)$, easier version of HF^-
- Manolescu-Ozsváth-Thurston: $\widehat{HF}^{\pm,\infty}$, 4-manifold invariant

$\widehat{HFK}$ - What will we get?

$\dim\left(\widehat{HFK}_M(K,A)\right)$ (for $K = 10_{132}$)



- bigraded;
- Euler characteristic is the Alexander polynomial;
- Max grading is knot genus; (Ozsváth-Szabó, 2001)
- Determines knot fibration (Ghiggini, Ni 2006)
- Gives an effective transverse & Legendrian knot invariant (Baldwin, Khandawit, Lisca, Ng, Ozsváth, Stipsicz, Szabó, Thurston, Vela-Vick, V)

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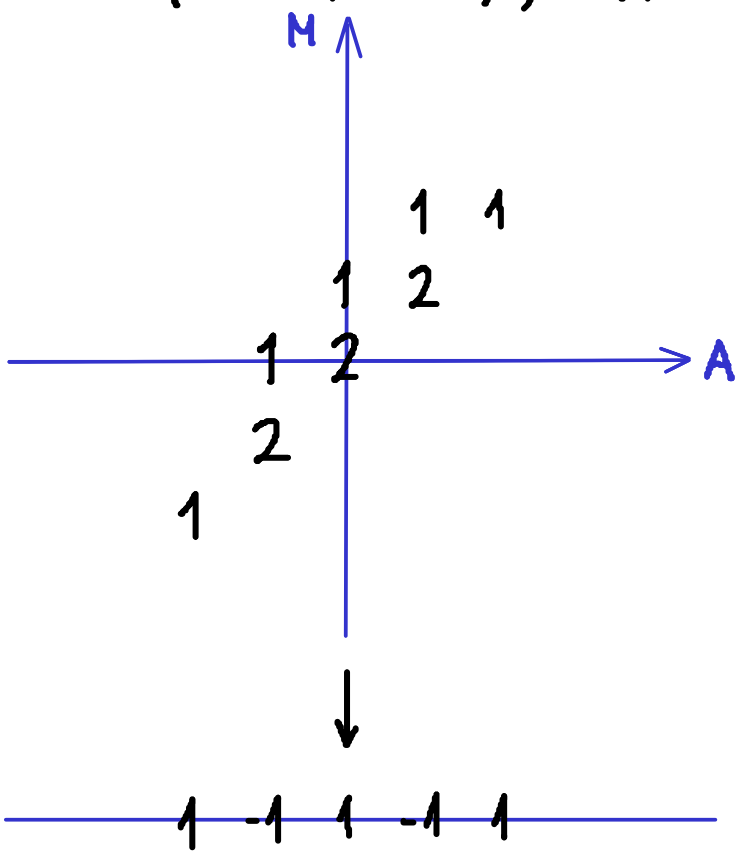

$\Delta_K(t) = t^{-2} - t^{-1} + 1 - t + t^2$

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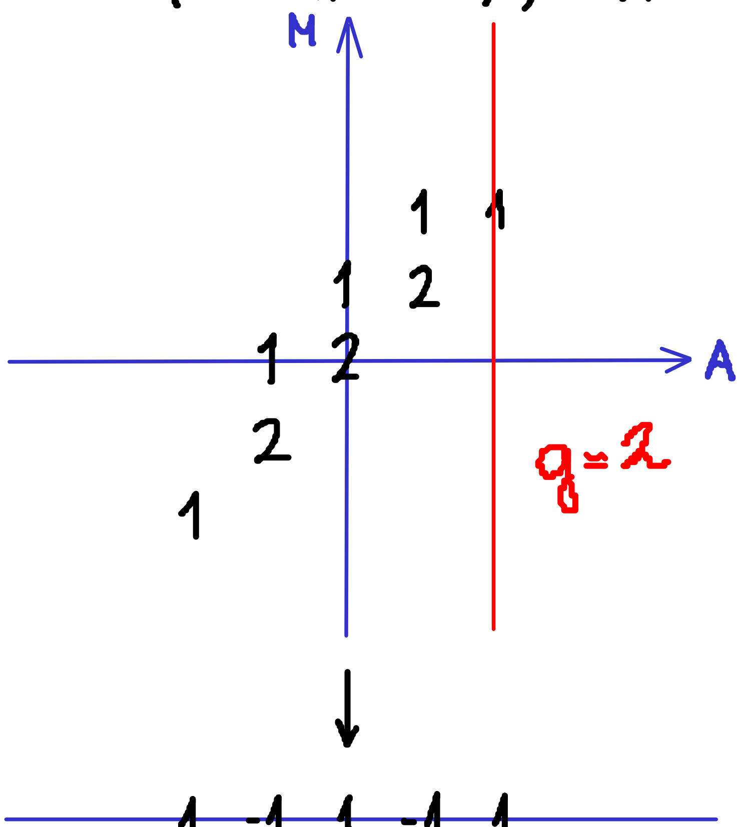

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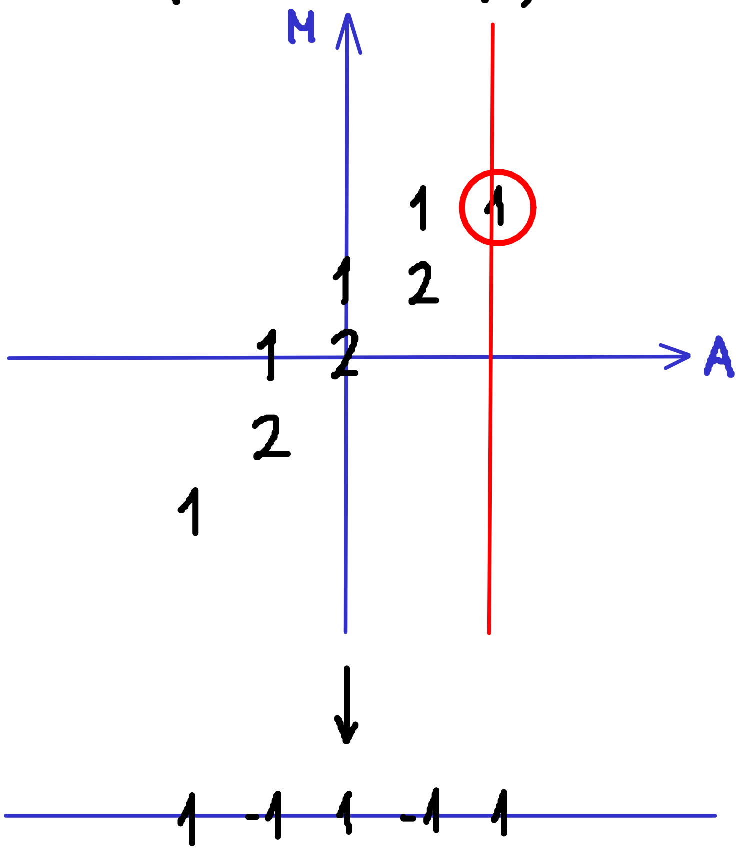

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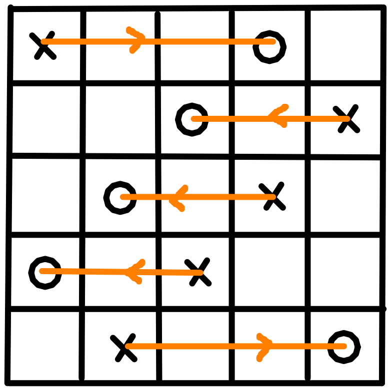
Grid Diagrams for Knots

X			O	
		O		X
	O		X	
O		X		
	X			O

- every row/column contains one "X" and one "O"

Grid Diagrams for Knots



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- connect "X" to "O" horizontally

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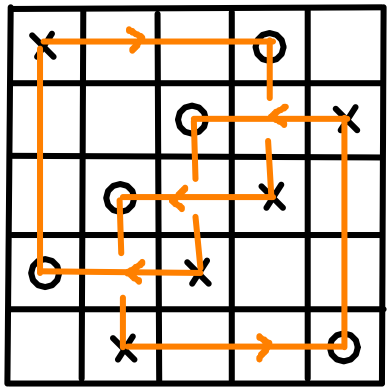

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- connect "O" to "X" vertically
- vertical strands are OVER the horizontal strands

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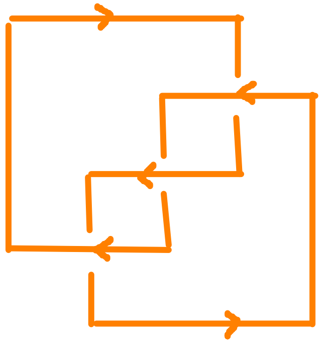

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$\rightsquigarrow$ knot (or link) in $\mathbb{R}^3$

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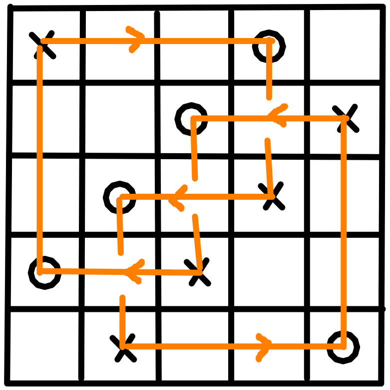

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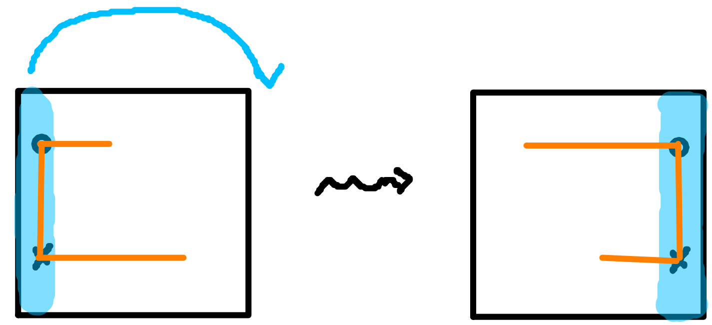
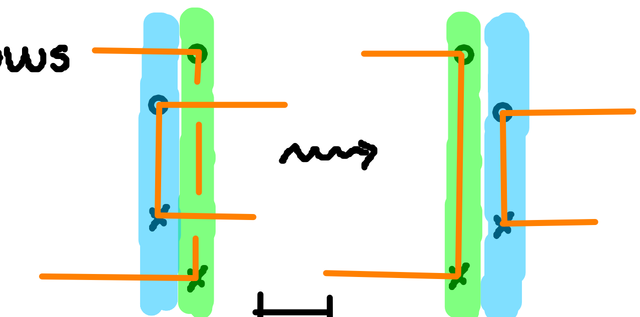
<u>grid moves</u>:

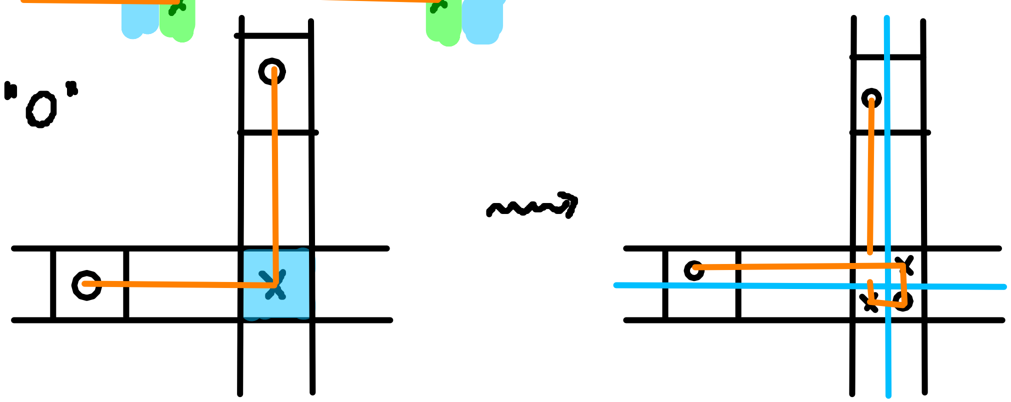
- cyclic permutation of columns and rows



- commutation of columns and rows whose "X"s and "O"s do not overlap

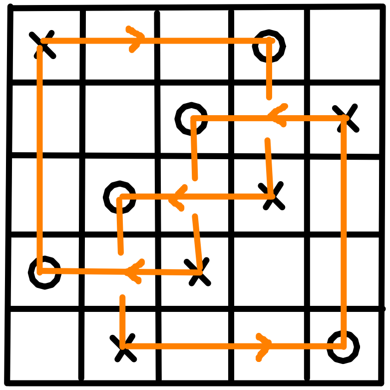


- stabilization of an "X" or an "O"



these moves give isotopic knots

Grid Diagrams for Knots



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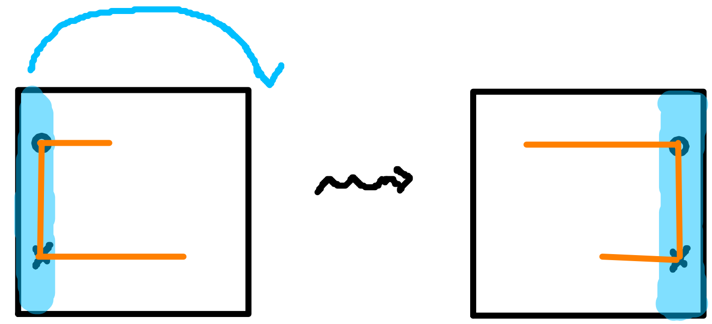

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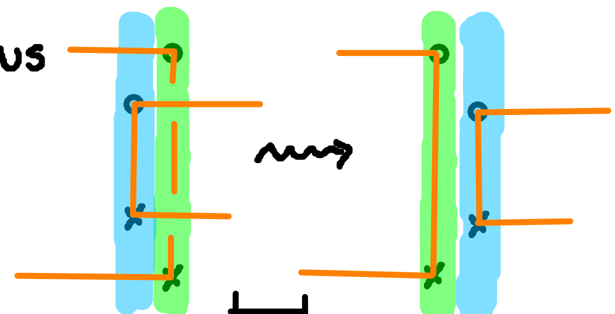

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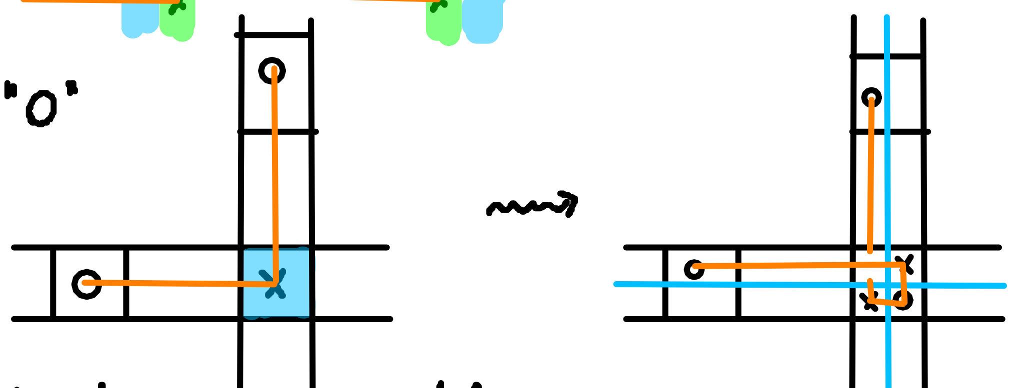
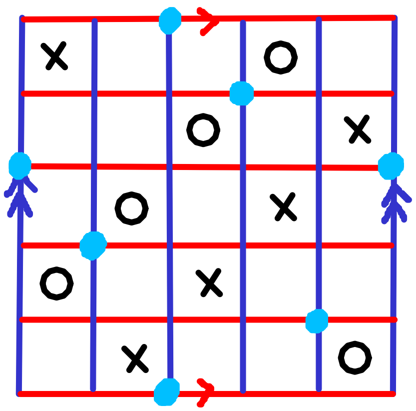


Thm (Cromwell, Dynnikov) two grid diagrams define isotopic knots if they are related by a sequence of grid moves.

KNOT FLOER HOMOLOGY

Strategy: define a homology for any grid diagram
then prove it is independent of grid moves



- identify the left and right, and top and bottom of the grid

generators: N-tuples of intersection points such that
- every horizontal line contains 1 point
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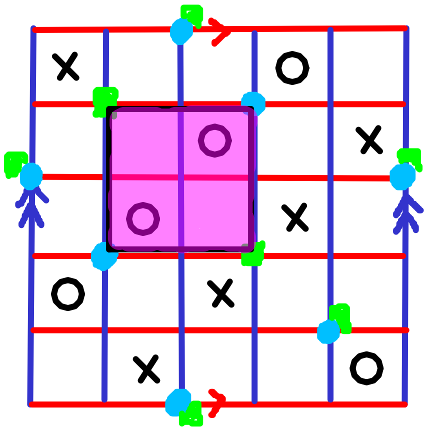

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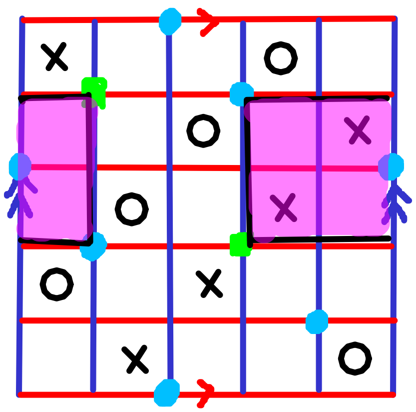

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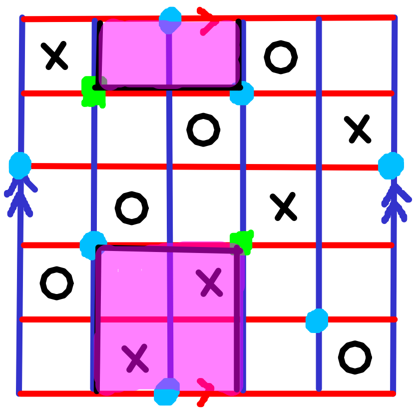

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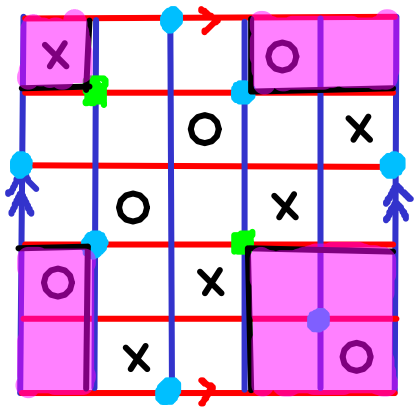

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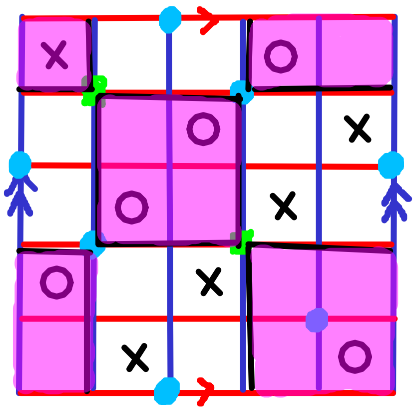

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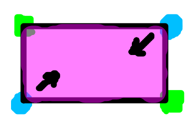
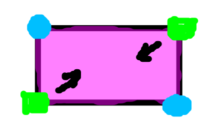
- 2 of which goes from to :  the other 2 from to 

- a rectangle is emty if there is no "X", "O", other point in its interior

$$\hat{\partial}\underline{x} = \sum_{\exists \text{ emty rectangle } \underline{x} \to y} y$$

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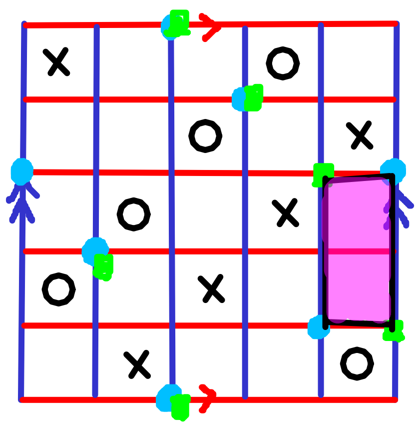

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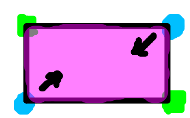
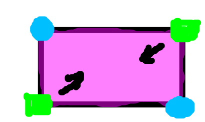
- 2 of which goes from • to ■ :  the other 2 from ■ to • 

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e.g. $\hat{\partial}$ • = ■

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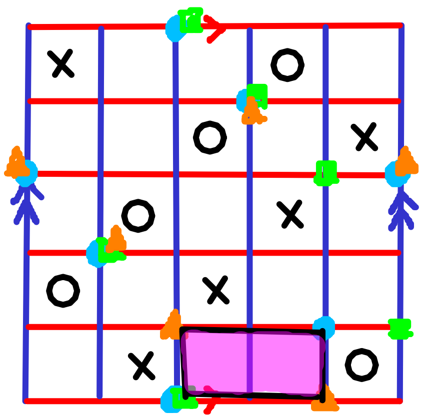

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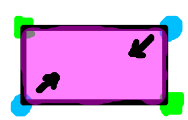
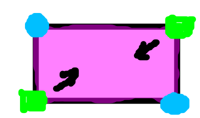
generators: N-tuples of intersection points such that
- every horizontal line contains 1 point
- every vertical line contains 1 point

$$\widehat{CFK}(G) = \bigoplus \widehat{CFK}_M(G)$$

differential: $\hat{\partial} : \widehat{CFK}(G) = \bigoplus \widehat{CFK}_M(G) \circlearrowleft -1$ given by emty rectangles:

- and differs in exactly 2 coordintes
- they span 4 rectangles on the torus
- 2 of which goes from to :  the other 2 from to 

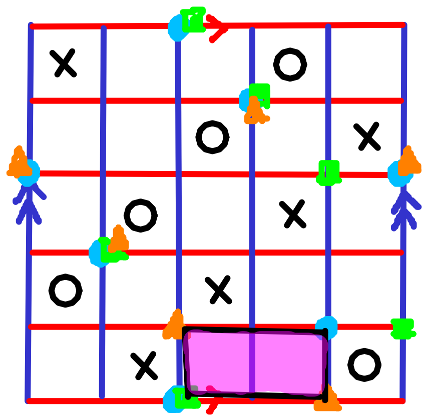
- a rectangle is emty if there is no "X", "O", other point in its interior

$$\hat{\partial} \underline{x} = \sum_{\exists \text{ emty rectangle } x \to y} y$$

e.g. $\hat{\partial} \, \bullet = \blacksquare + \blacktriangle$

Knot Floer Homology

Strategy: define a homology for any grid diagram
then prove it is independent of grid moves



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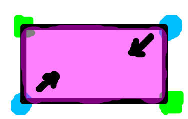
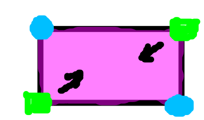
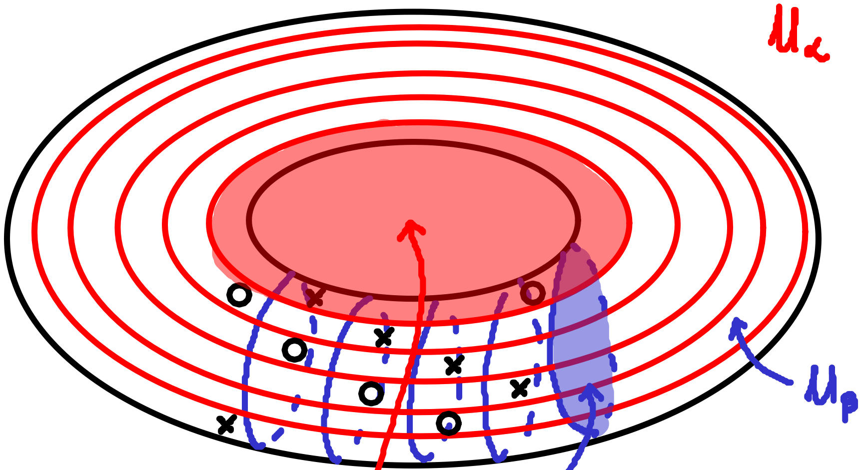
- ● and ■ differs in exactly 2 coordintes
- they span 4 rectangles on the torus
- 2 of which goes from ● to ■ :  the other 2 from ■ to ● 

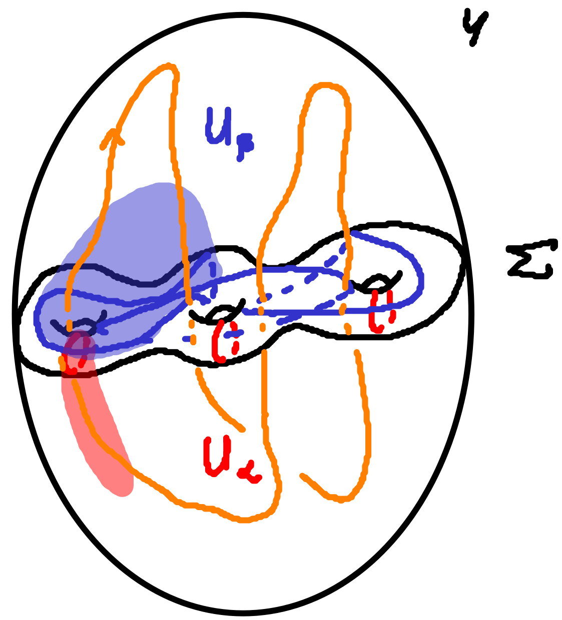
- a rectangle is emty if there is no "X", "O", other point ● = ■ in its interior

$$\hat{\partial} \underline{x} = \sum_{\exists \text{ emty rectangle } \underline{x} \to \underline{y}} \underline{y}$$

Thm (Manolescu - Ozsváth - Sarkar) $H_*(\widehat{CFK}, \hat{\partial})$ gives an invariant of K

GENERALIZATION





Heegaard decomposition of S^3 $\longrightarrow$ Heegaard decomposition of Y

α-curve bound discs in U_α $\longrightarrow$ α-curves on Σ

β-curve bound disc in U_β $\longrightarrow$ β-curves on Σ

X + intersection of K w/ T^2 $\longrightarrow$ one X and one O

O + intersection of K w/ T^2 $\longrightarrow$ in each comp of $\Sigma - \alpha$ & $\Sigma - \beta$

generators: N-tuples of α & β curves; one on each $\longrightarrow$

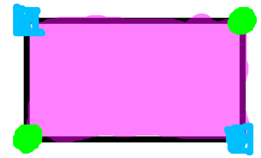
boundary map: rectangles - - - - - -> holomorphic curves in $\Sigma \times D^2$

Thm (Ozsváth-Szabó): the homology of gives an invariant $\widehat{HFK}(Y,K)$

Merci de
votre
Attention !

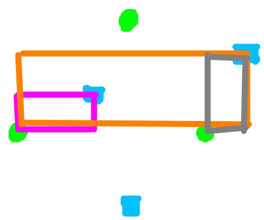
GRADINGS

Remember: $\hat{\partial}$ removed an inversion



Notation $\mathcal{J}(A,B) := \#\{\, {}_a\square^b : a \in A,\ b \in B \}$

e.g.:



$\mathcal{J}(A,B) = 3$

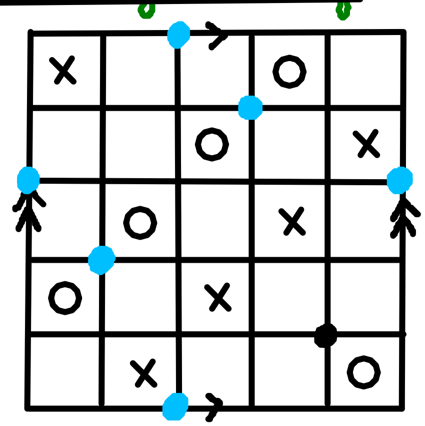
$$\mathcal{J}(A-B,C) := \mathcal{J}(A,C) - \mathcal{J}(B,C)$$

$\underline{x}$ generator, $\underline{X}$ set of "X"s, $\underline{O}$ set of "O"s on the grid

Maslov grading: $M(\underline{x}) = \mathcal{J}(\underline{x}-\underline{O},\underline{x}-\underline{O}) + 1$

Alexander grading: $A(\underline{x}) = \frac{1}{2}\left(\mathcal{J}(\underline{x}-\underline{O},\underline{x}-\underline{O}) - \mathcal{J}(\underline{x}-\underline{X},\underline{x}-\underline{X}) - (N-1)\right)$

e.g.:



$M(\underline{x}) = 1$

$A(\underline{x}) = -3$

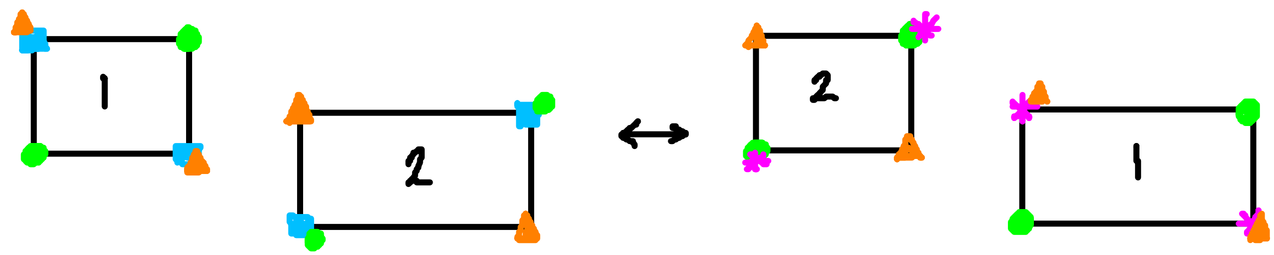
$(\widehat{CFK}, \hat{\partial})$ IS A CHAIN COMPLEX

Need to prove $\partial^2 = 0$

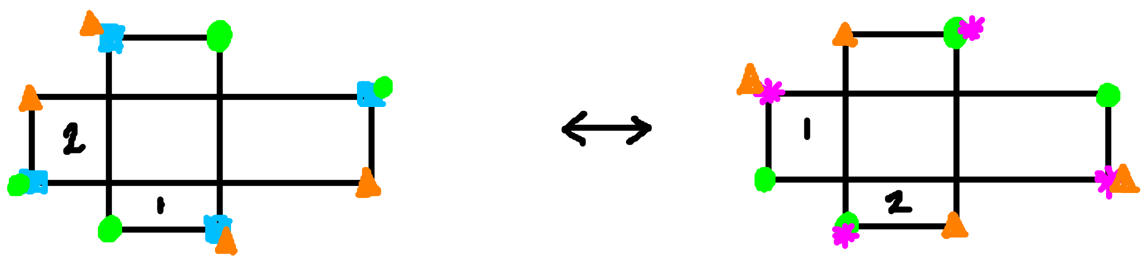
$$\partial^2 \underline{x} = \partial\Big(\sum_{\exists \text{ emty rectangle } \underline{x}\to\underline{y}} \underline{y}\Big) = \sum_{\exists \text{ emty rectangle } \underline{x}\to\underline{y}} \Big(\sum_{\exists \text{ emty rectangle } \underline{y}\to\underline{z}} \underline{z}\Big)$$

the coefficient of $\underline{z}$ is given by # 2 empty rectangles $\underline{x} \to \underline{y} \to \underline{z}$

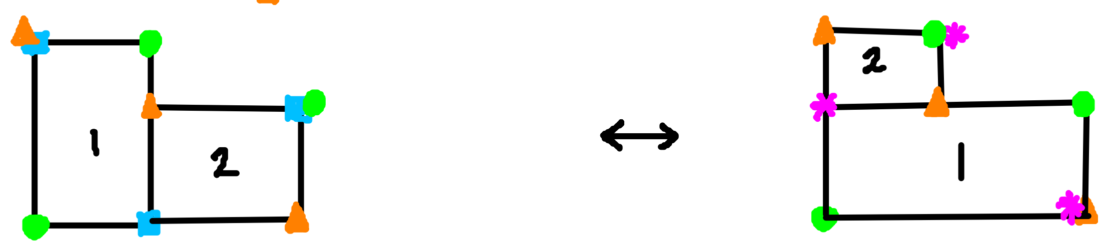
→ disjoint rectangles



→ the interiors intersect



→ common corner



$\Rightarrow \partial^2 = 0$ over $\mathbb{F}_2$

Thm (Manolescu - Ozsváth - Sarkar) $H_*(\widehat{CFK}, \hat{\partial}) \cong \widehat{HFK}(K) \otimes V^{\otimes N-1}$

where $V = (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{(-1,-1)}$

invariant of the knot